An Introduction to the Hardness vs. Randomness Paradigm

Tutorial for the Simons HDX and Pseudorandomness Workshop

Ronen Shaltiel

University of Haifa

August 2026

Outline

About these talks

- Who is this talk for?

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- Who is this talk for? I am not sure...

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Let's see how this goes...

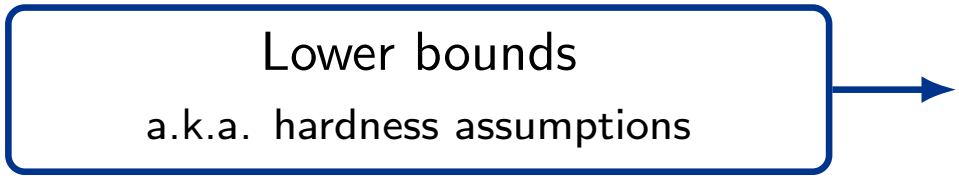
The hardness vs. randomness paradigm at a glance

The hardness vs. randomness paradigm at a glance

Lower bounds

a.k.a. hardness assumptions

The hardness vs. randomness paradigm at a glance



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Thm: [IW97]

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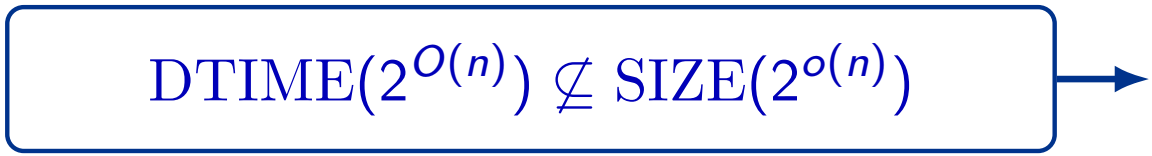

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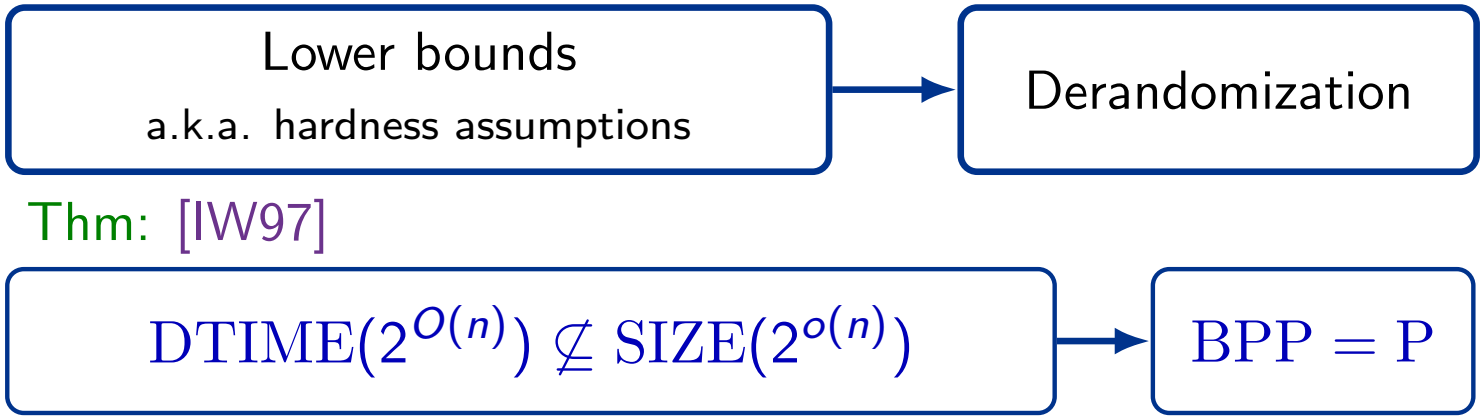

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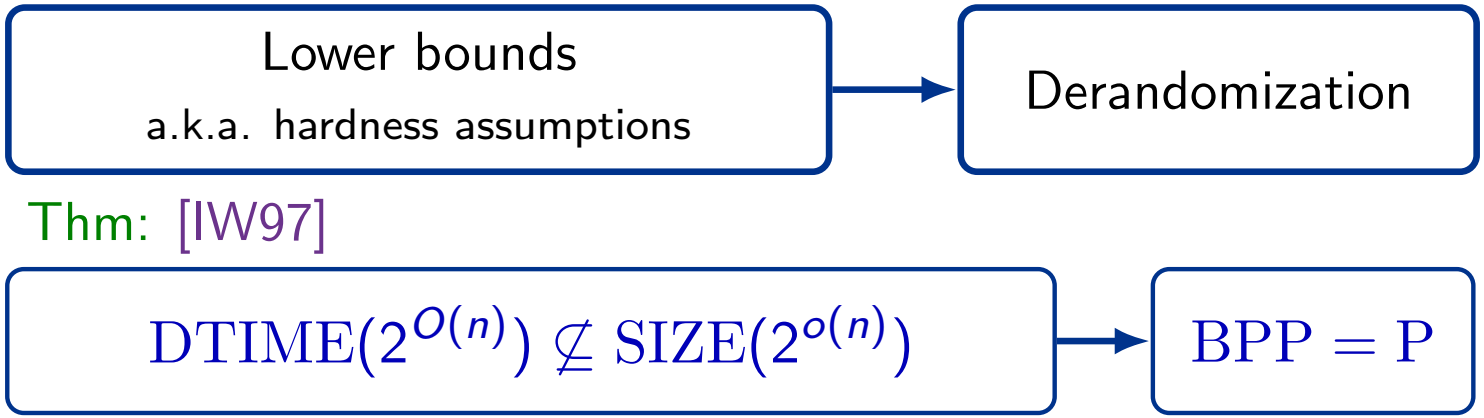

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Thm: [IW97]

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Interpretation: *CLIQUE* easy $\Rightarrow$ $3-SAT$ easy

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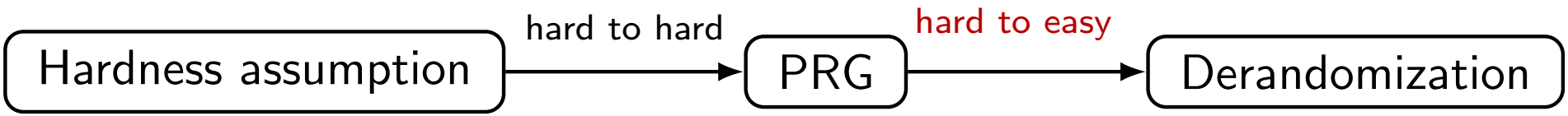

Plan for talks

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PRGs

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Derandomization
$\mathrm{BPP} = \mathrm{P}$

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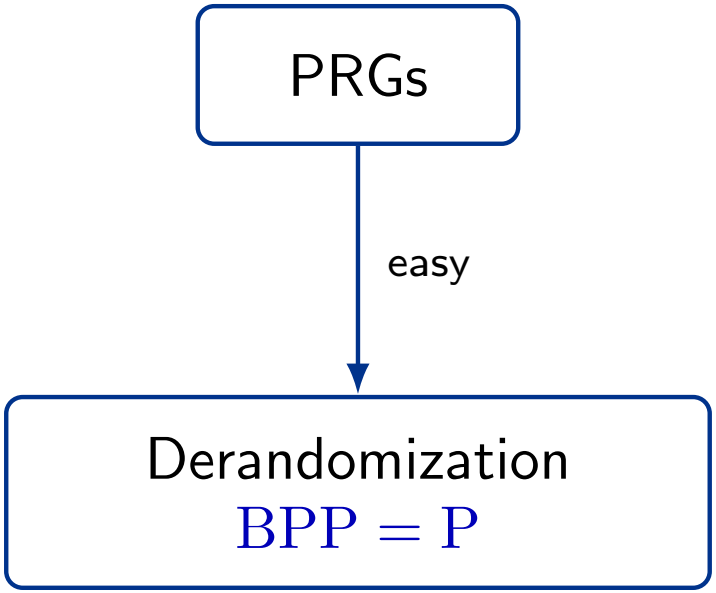

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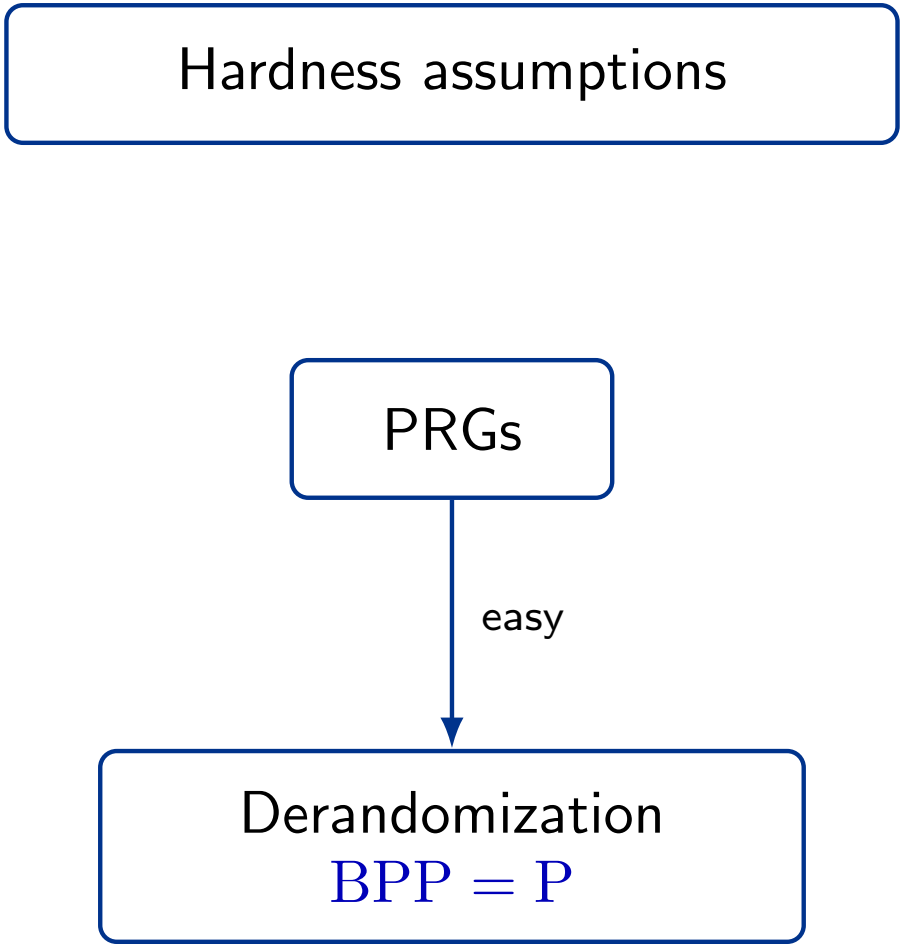

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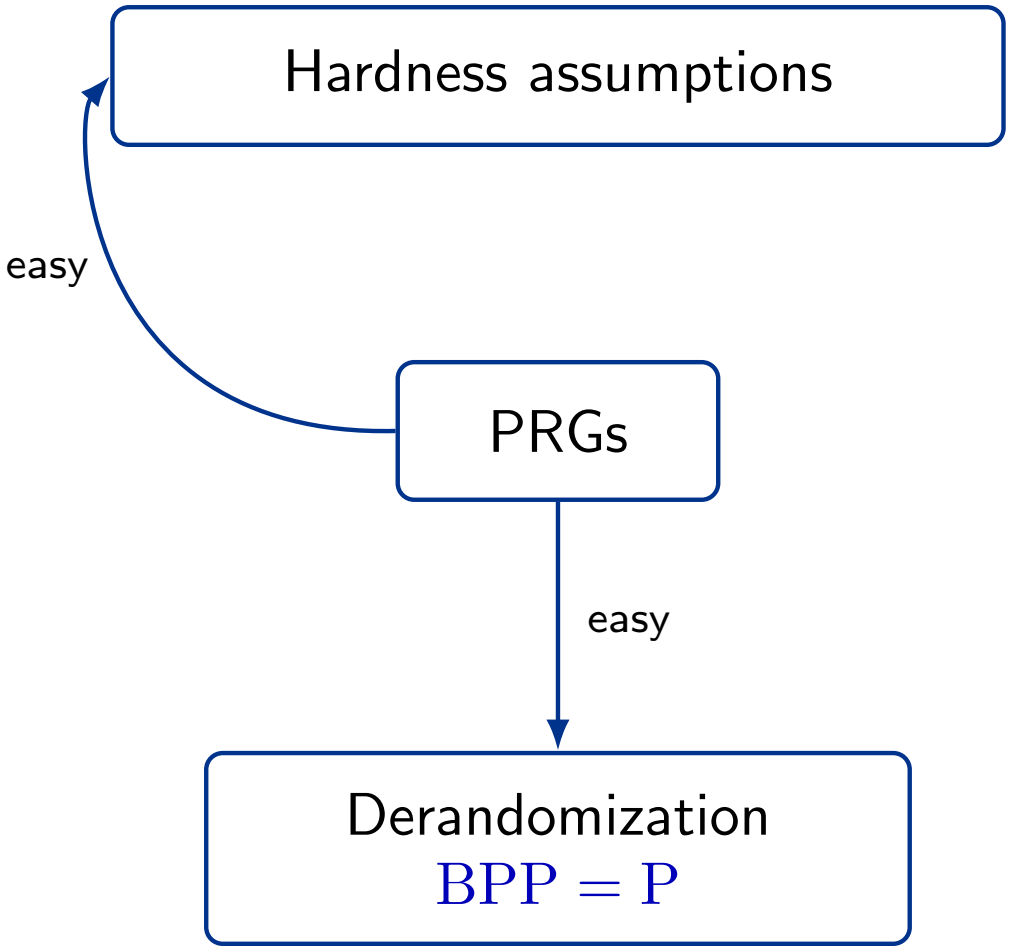

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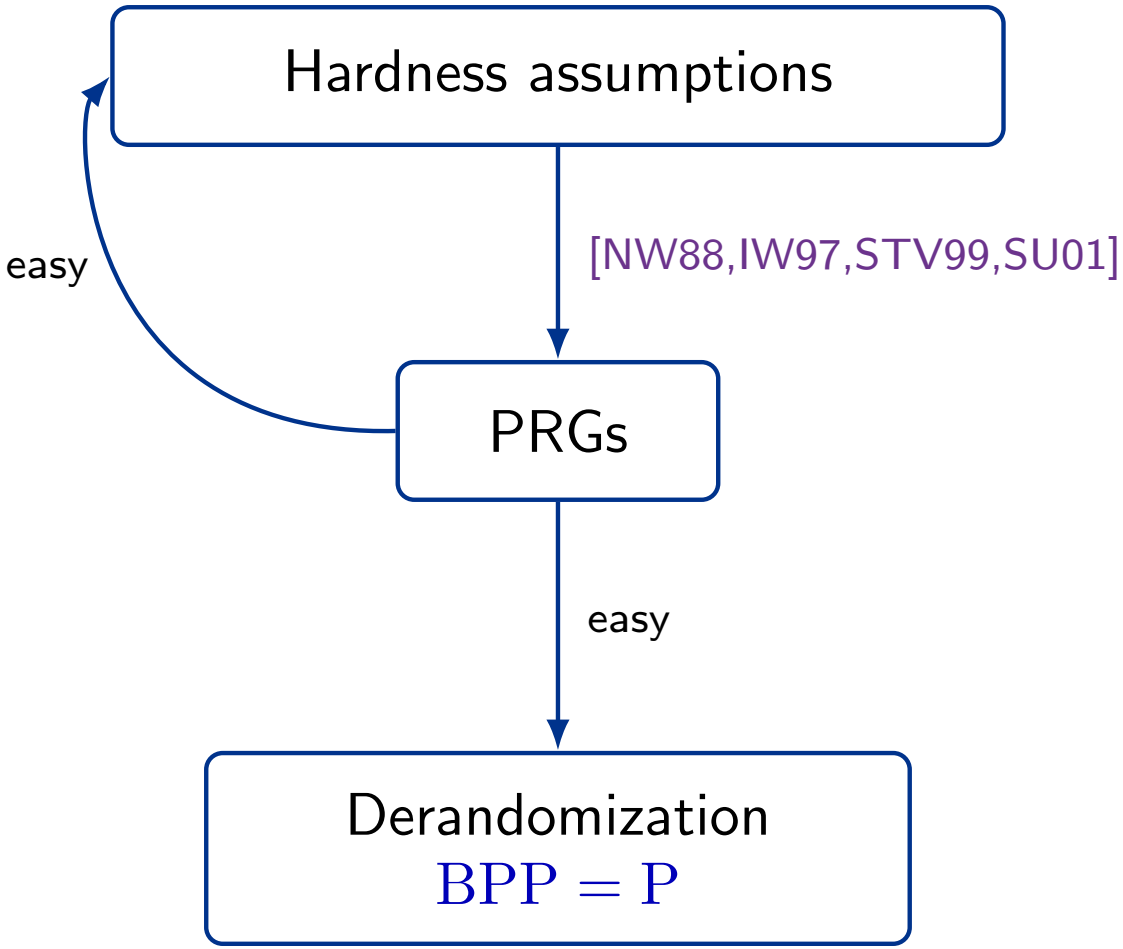

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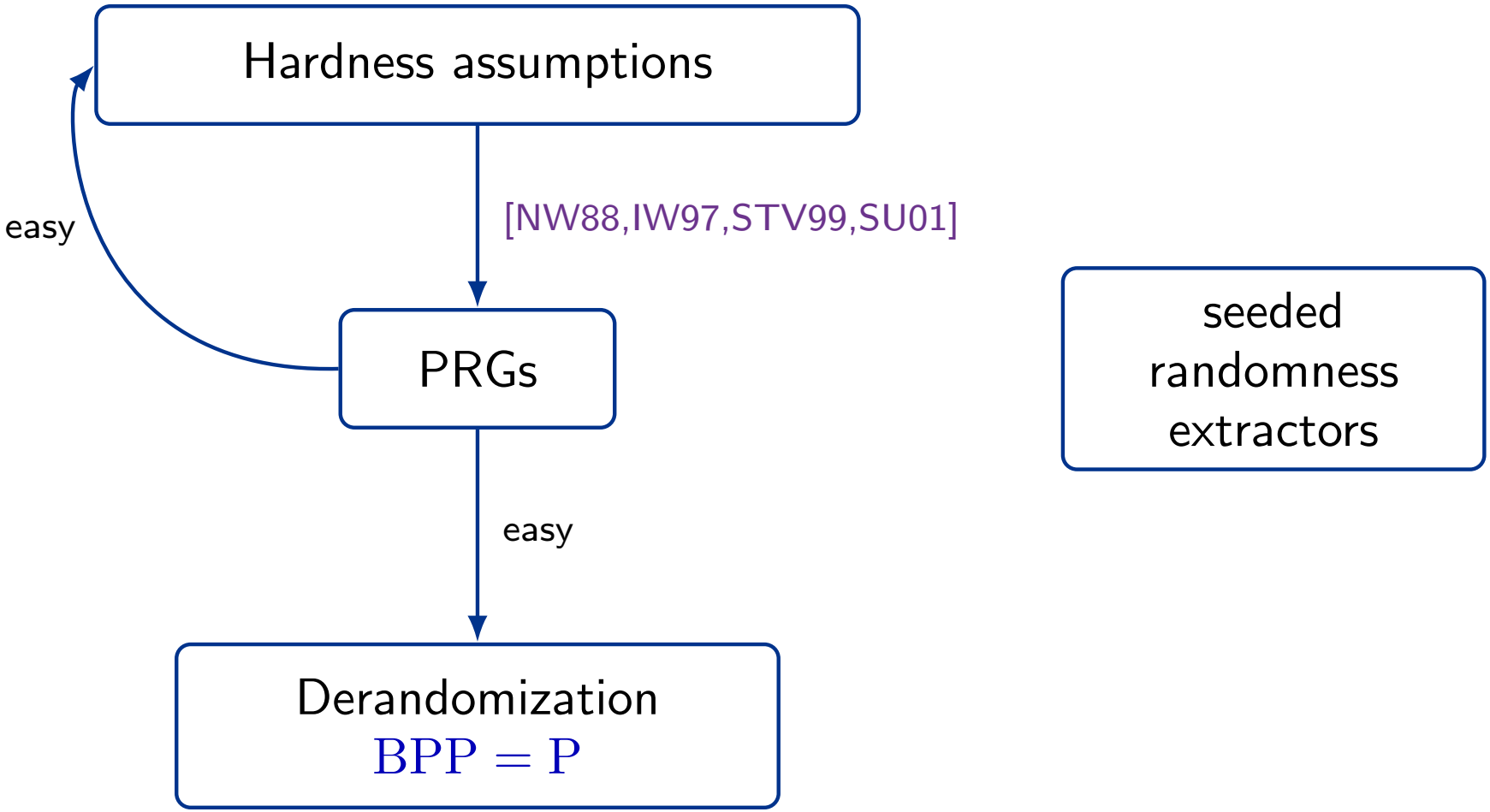

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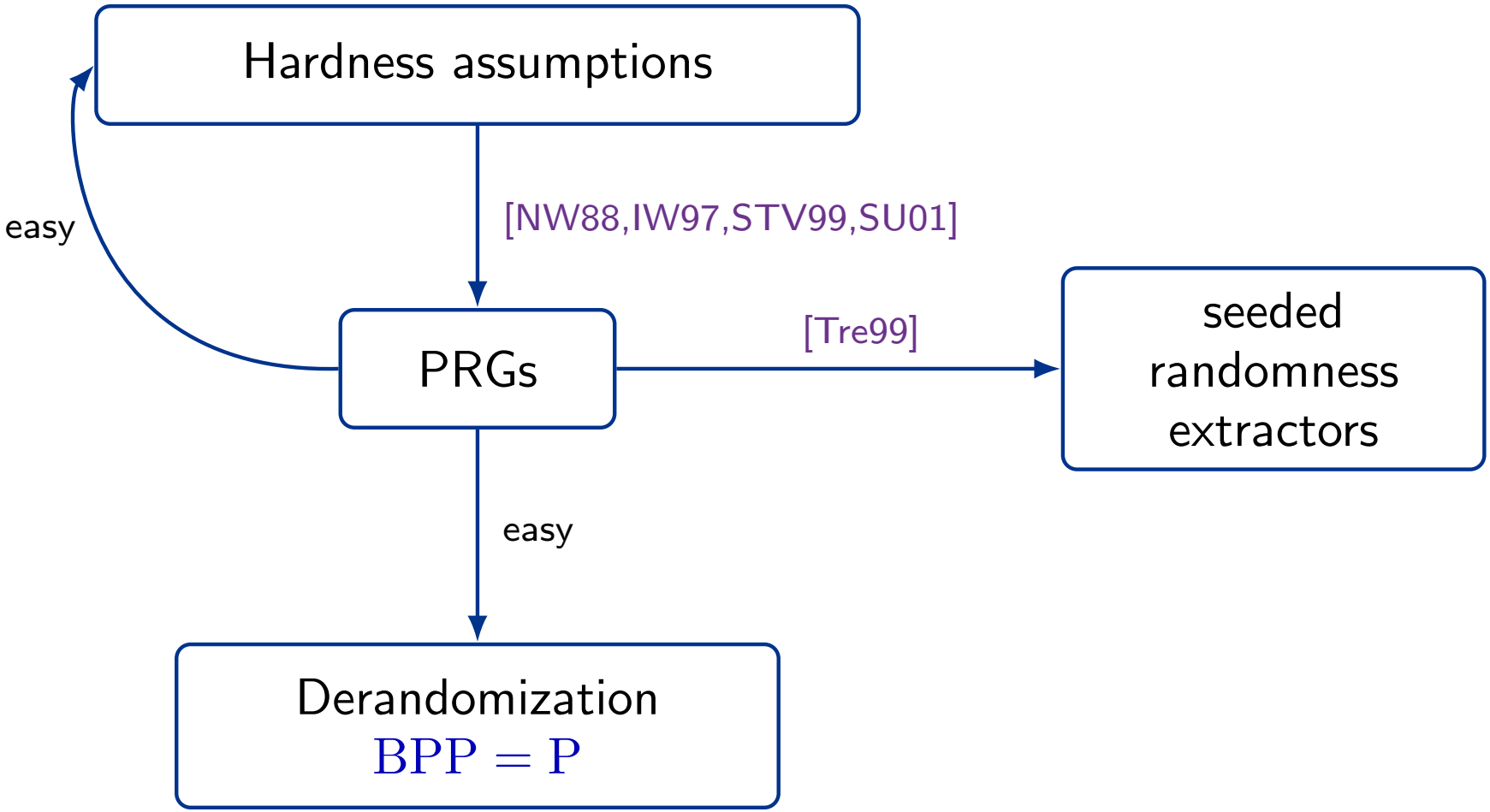
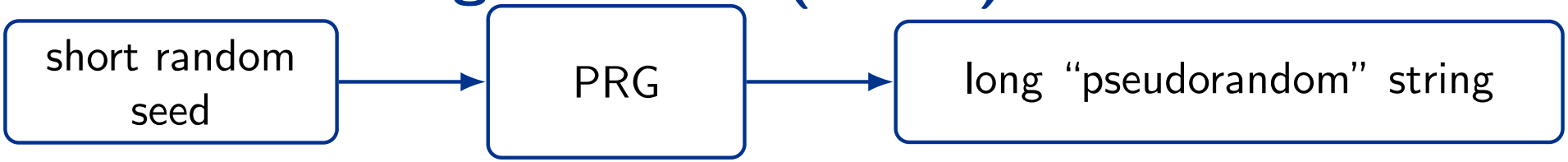


Pseudorandom generators (PRGs)



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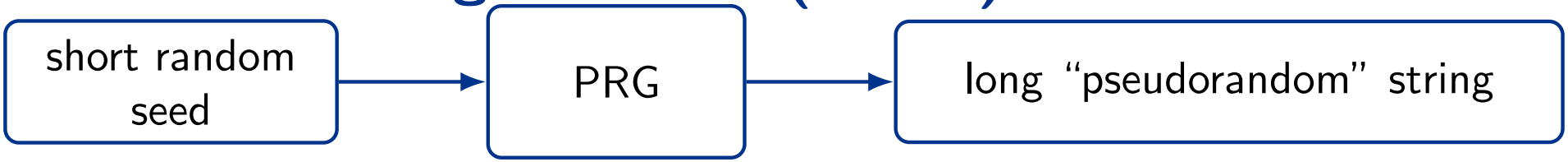

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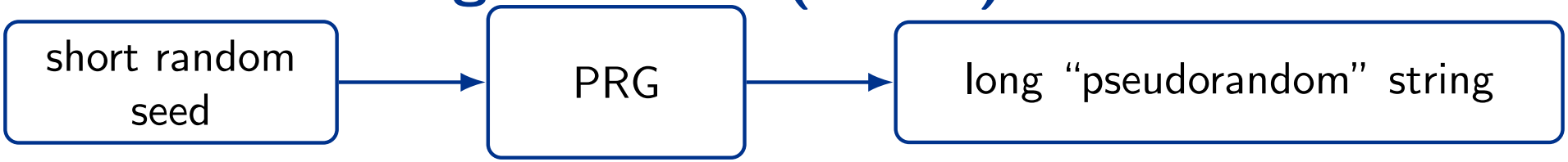

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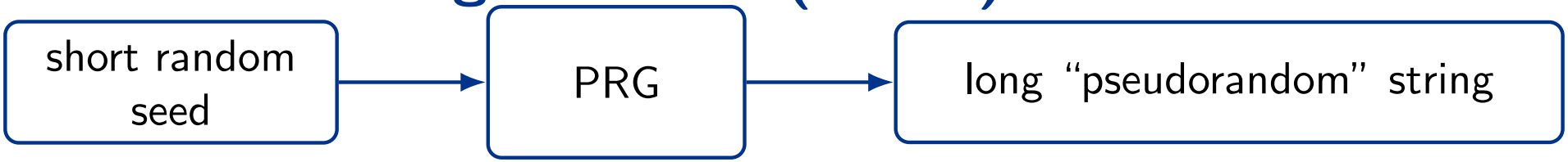

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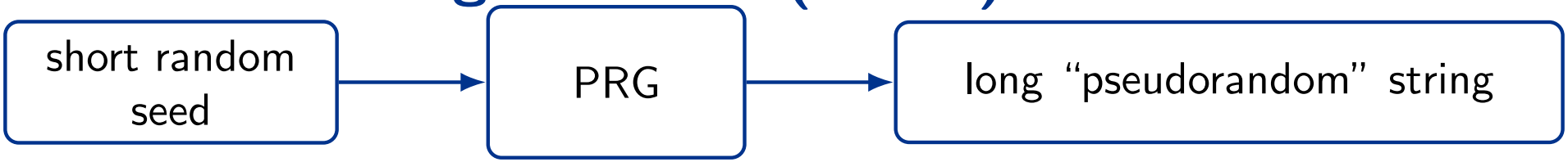

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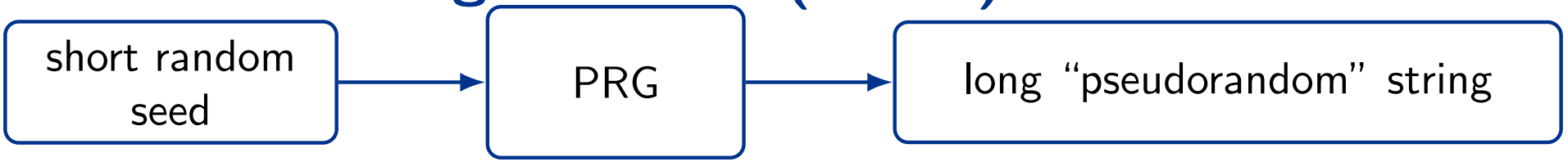

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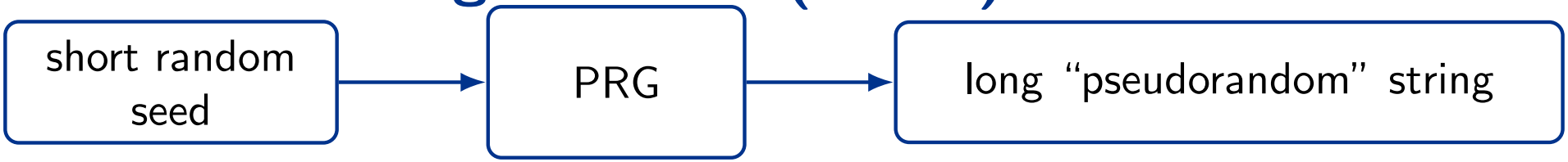

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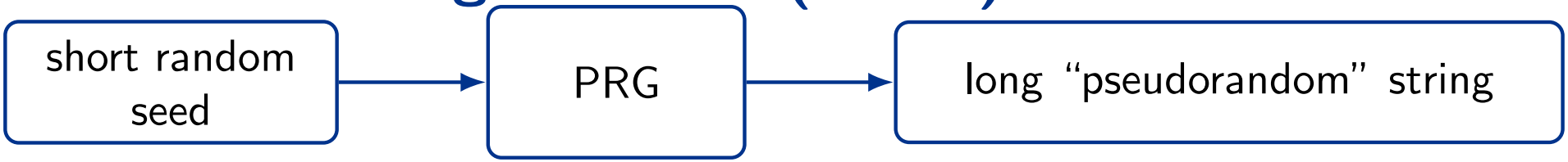

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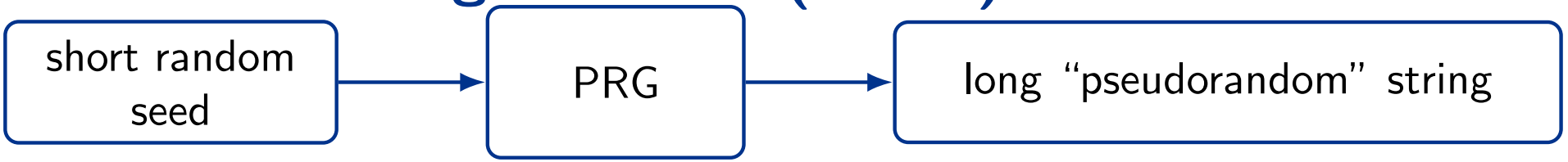

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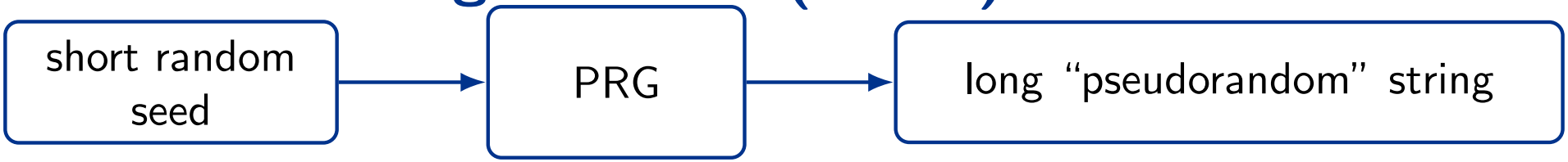

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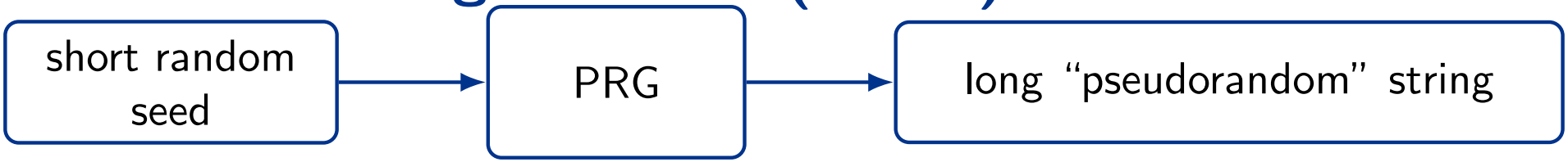

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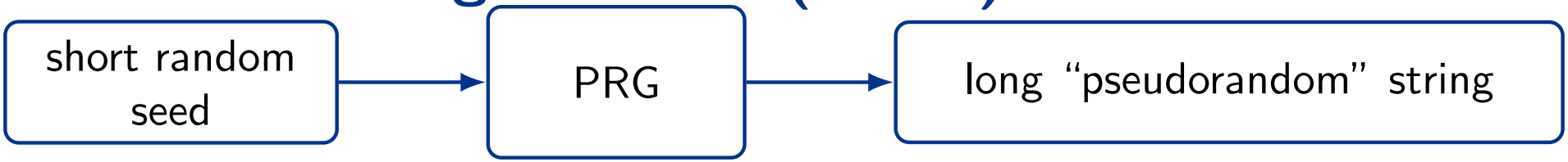

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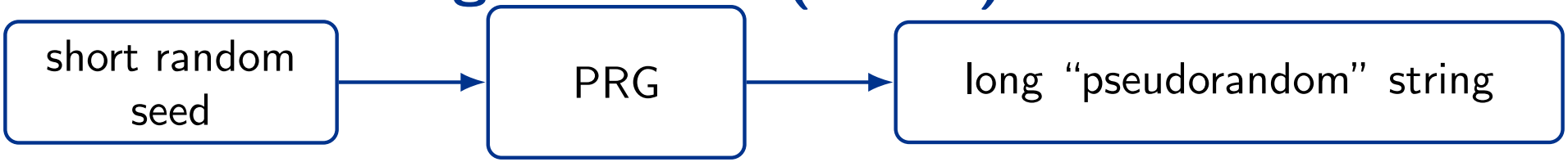
Pseudorandom generators (PRGs)



Definition (Pseudorandom generator (PRG))

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Throughout this talk, $\epsilon = 1/10$.

Big idea: "Randomness is in the eye of the beholder".

- Typically interested in a class of distinguishers. This talk: circuits of size m.
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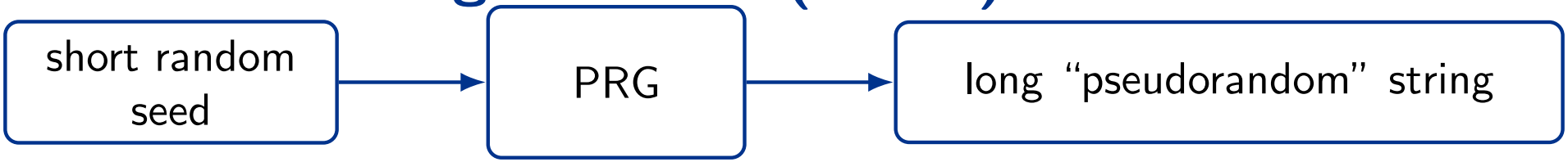

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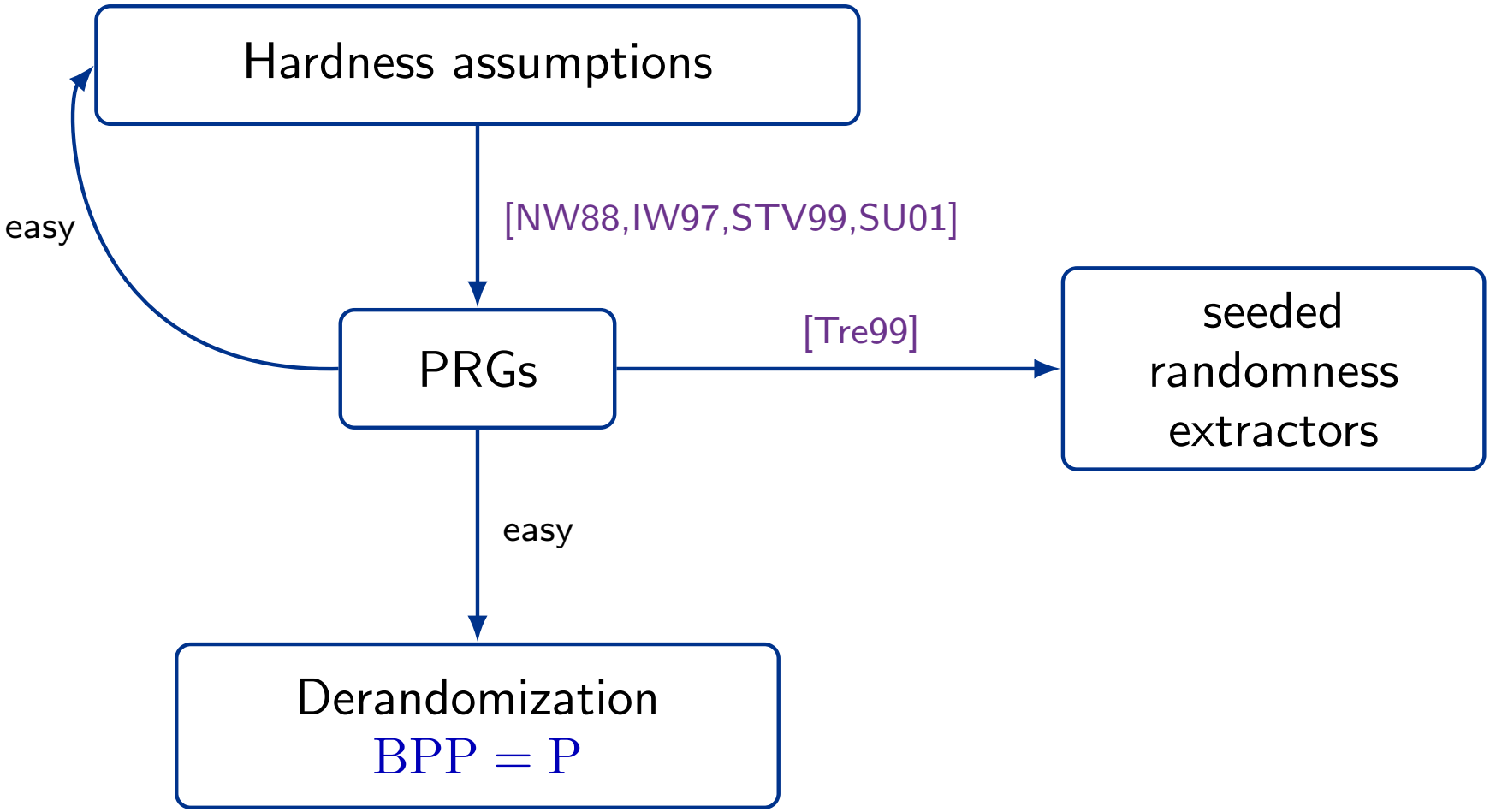
- Comment: Recent work replaces PRGs with "targeted PRGs".

Plan for talks



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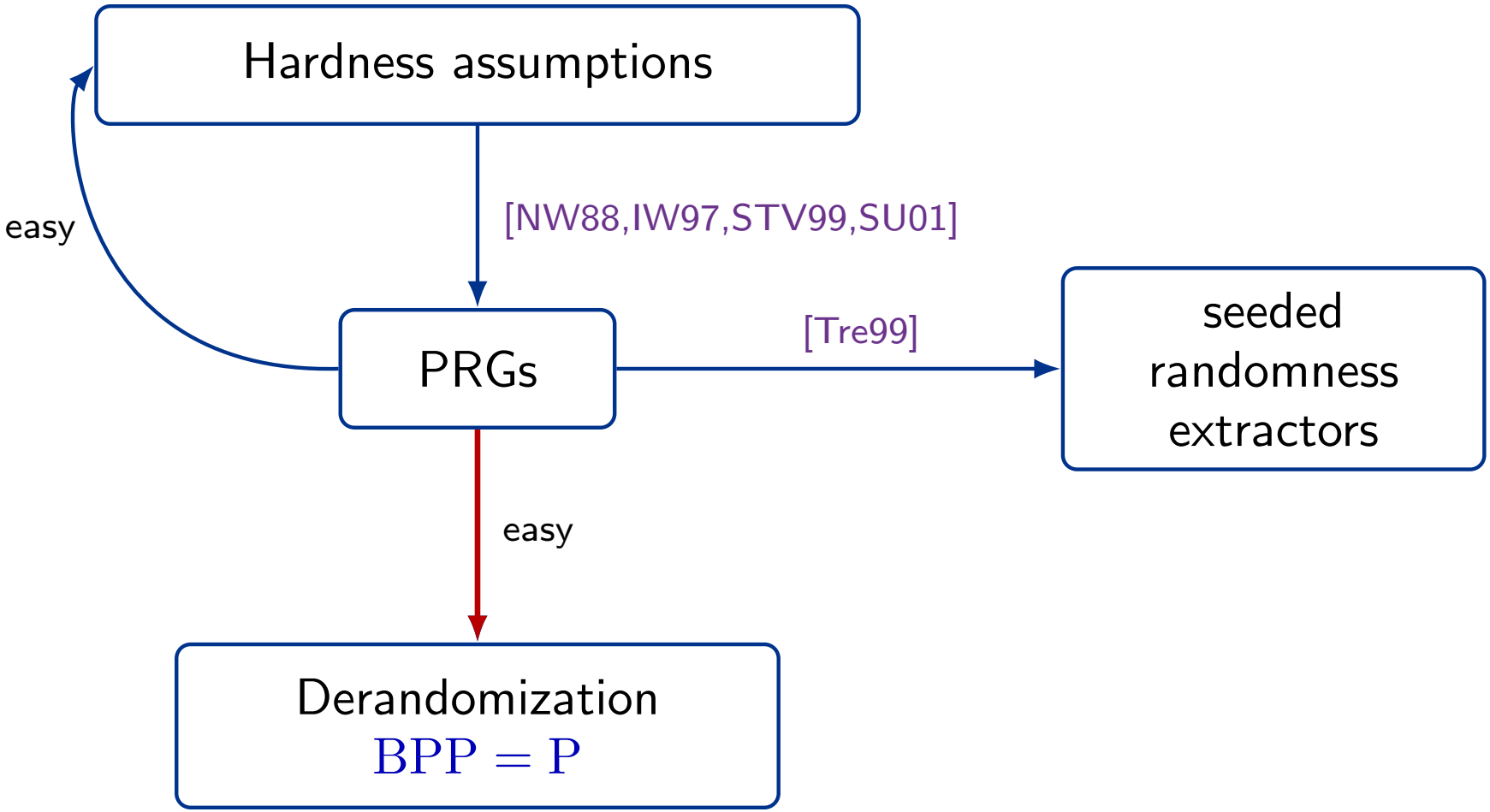

PRGs $\Rightarrow$ BPP $=$ P

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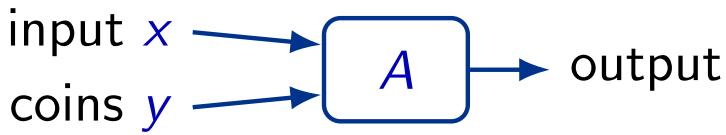

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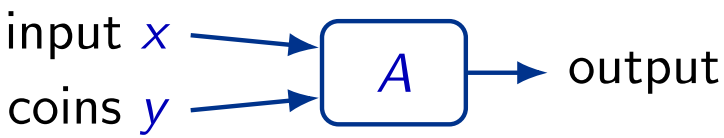

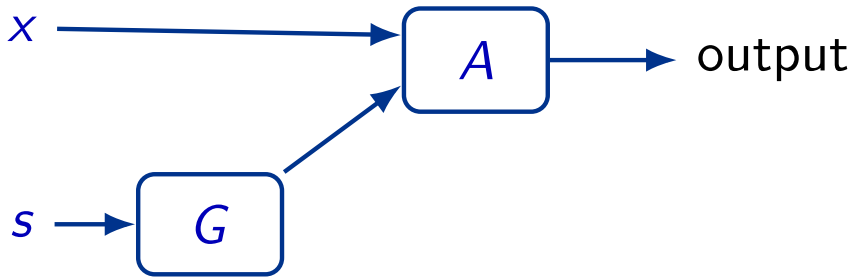

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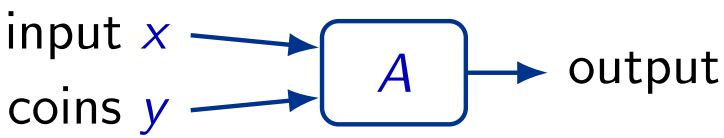

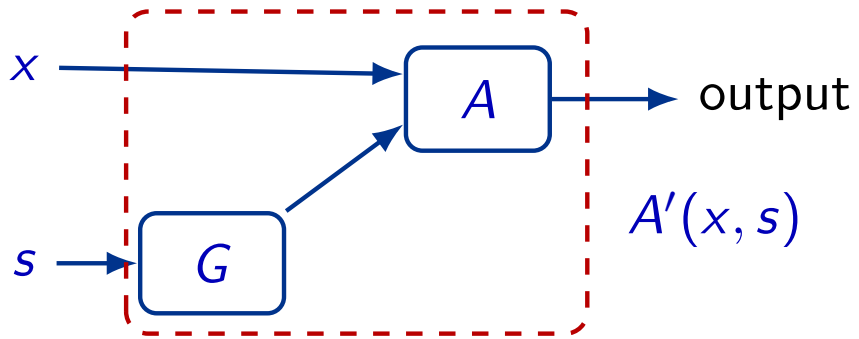

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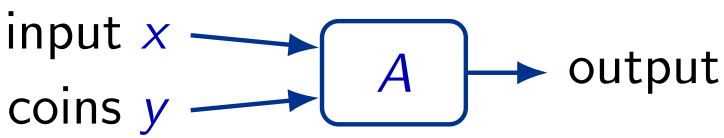

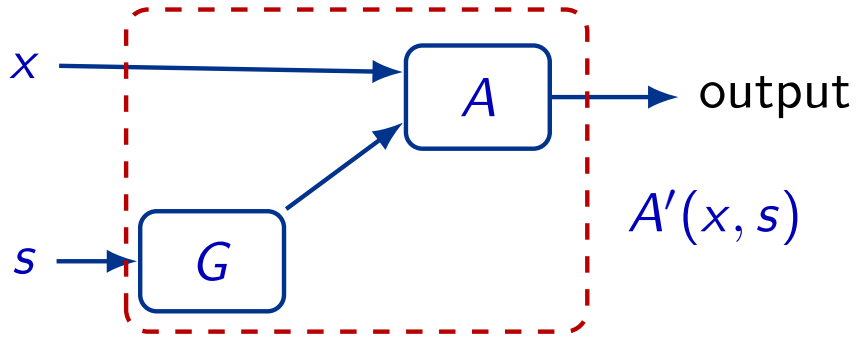

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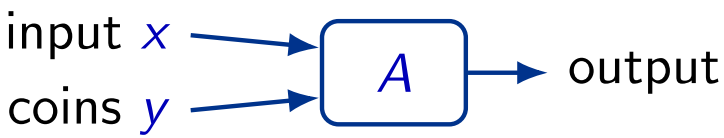

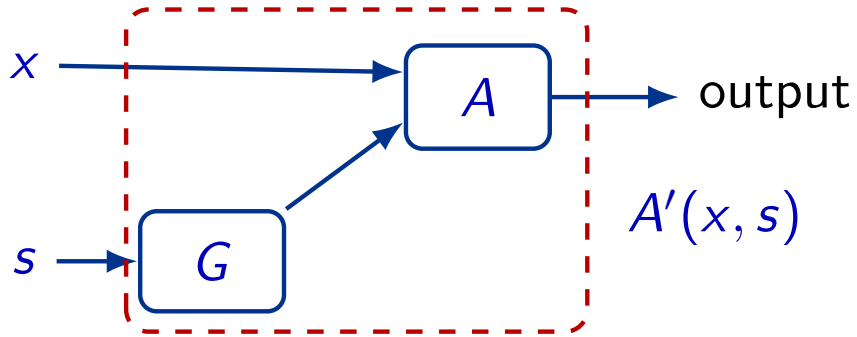

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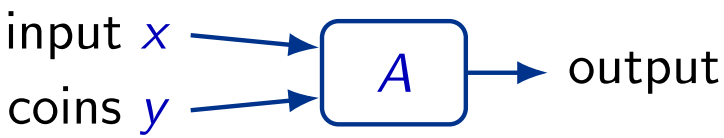

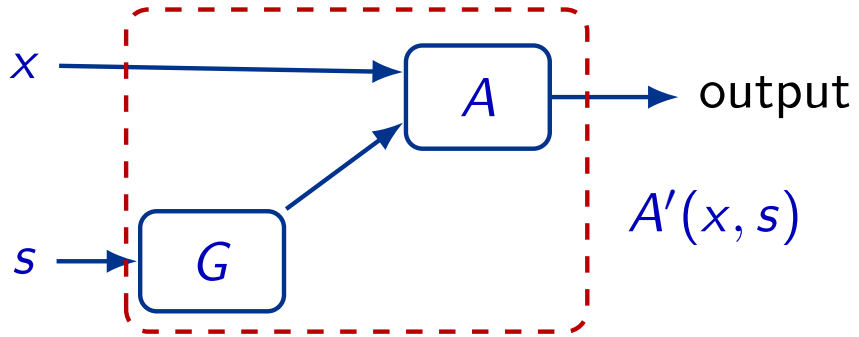

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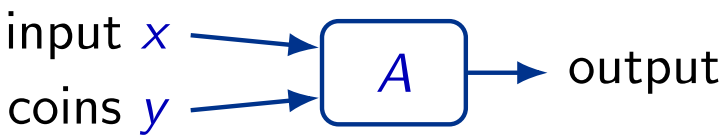

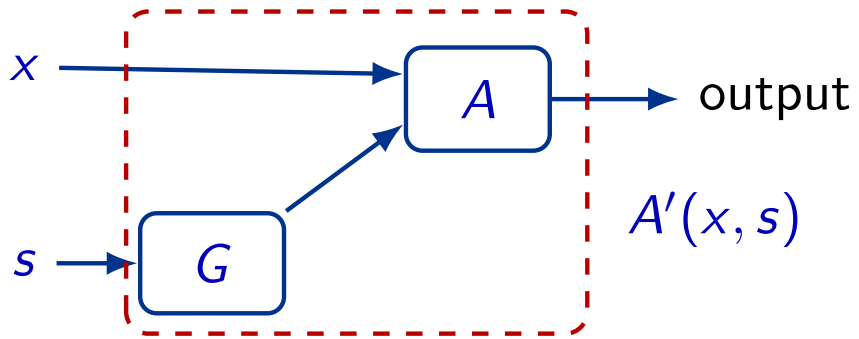

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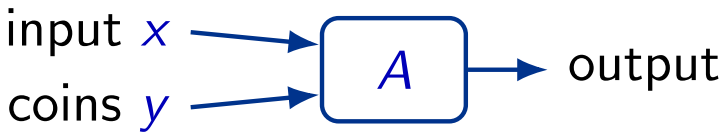

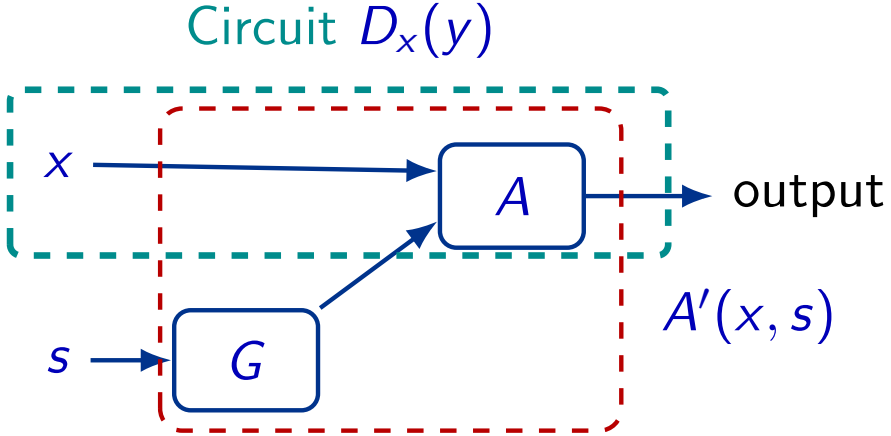

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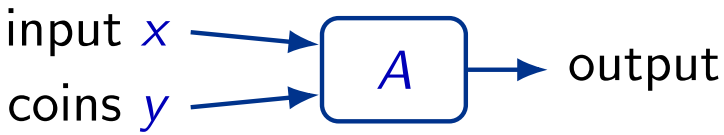

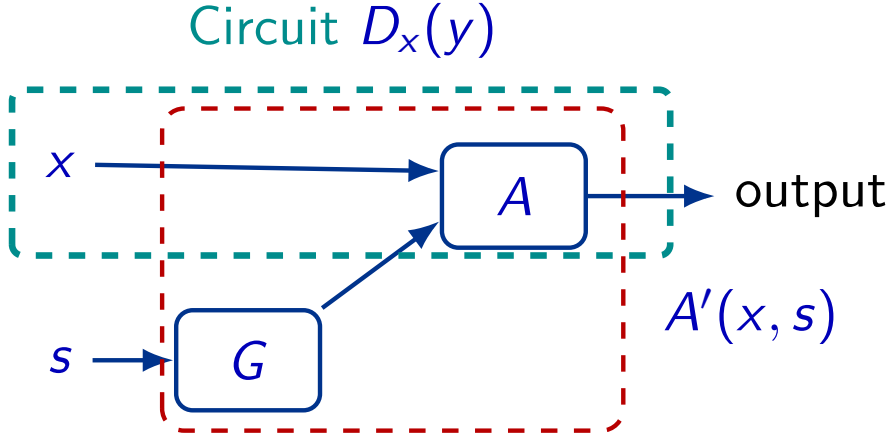

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Uniform derandomization is extensively studied [IW98,Kab00,Lu00,TV02,GW02, GST03,SU07,Sha09,KvMS09,CIS18,CRTY20,CT21,CRT22,BCT25...].

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Recent work: Keep assumption uniform, shoot for derandomization on every input.

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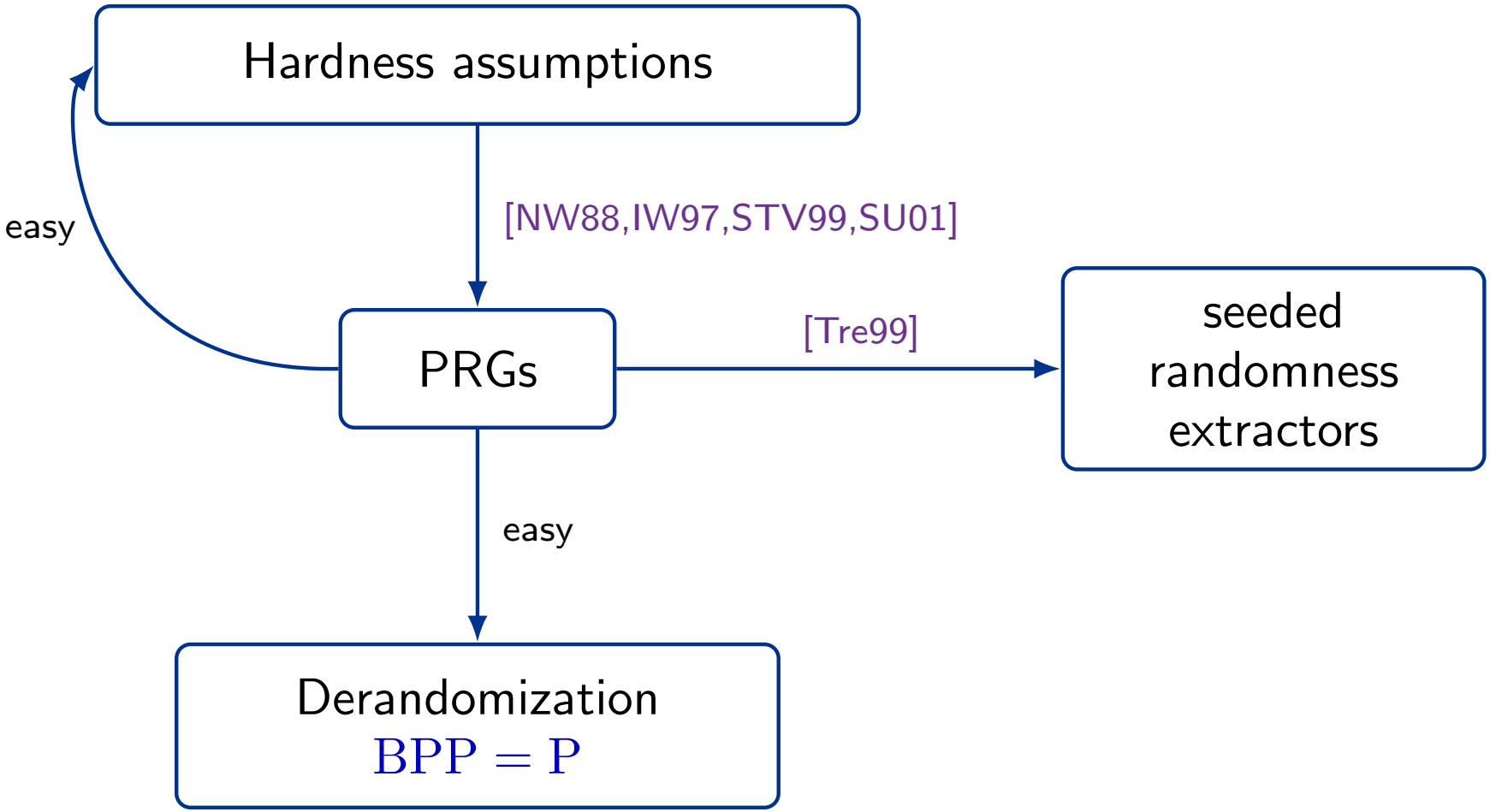
Comment: Some applications of PRGs (other than derandomization) require the full power of the definition, and targetd PRGs do not suffice.

Plan for talks



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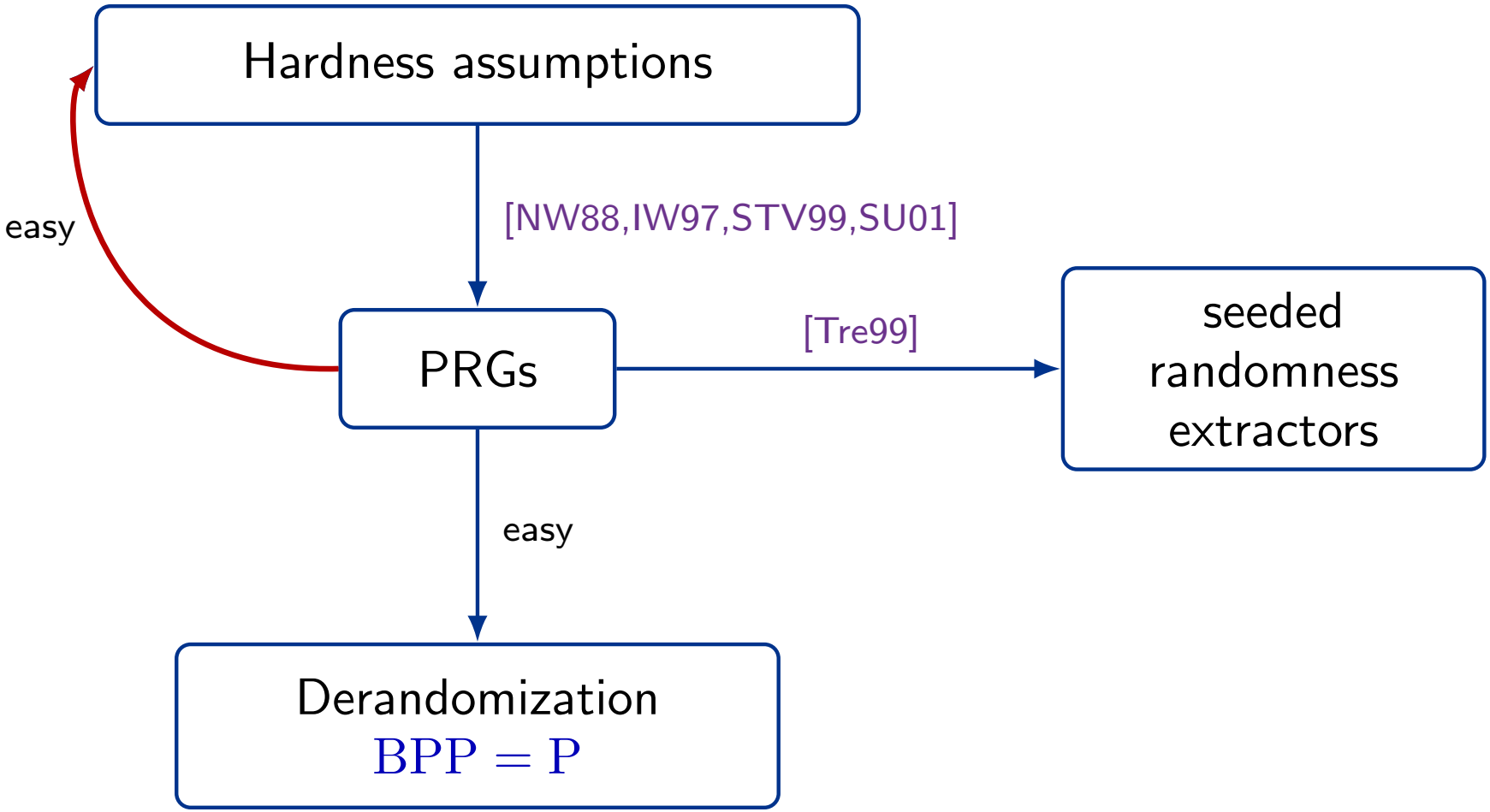

Hardness assumptions

Definition (Hard functions)

$f : \{0,1\}^n \to \{0,1\}$ is

- s-hard if $\forall$size s circuit C, $\exists x \in \{0,1\}^n$ s.t. $C(x) \neq f(x)$.
- s-average-case-hard if $\forall$size s circuit C, $\Pr_{x \leftarrow U_n}[C(x) = f(x)] \leq \frac{1}{2} + \frac{1}{s}$.

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Theorem (PRGs $\Rightarrow$ worst-case hardness (appears as an exercise))

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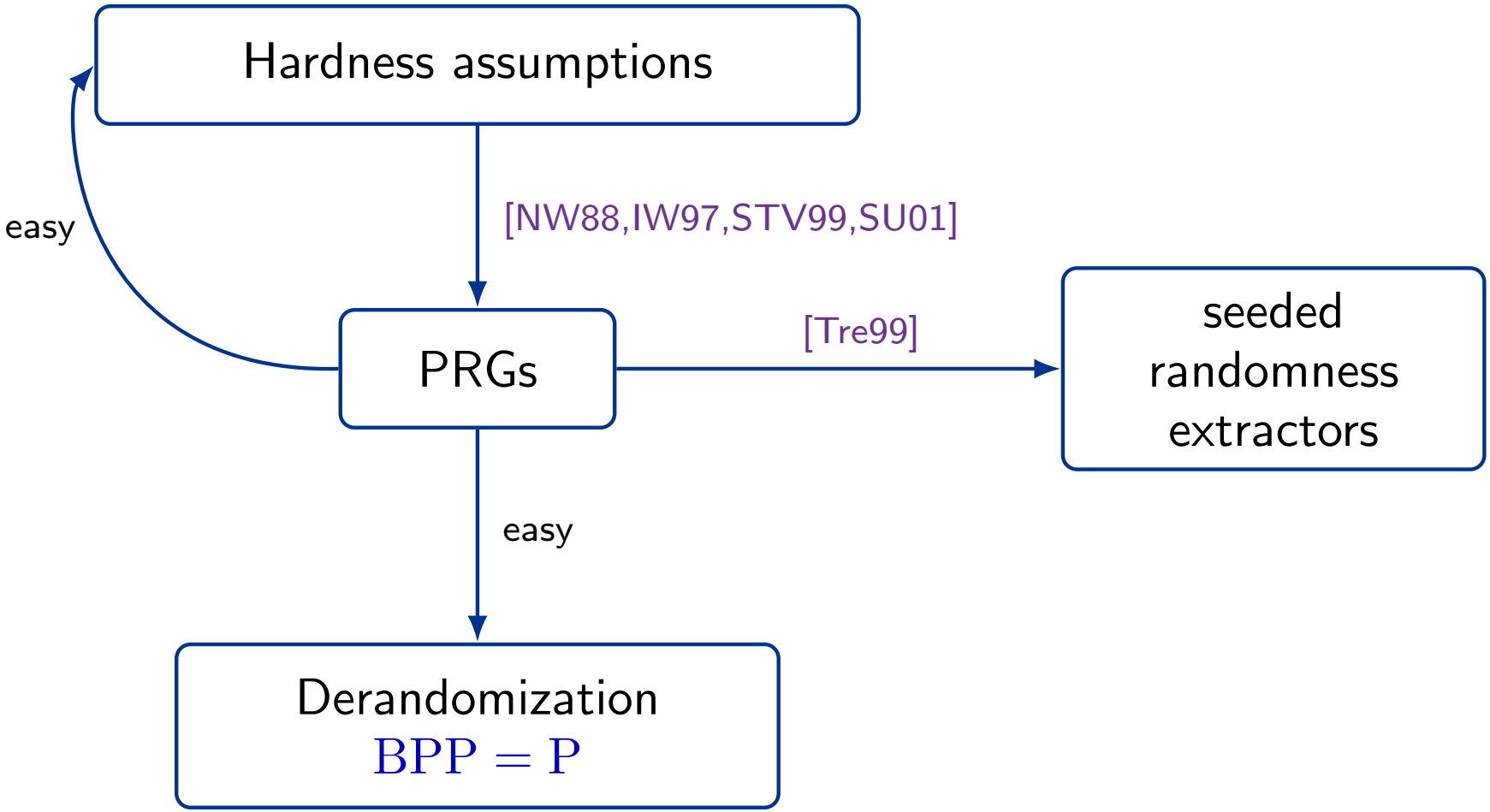
We don't have such lower bounds $\Rightarrow$ We cannot expect to construct explicit PRGs, unless we assume that E is hard for exponential size circuits.

Does derandomization imply hardness assumptions?



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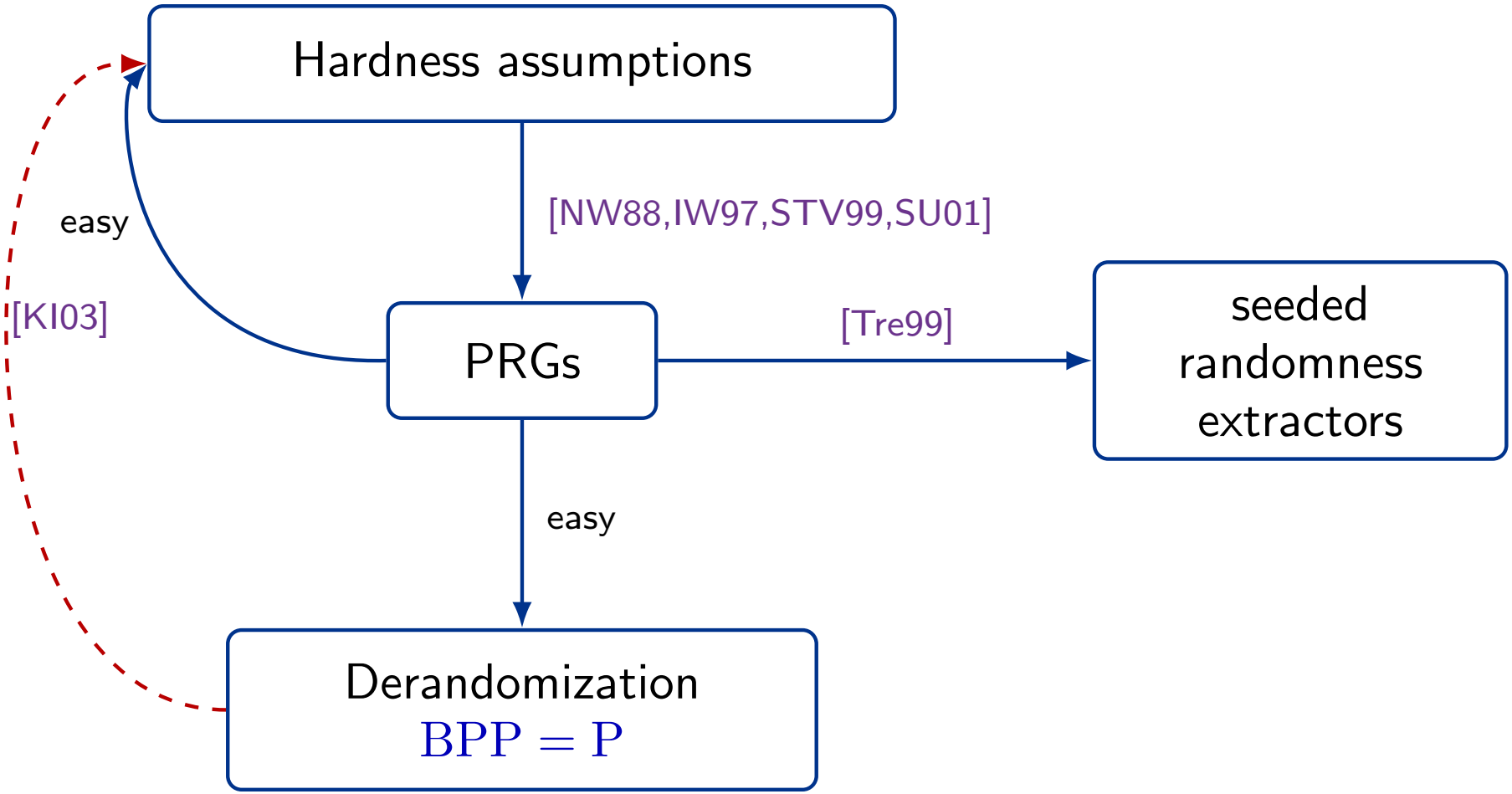

* The hardness assumption implied by $\mathrm{BPP} = \mathrm{P}$ in [KI03] is different and less clean than the one we considered.

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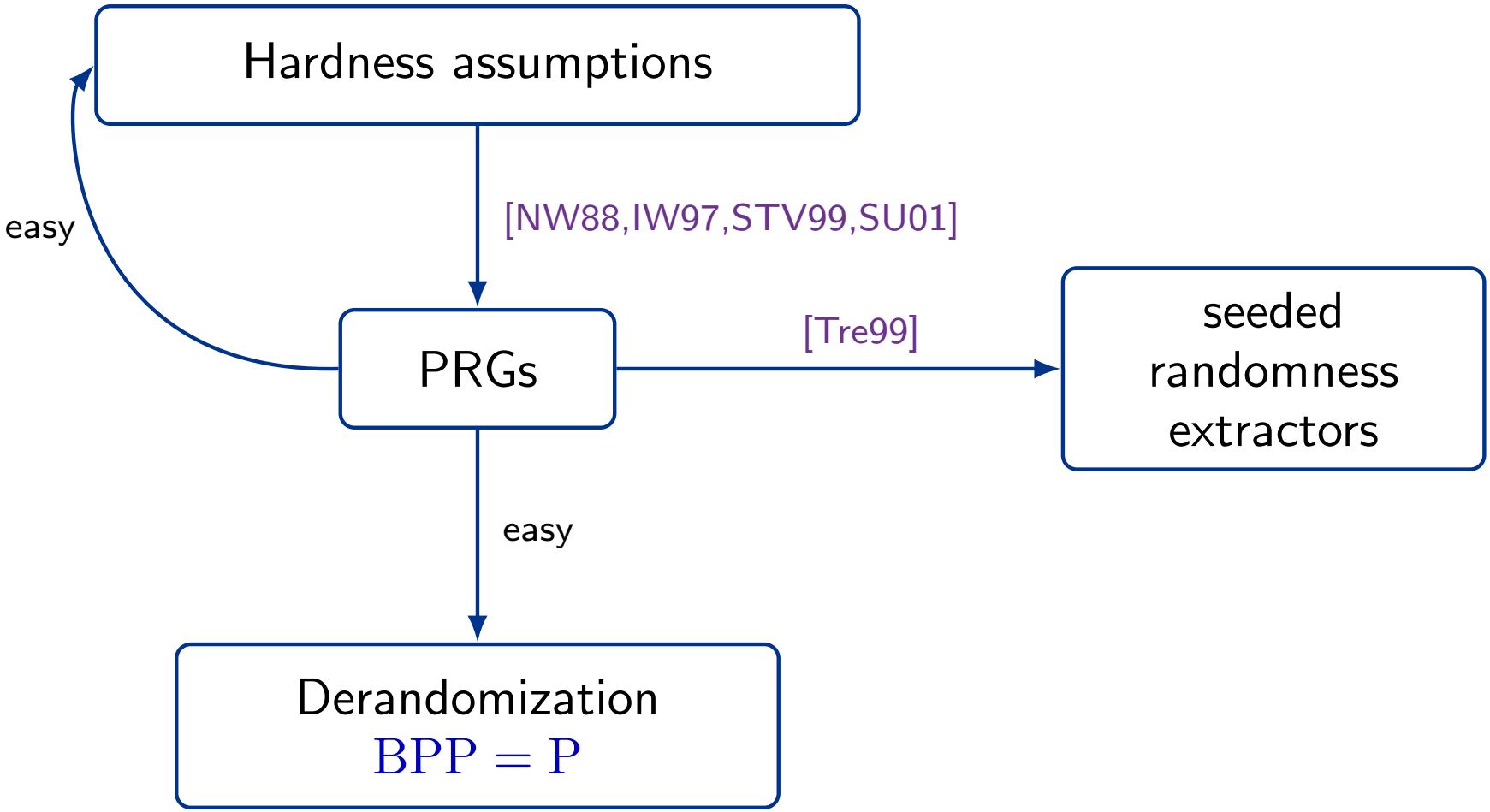

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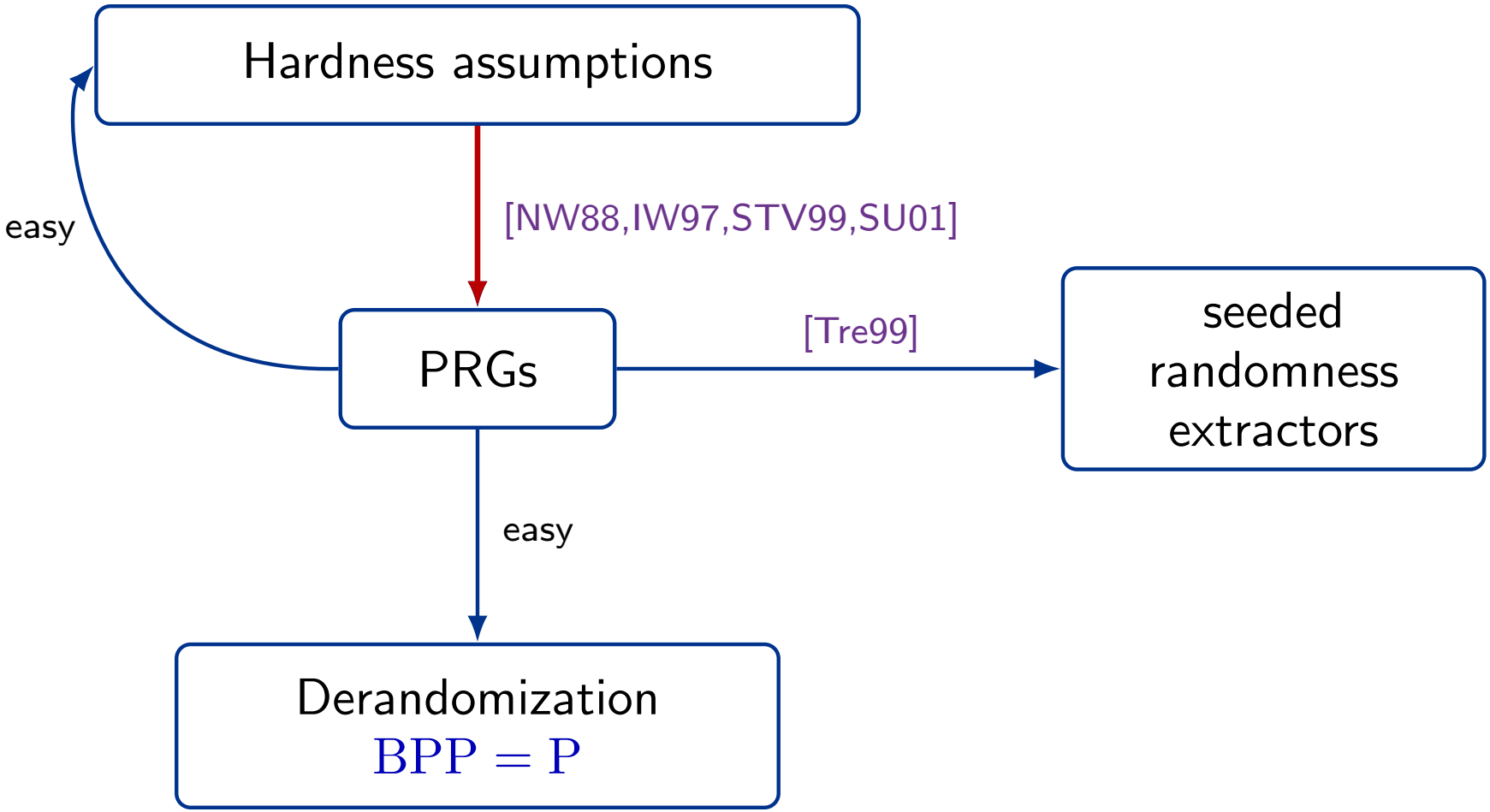

Hardness assumptions $\Rightarrow$ PRGs

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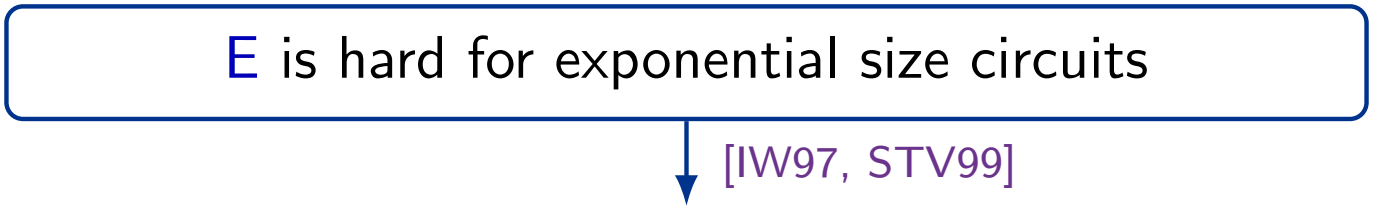

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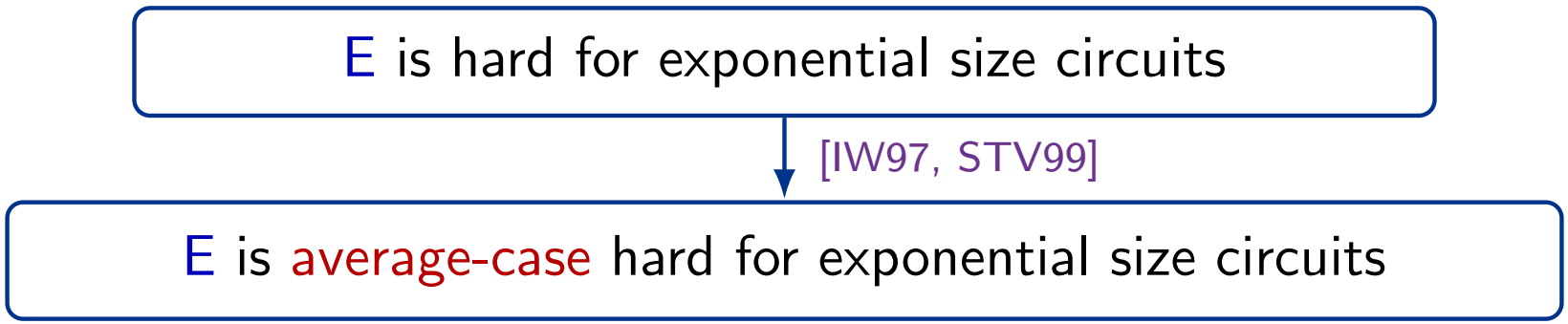

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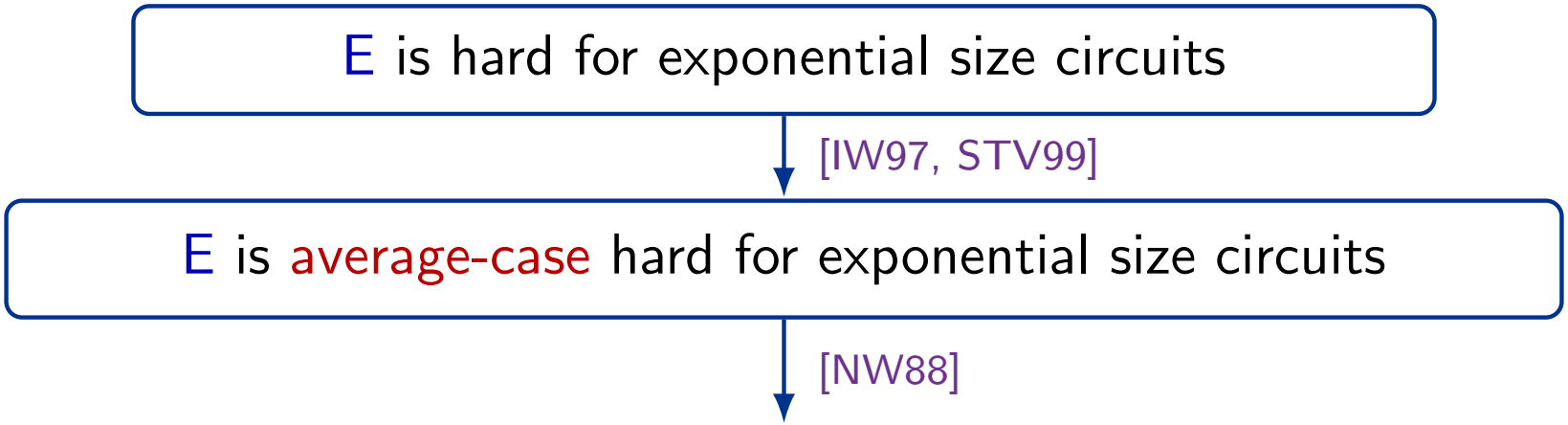

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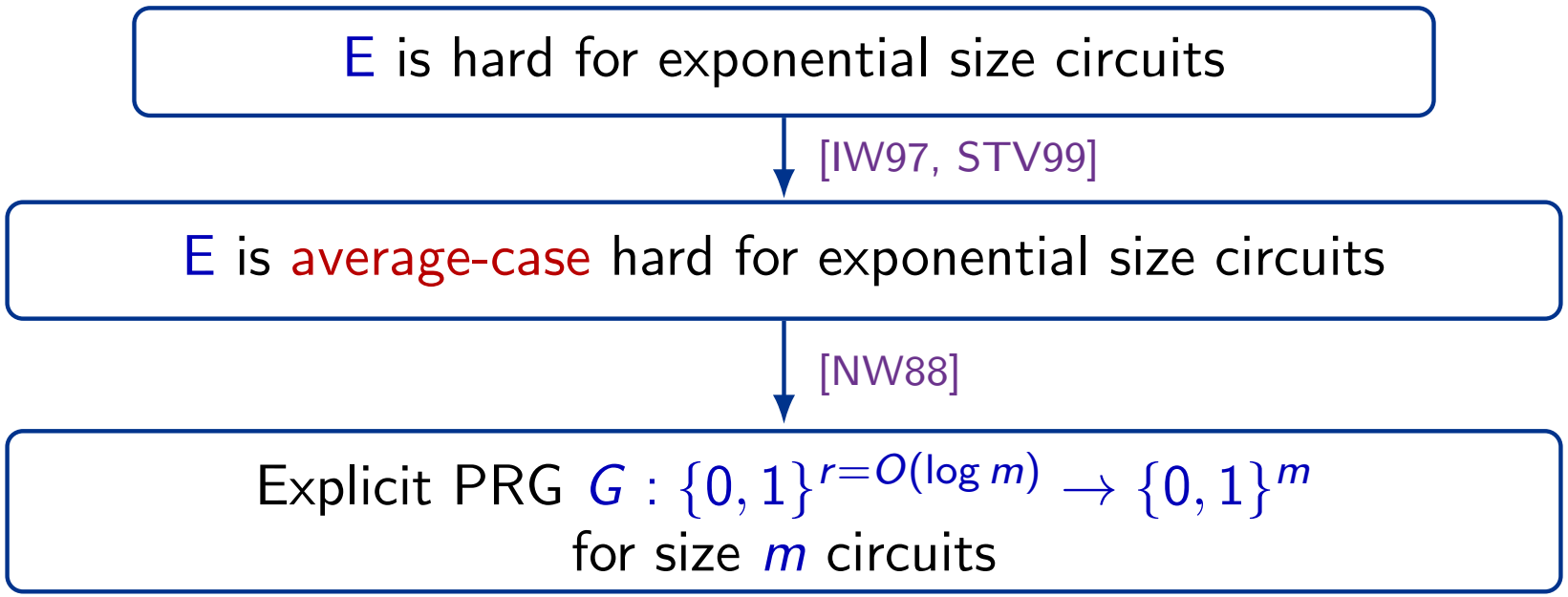

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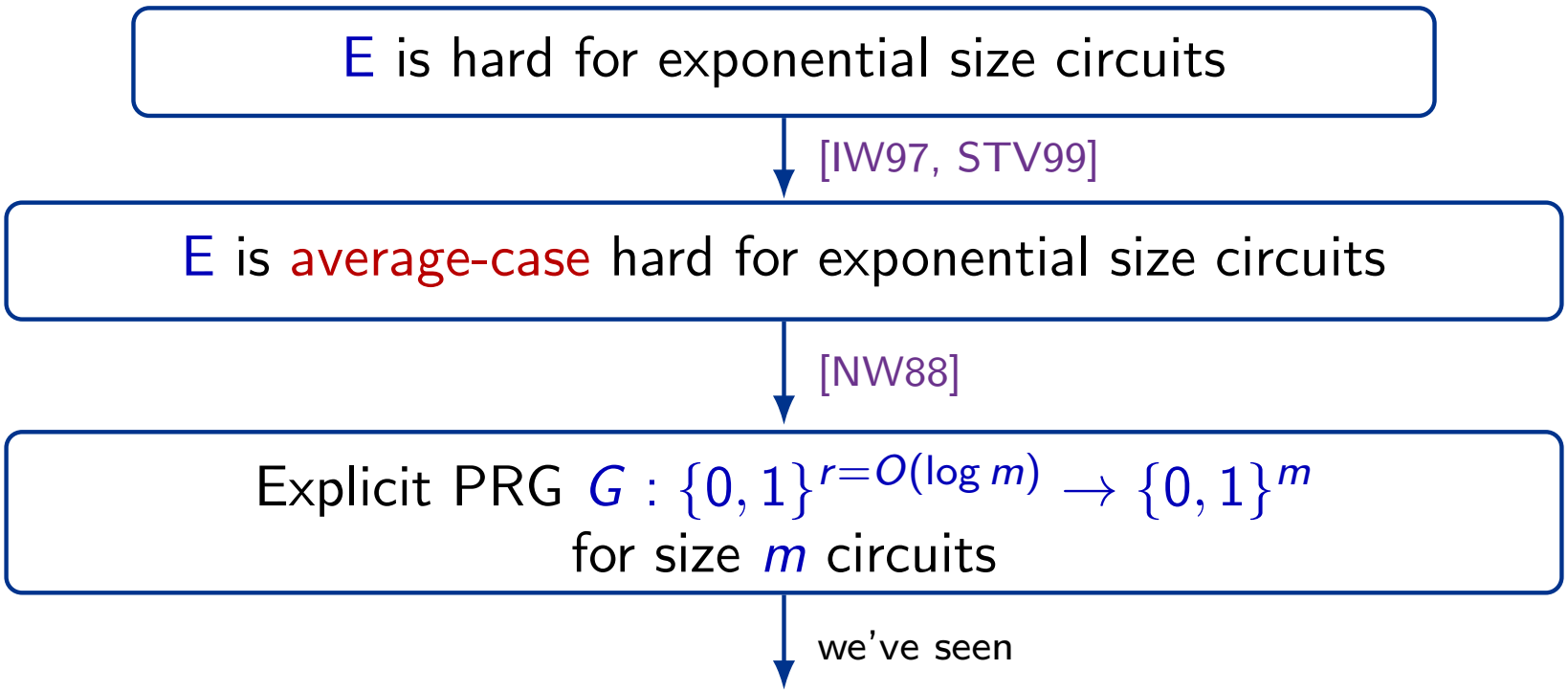

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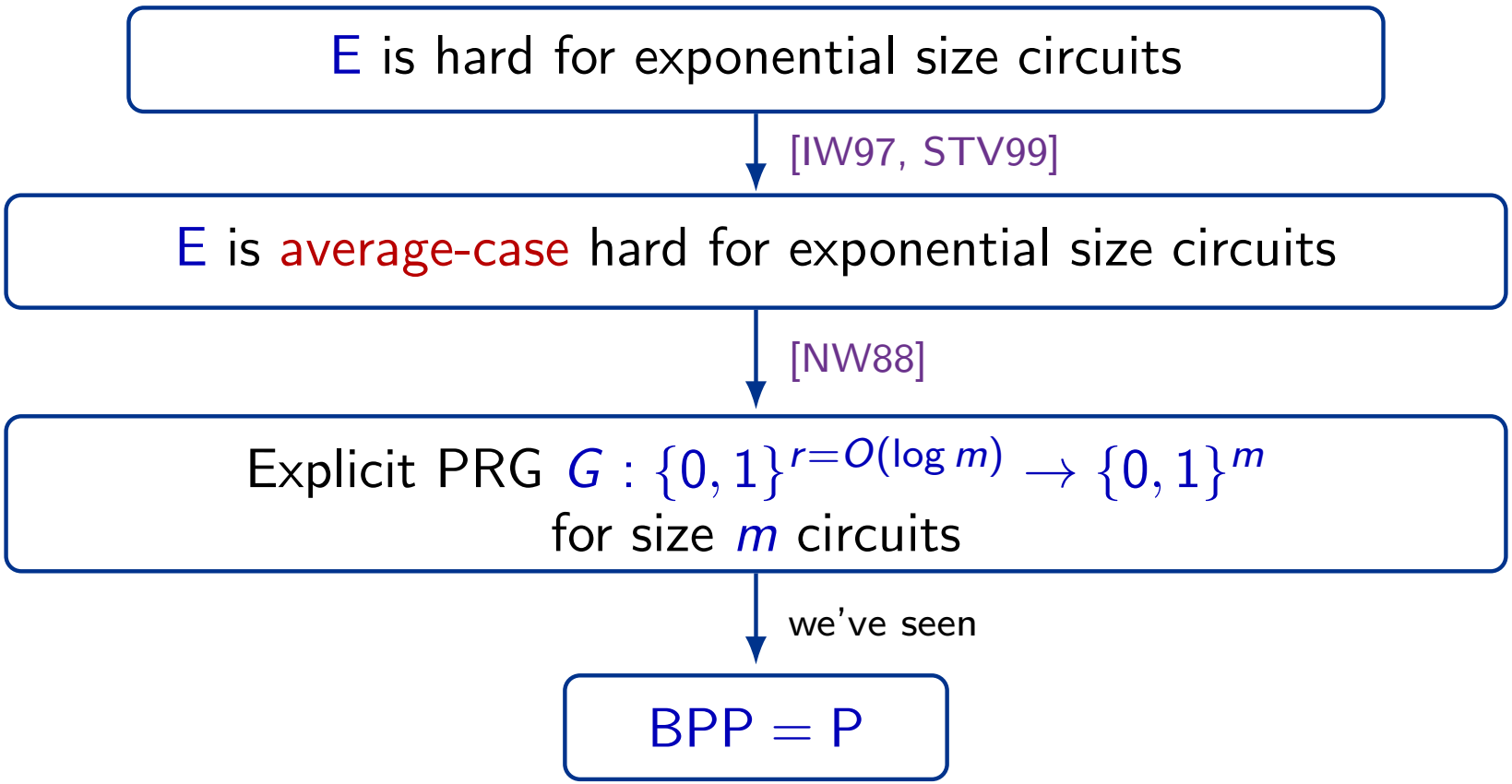

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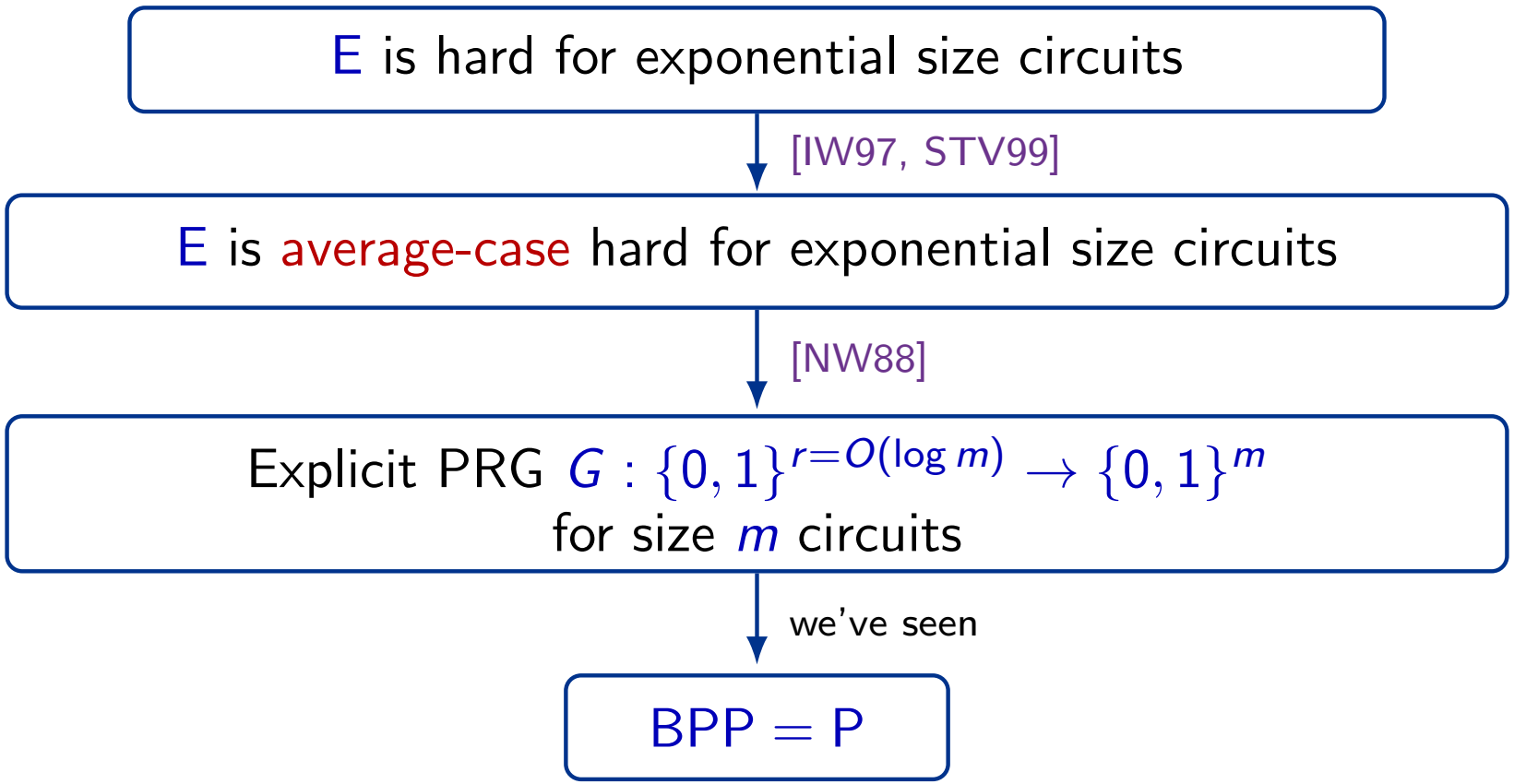

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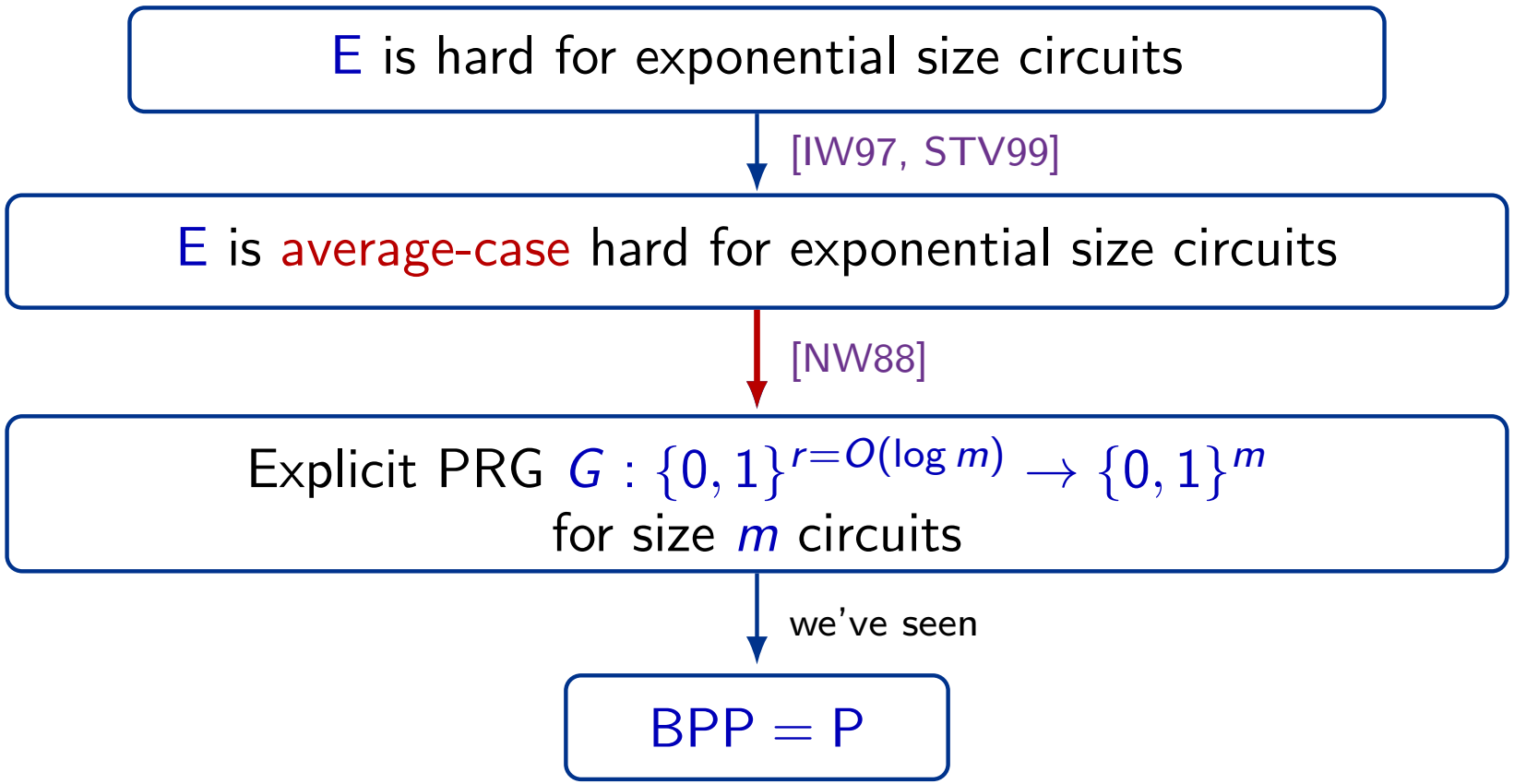

PRGs from average-case hardness

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Theorem (Distinguisher to Predictor (toy version appears as an exercise))

$G : \{0,1\}^r \to \{0,1\}^m$ is a PRG for size m circuits,
if for every $i \in [m]$, and every size m circuit $P : \{0,1\}^{i-1} \to \{0,1\}$,

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which contradicts the average-case hardness of f. □

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Claim

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- For every $1 \leq i \leq m$, $|S_i| = \ell$.
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Theorem (Left as an exercise)

$\forall \gamma > 0, \exists a > 1$ s.t. for every sufficiently large ℓ, there is an explicit $(\ell, \gamma \cdot \ell)$-design with $m = 2^{\Omega(\ell)}$ sets, in a universe of size $r = a \cdot \ell$.
$\Rightarrow NW^f : \{0,1\}^{r=O(\ell)=O(\log m)} \to \{0,1\}^m$.

NW PRG analysis

Claim (NW analysis)

$f : \{0,1\}^\ell \to \{0,1\}$ is m-average-case hard $\Rightarrow$ $NW^f : \{0,1\}^r \to \{0,1\}^{r+m}$ defined by $NW^f(x) = x, f(x|_{S_1}), \ldots, f(x|_{S_m})$ is a PRG for circuits of size m.

Proof.

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Comment: Even if you're happy with $r = \ell^2$, designs have other costs, that can be reduced with "weak designs" [RRV99].

NW PRG (reflection: cost of hybrid argument)

We relied on the following theorem:

Theorem (Distinguisher to Predictor)

$G : \{0,1\}^r \to \{0,1\}^m$ is a PRG for size m circuits,
if for every $i \in [m]$, and every size m circuit $P : \{0,1\}^{i-1} \to \{0,1\}$,

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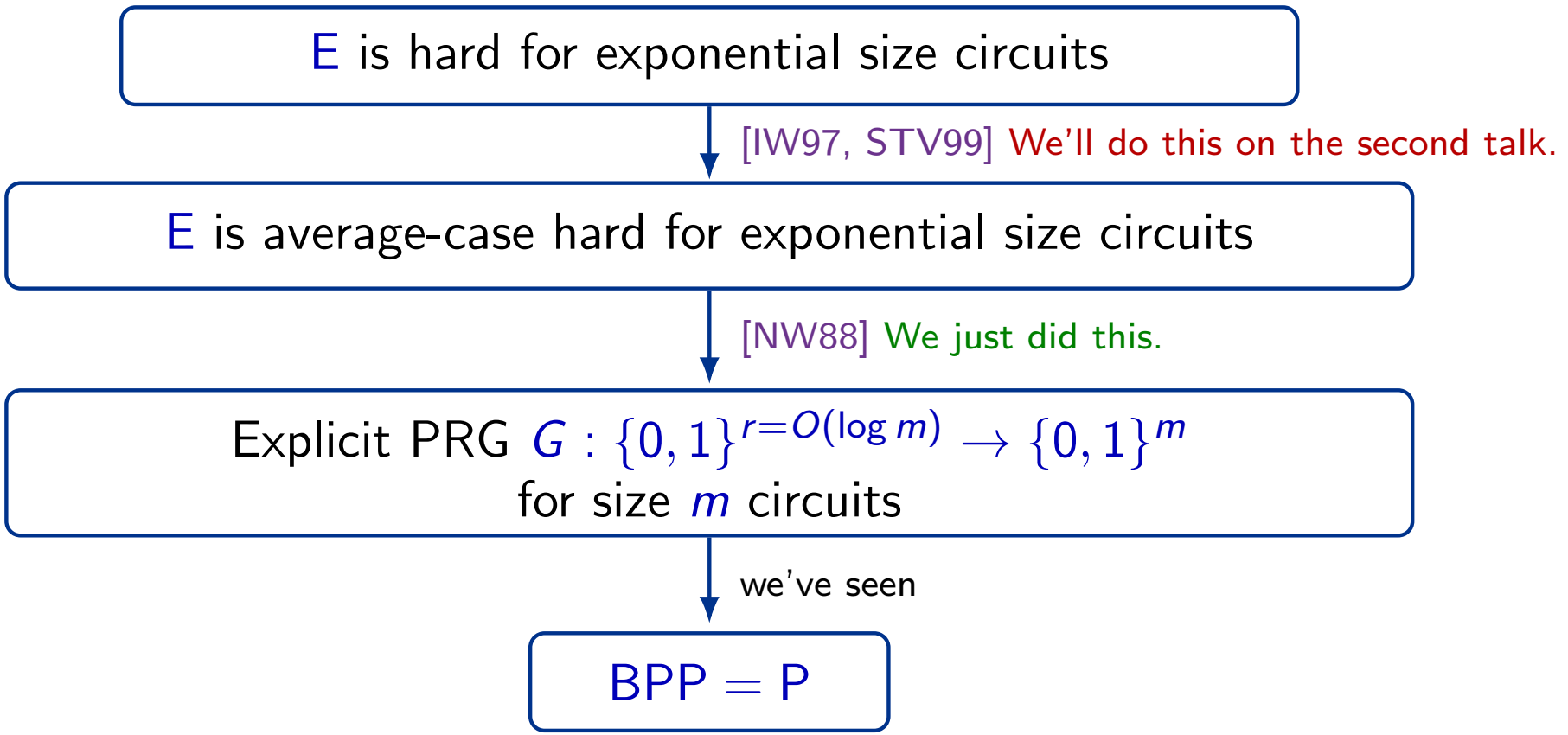
Exciting recent work on "fast derandomization" that "bypasses the hybrid argument" (typically using specialized and more elaborate hardness assumptions) [DMOZ20,CT21a,CT21b,CT23,BCT25,DMOZ26...].

Hardness $\Rightarrow$ PRGs (Where we are)

Theorem (IW97)

E *is hard for exponential size circuits* $\Rightarrow$ BPP $=$ P.



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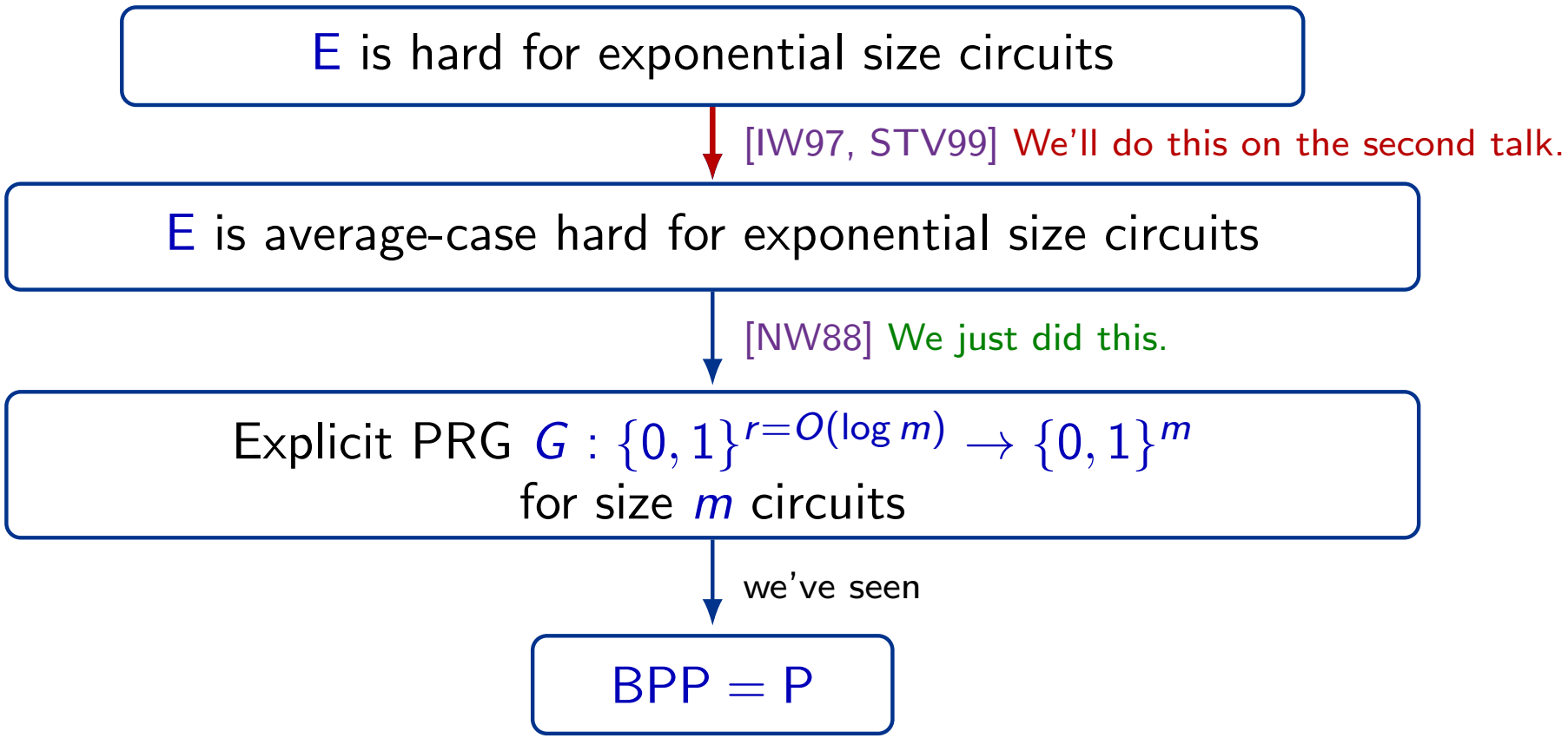
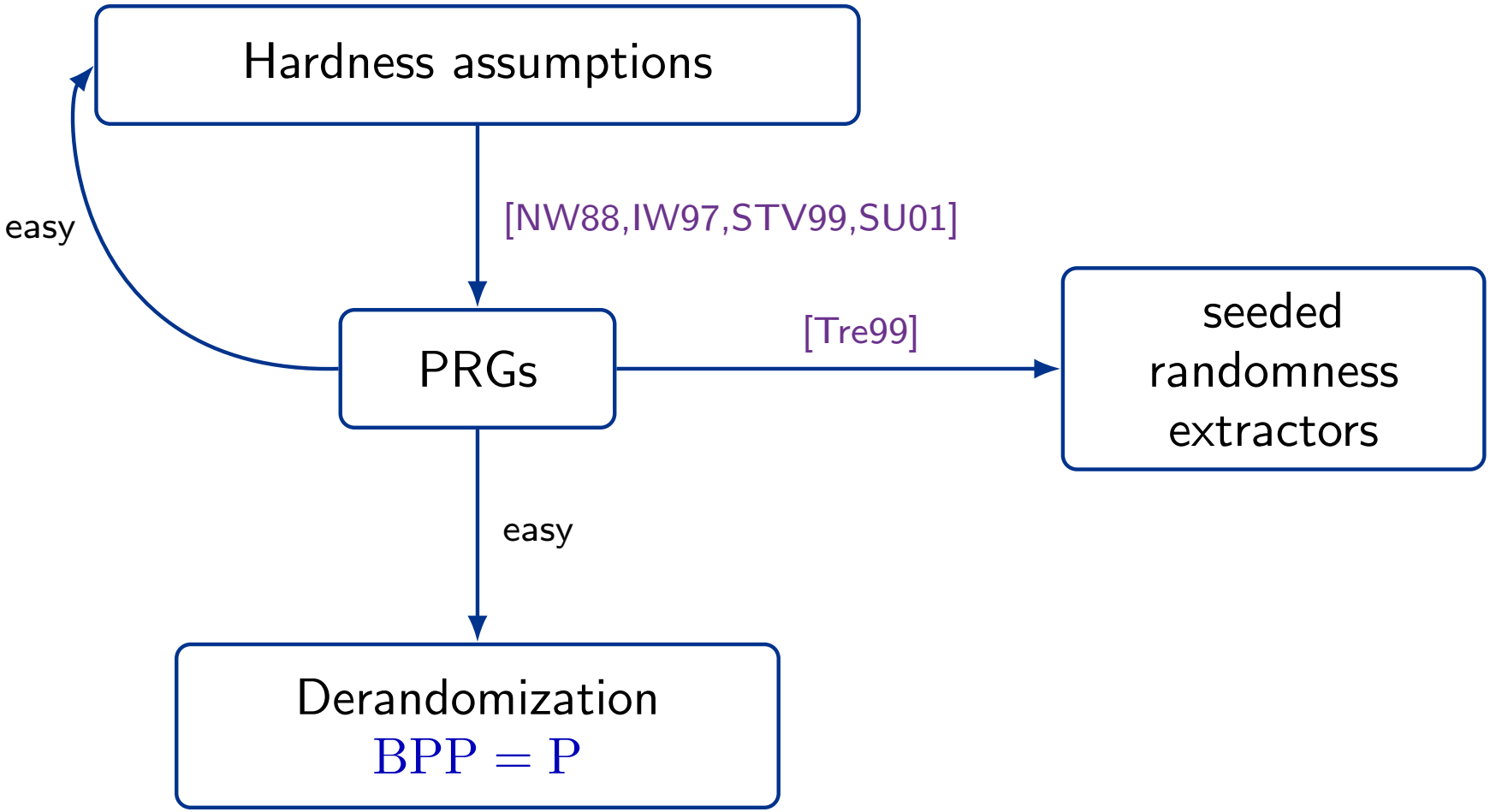


Plan for talks



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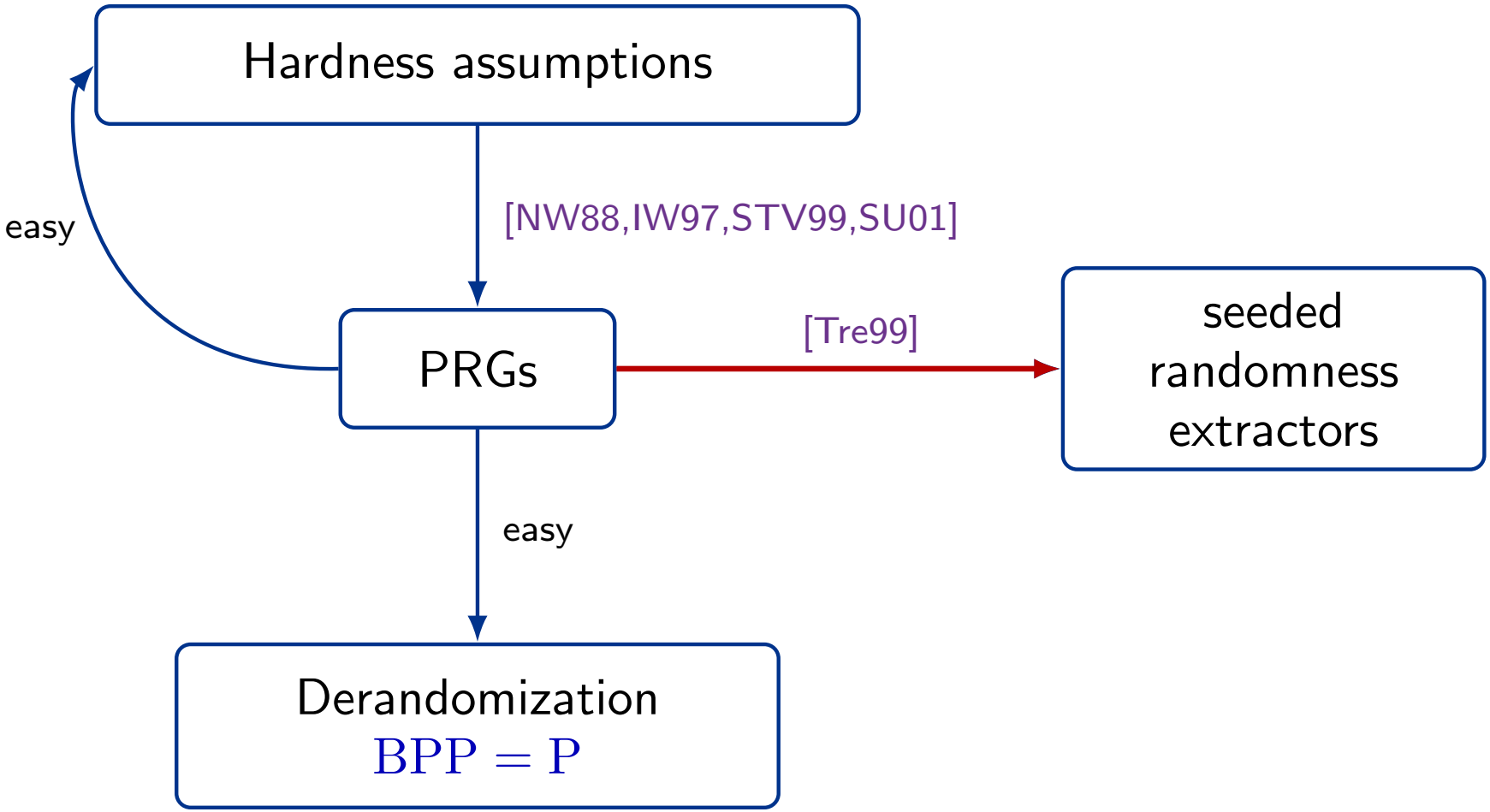

Randomness extractors (Eshan will give two talks)

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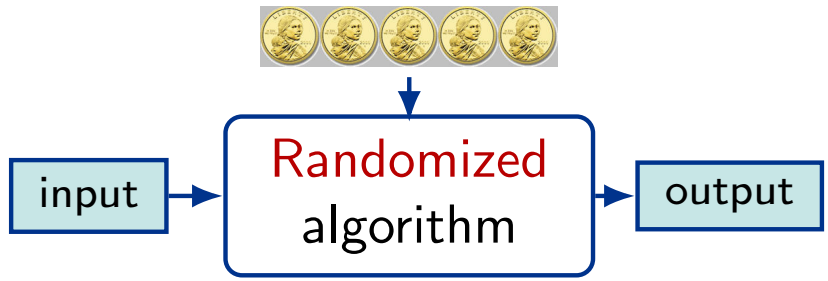
Eshan will give detailed talks on randomness extractors.



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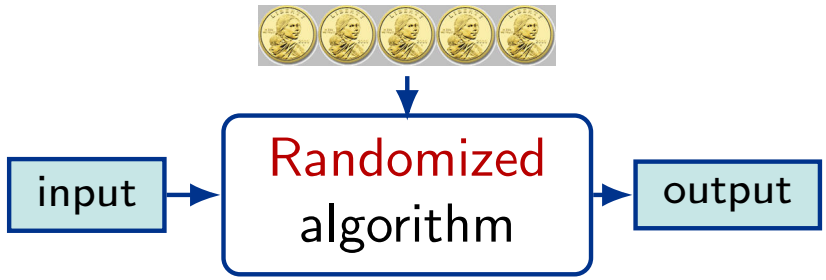

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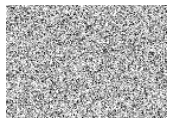

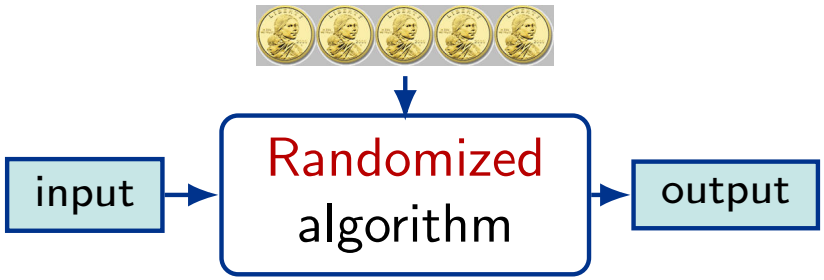

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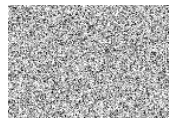

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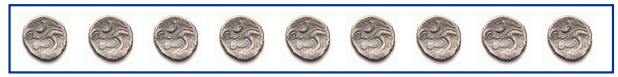

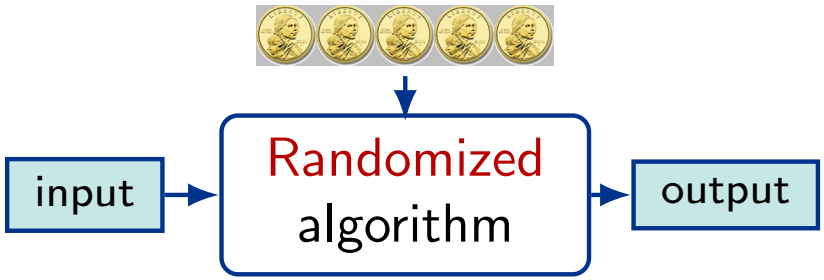

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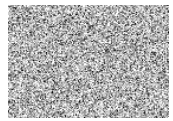

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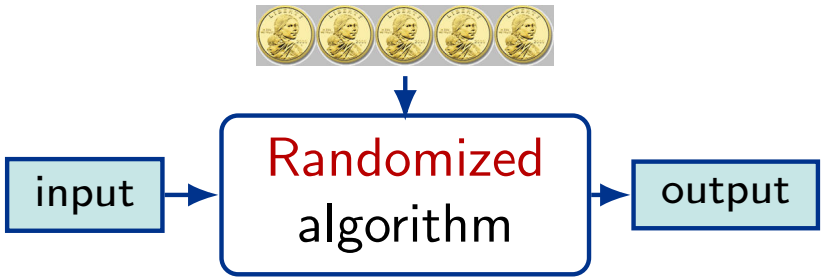

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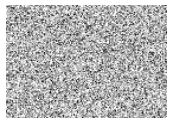

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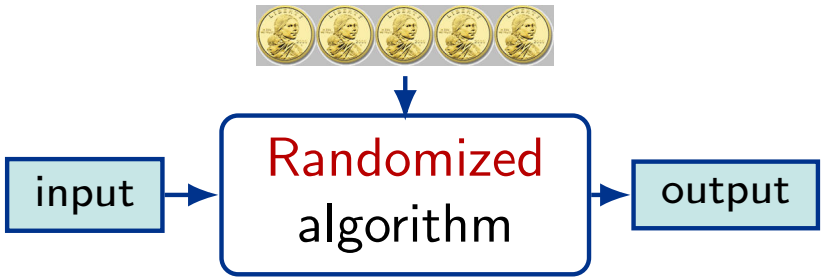

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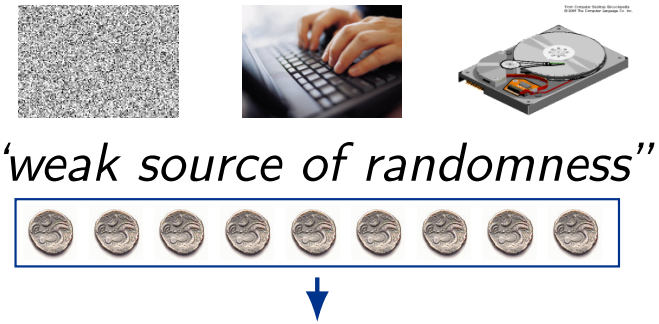

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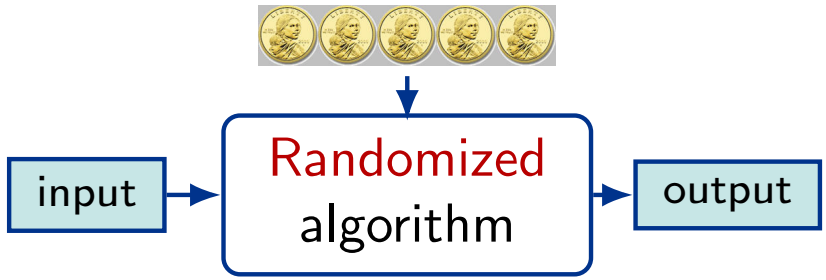

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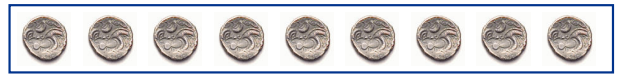

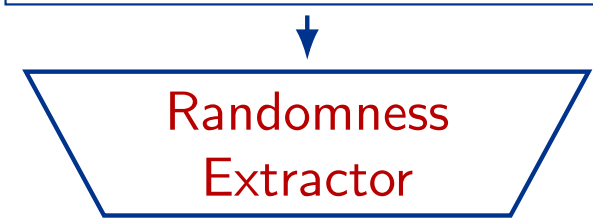

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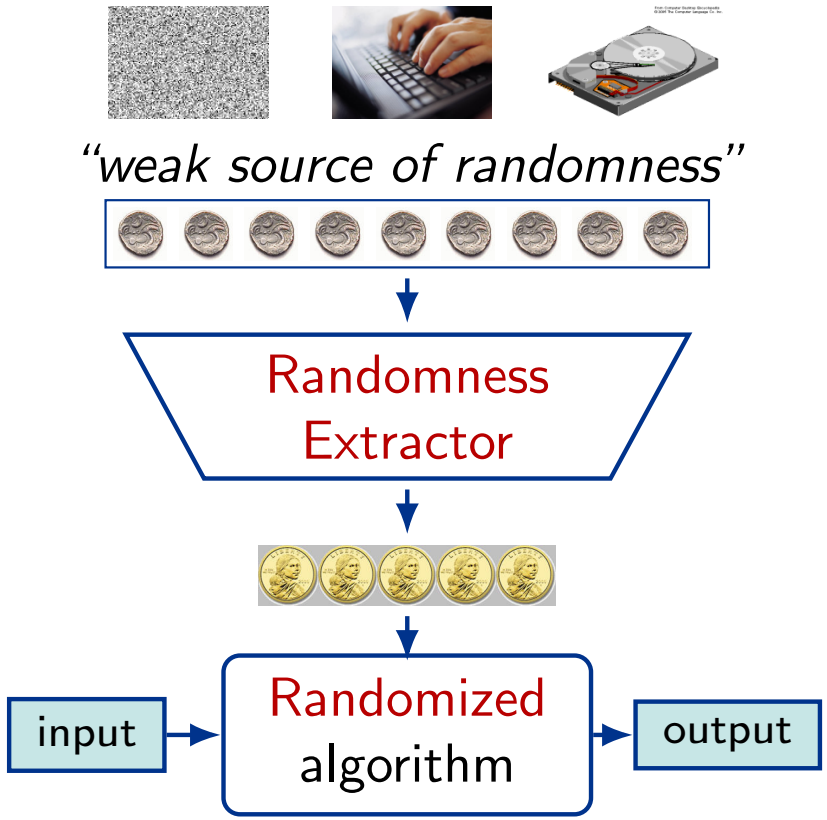

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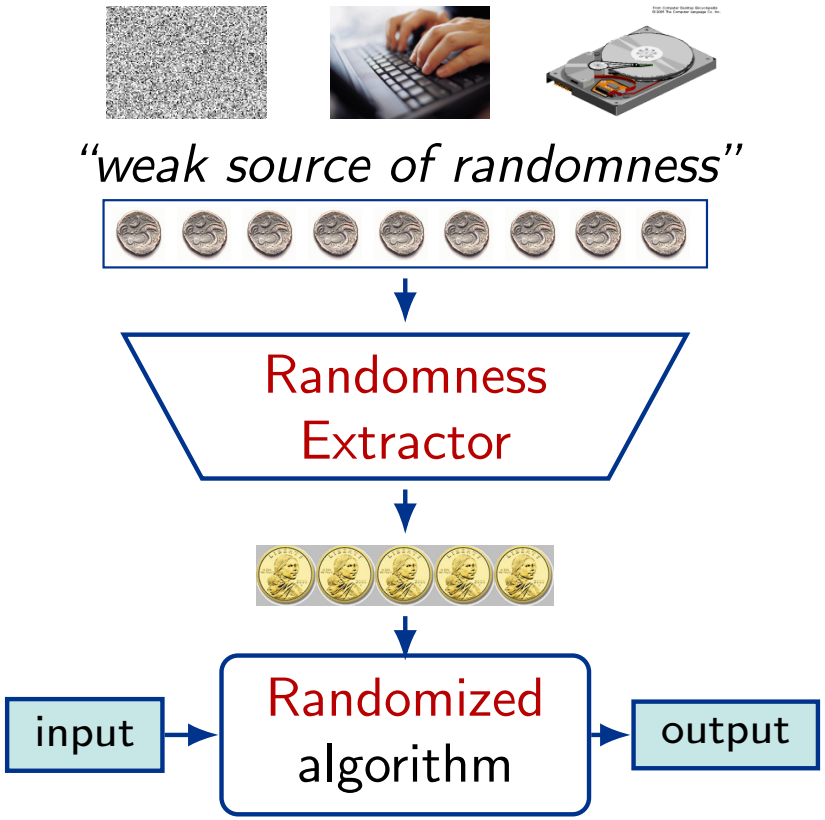

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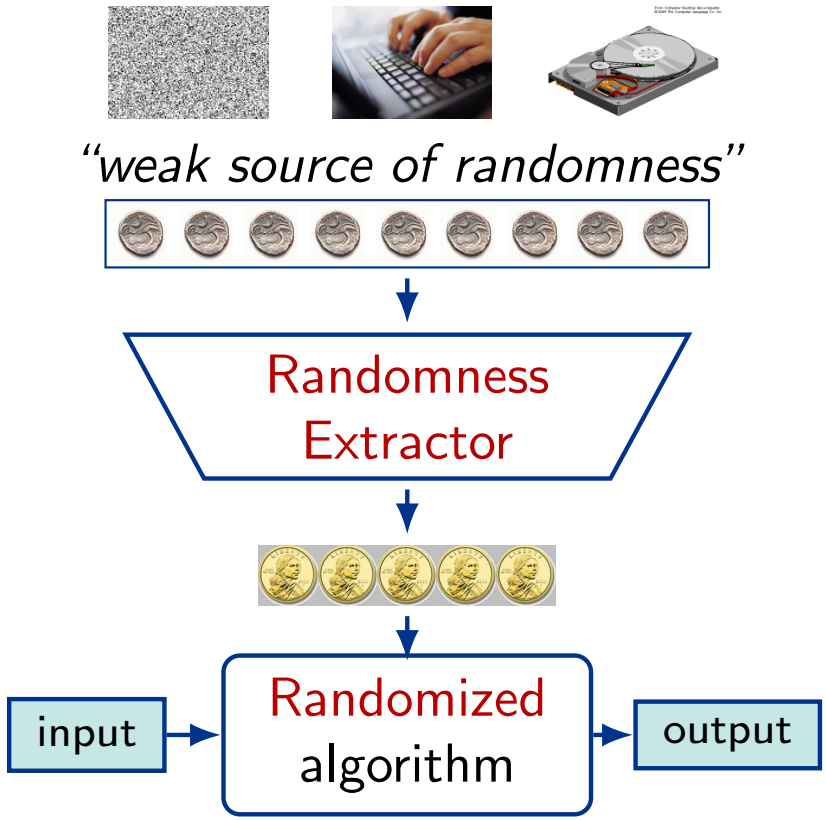

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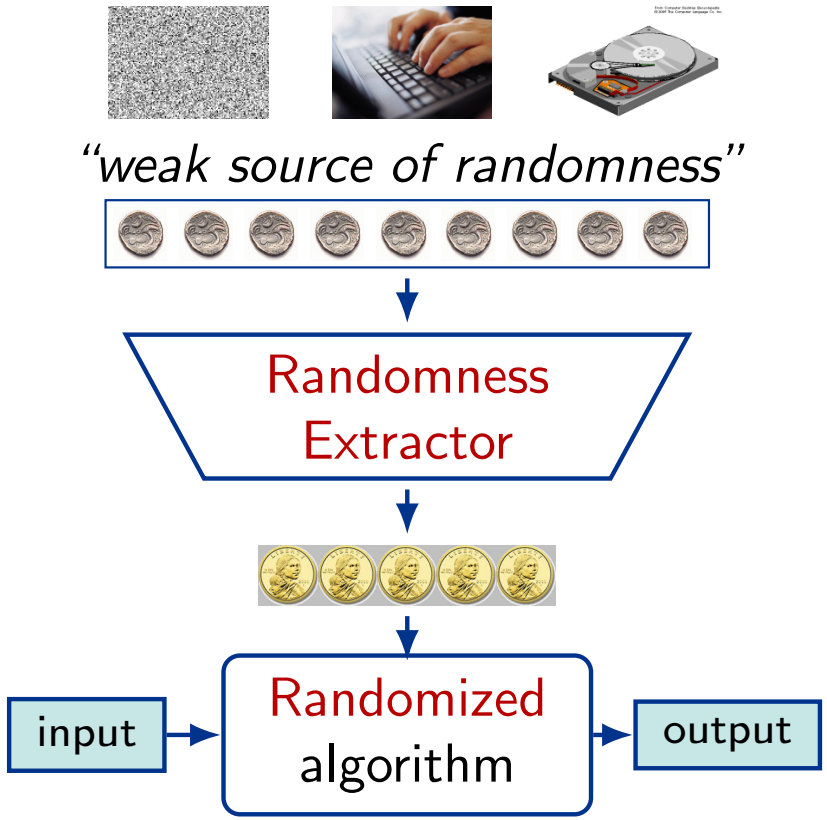

Extensively studied area, dates back to von-Neumann in 1951.

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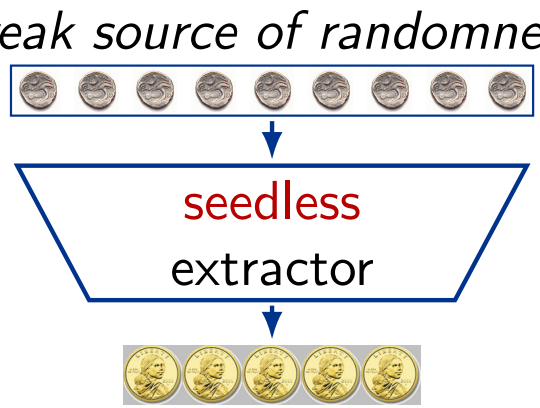

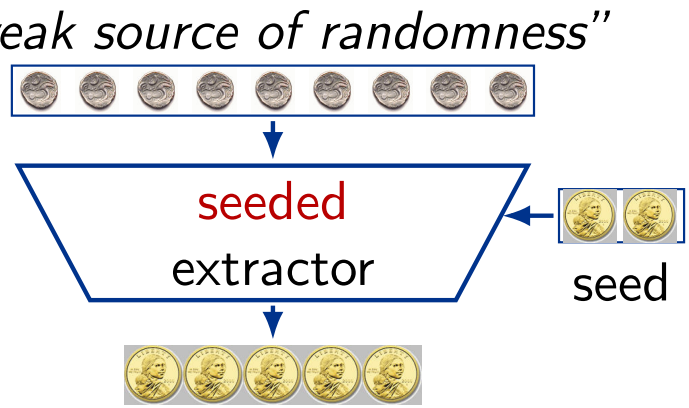

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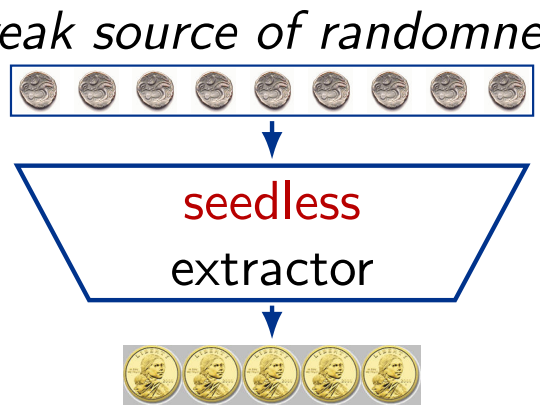

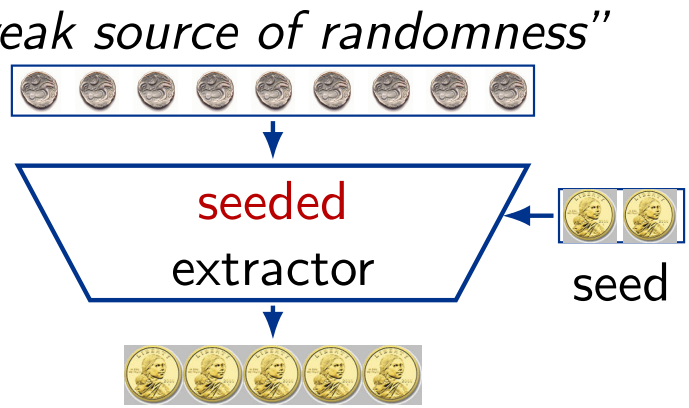

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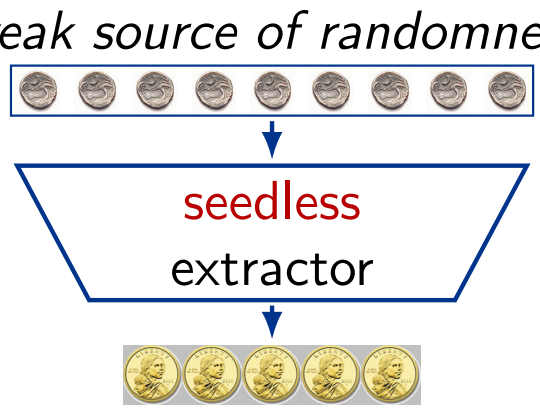

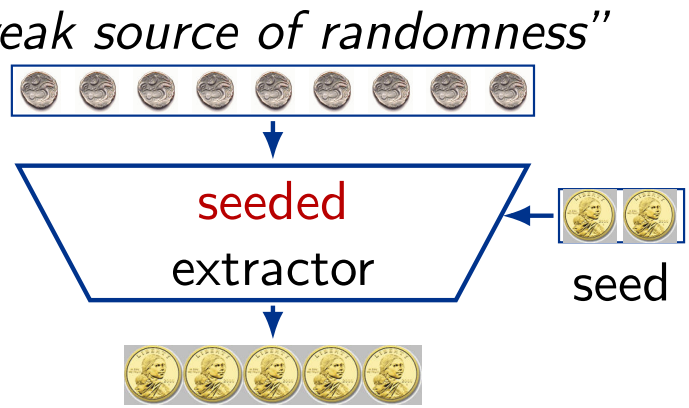

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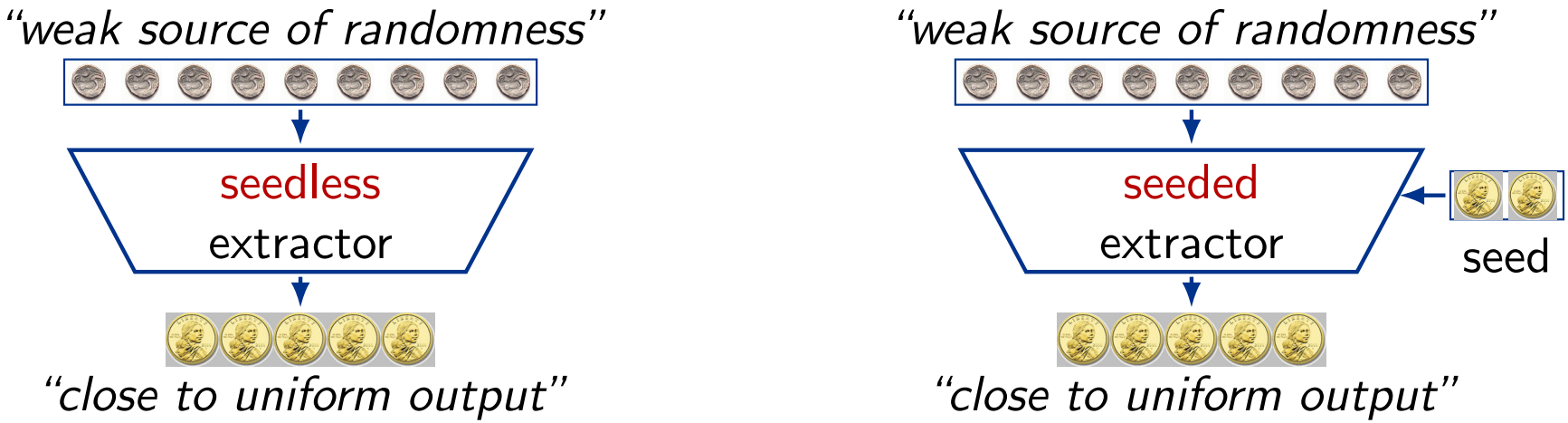

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Theorem (Simulating BPP with a weak random source, Informal)

Seed length $r = O(\log n)$ and output length $m = n^{\Omega(1)}$ $\Rightarrow$ seeded extractors can be used to simulate any poly-time randomized algorithm with access to an arbitrary weak random source.

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Theorem (Simulating BPP with a weak random source, Informal)

Seed length $r = O(\log n)$ and output length $m = n^{\Omega(1)}$ $\Rightarrow$ seeded extractors can be used to simulate any poly-time randomized algorithm with access to an arbitrary weak random source.

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▸ Conclude talk

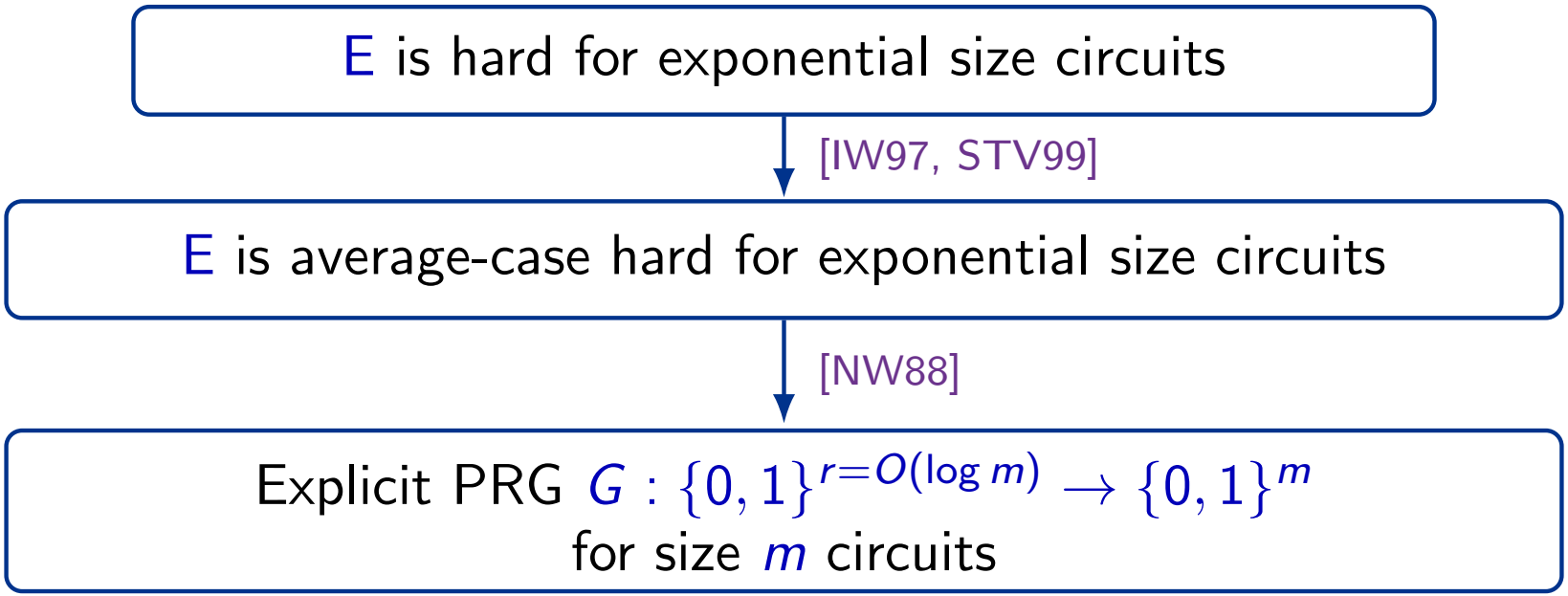
Hardness $\Rightarrow$ PRGs (review of argument):

Theorem (IW97)

E *is hard for exponential size circuits* $\Rightarrow$ *there is an explicit PRG*

$$G : \{0,1\}^{r=O(\log m)} \to \{0,1\}^m$$

for size m *circuits.*



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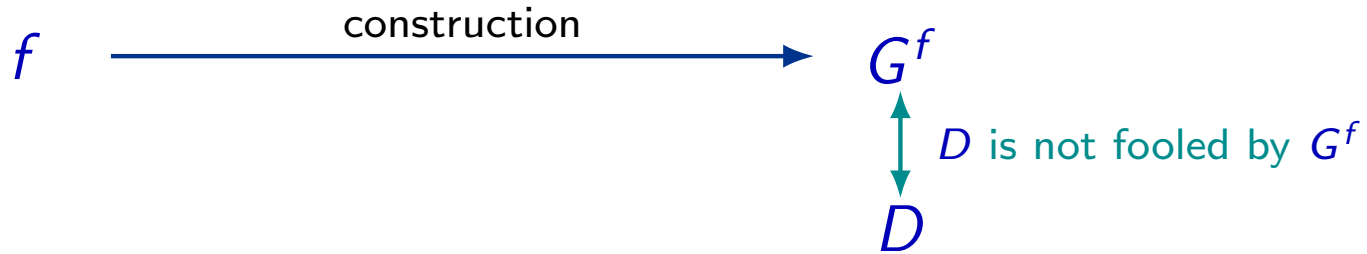

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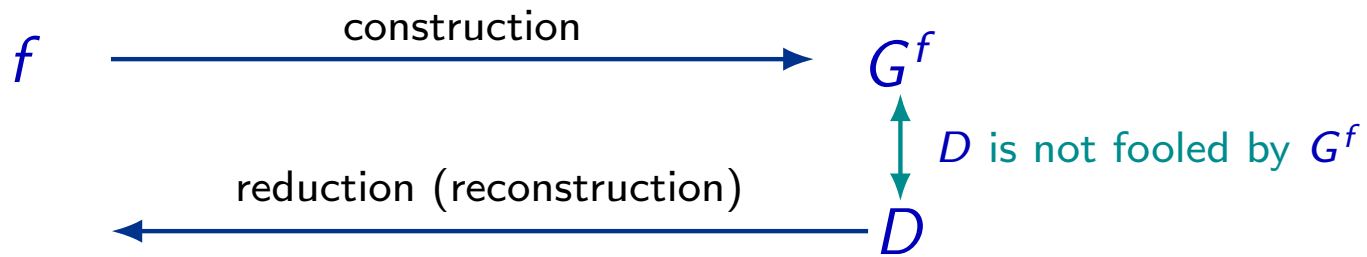

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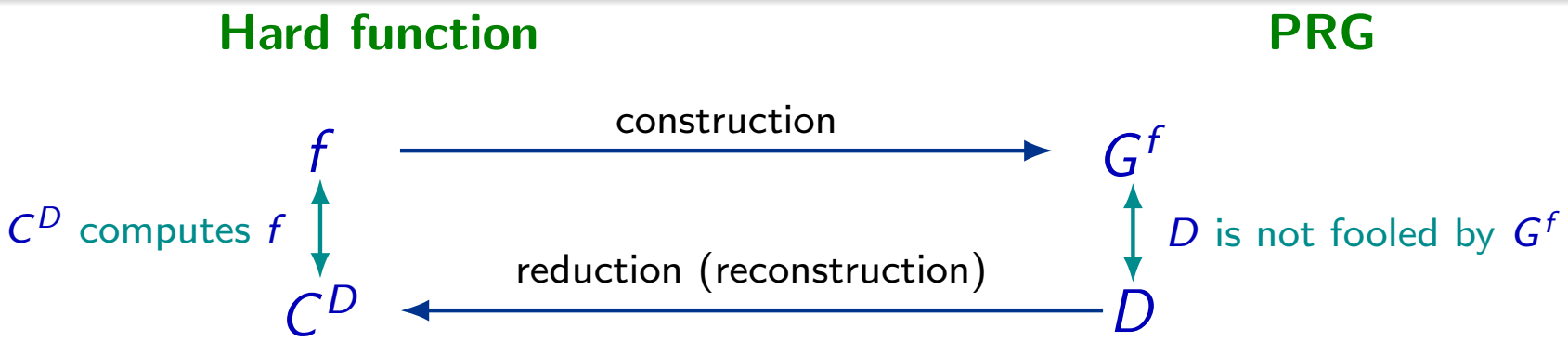

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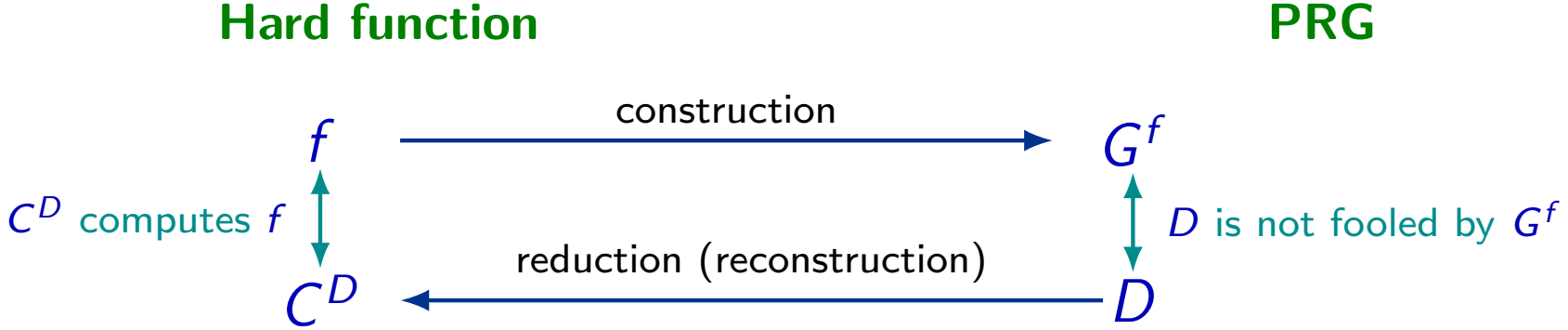

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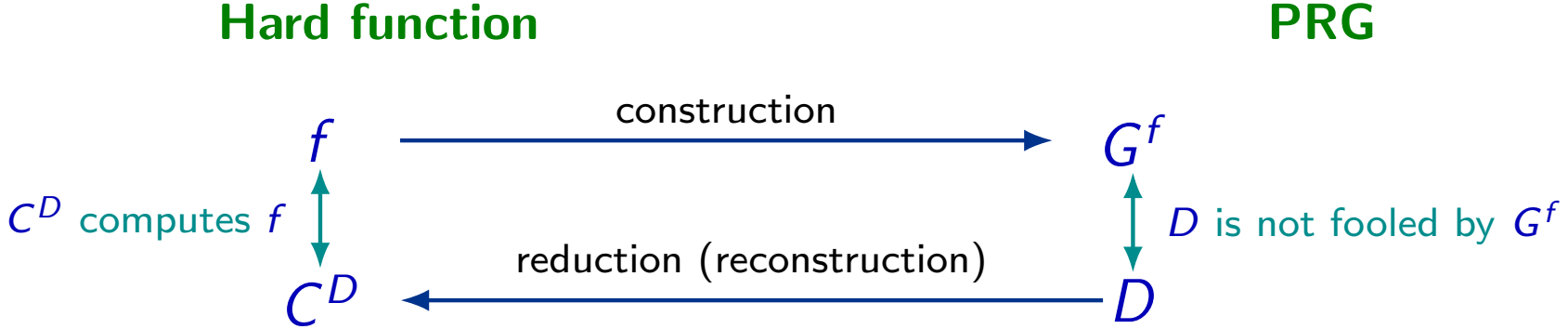

Key point: We do not assume that f or D are efficient (except for the consequently clauses).

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Theorem (Trevisan (informal))

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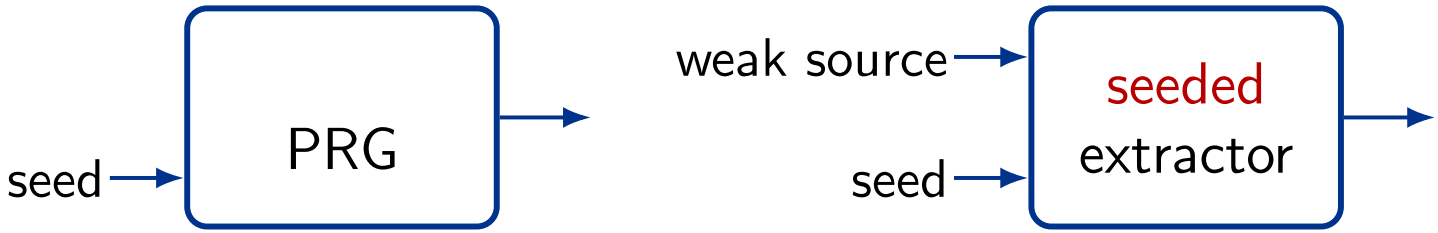

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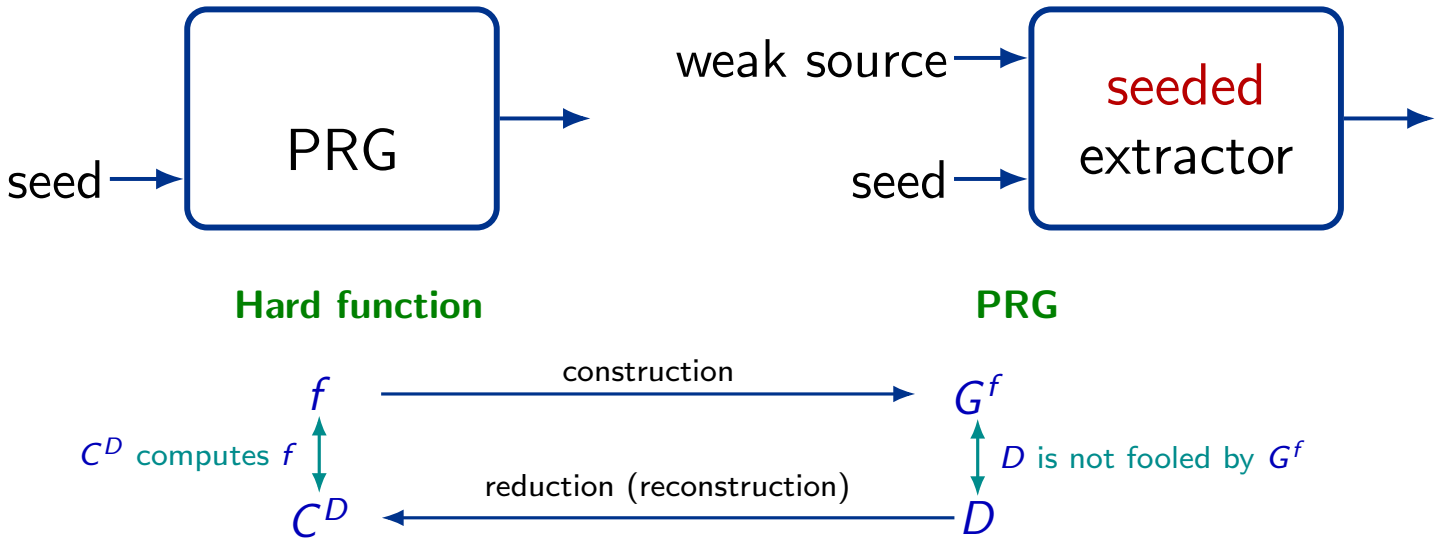

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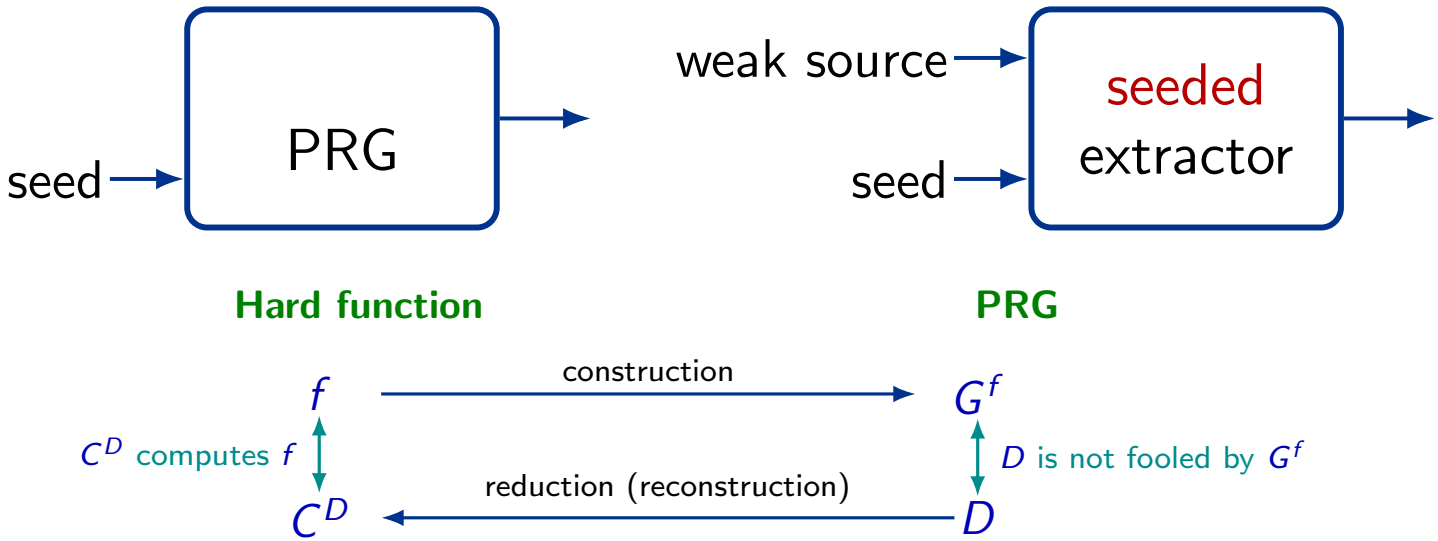

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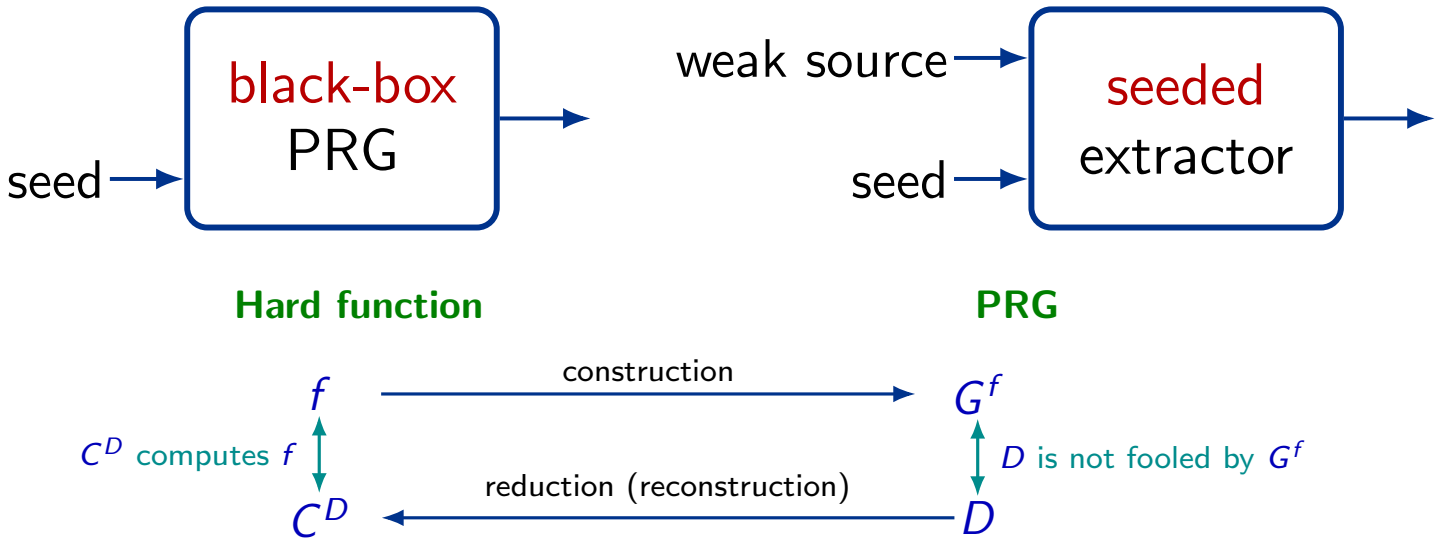

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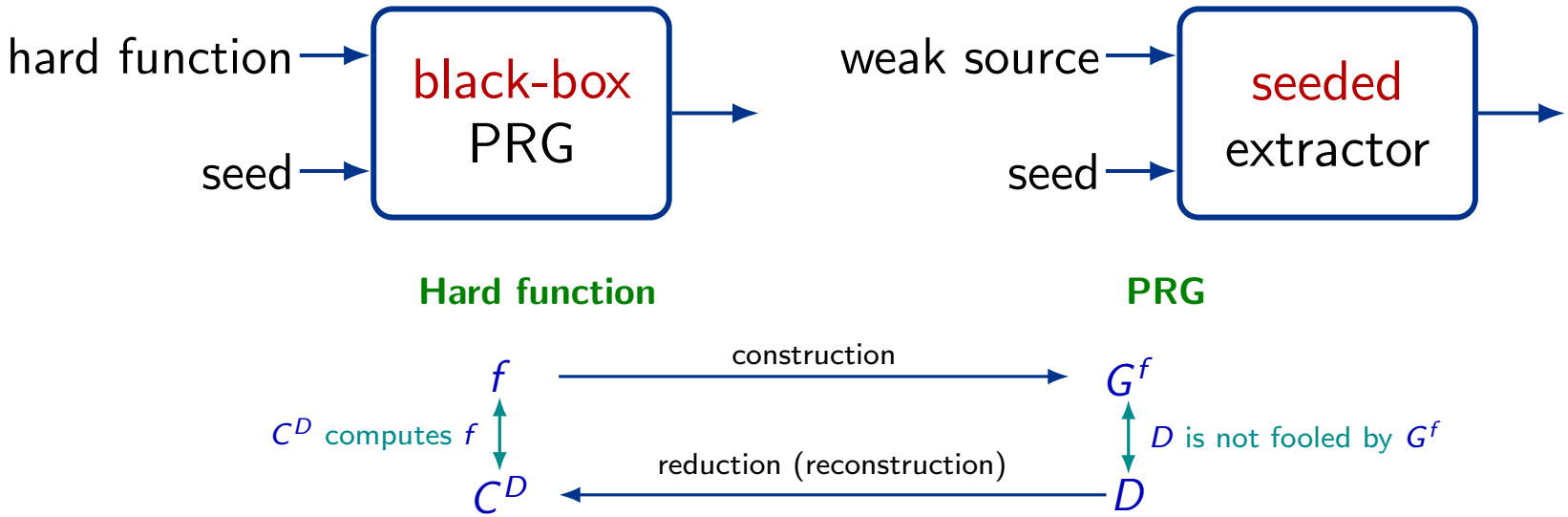

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then $\exists$size $2^{o(\ell)}$ oracle-circuit $C^{(\cdot)}$ s.t. $\forall x \in \{0,1\}^\ell$, $C^D(x) = f(x)$.

Define: $E(f,x) = G^f(x)$. (f of length $n = 2^\ell$ is "source" and x is "seed").

If E is not an extractor, $\exists$distribution F with $H_\infty(F) \geq k := n^{\Omega(1)}$, and
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For every such f, $\exists$ size $2^{o(\ell)}$ oracle-circuit $C^{(\cdot)}$ s.t. $\forall x \in \{0,1\}^\ell$, $C^D(x) = f(x)$.

D is fixed $\Rightarrow$ every such f is "described" by C using $2^{o(\ell)} = m^{o(1)} = n^{o(1)}$ bits.

Trevisan's argument

Theorem (Trevisan (informal))

Fully-black box PRG construction $\Rightarrow$ *seeded extractor.*

Definition (Fully black-box PRG)

oracle TM $G^{(\cdot)}$ s.t. $\forall f : \{0,1\}^\ell \to \{0,1\}$, setting $m = 2^{\Omega(\ell)}$:
$G^f : \{0,1\}^{r=O(\log m)} \to \{0,1\}^m$ is computable in E^f.
s.t. $\forall D : \{0,1\}^m \to \{0,1\}$ (not necessarily a small circuit), if

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Impossible, because an $f \leftarrow F$ is very unlikely to have such a short description C.

Reflection: usefulness of black-box view of PRGs

Trevisan's principle:

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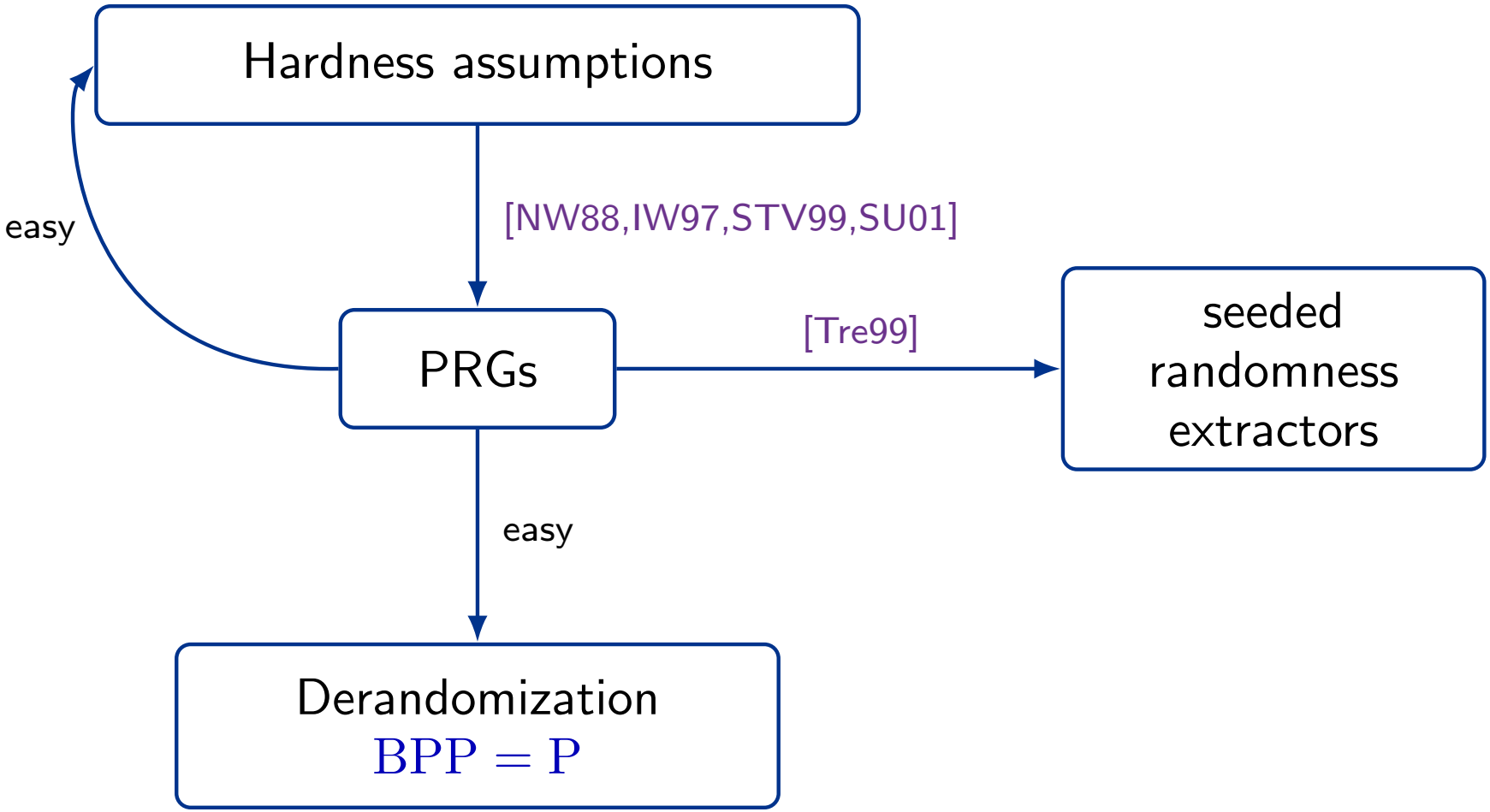
▸ Conclude talk

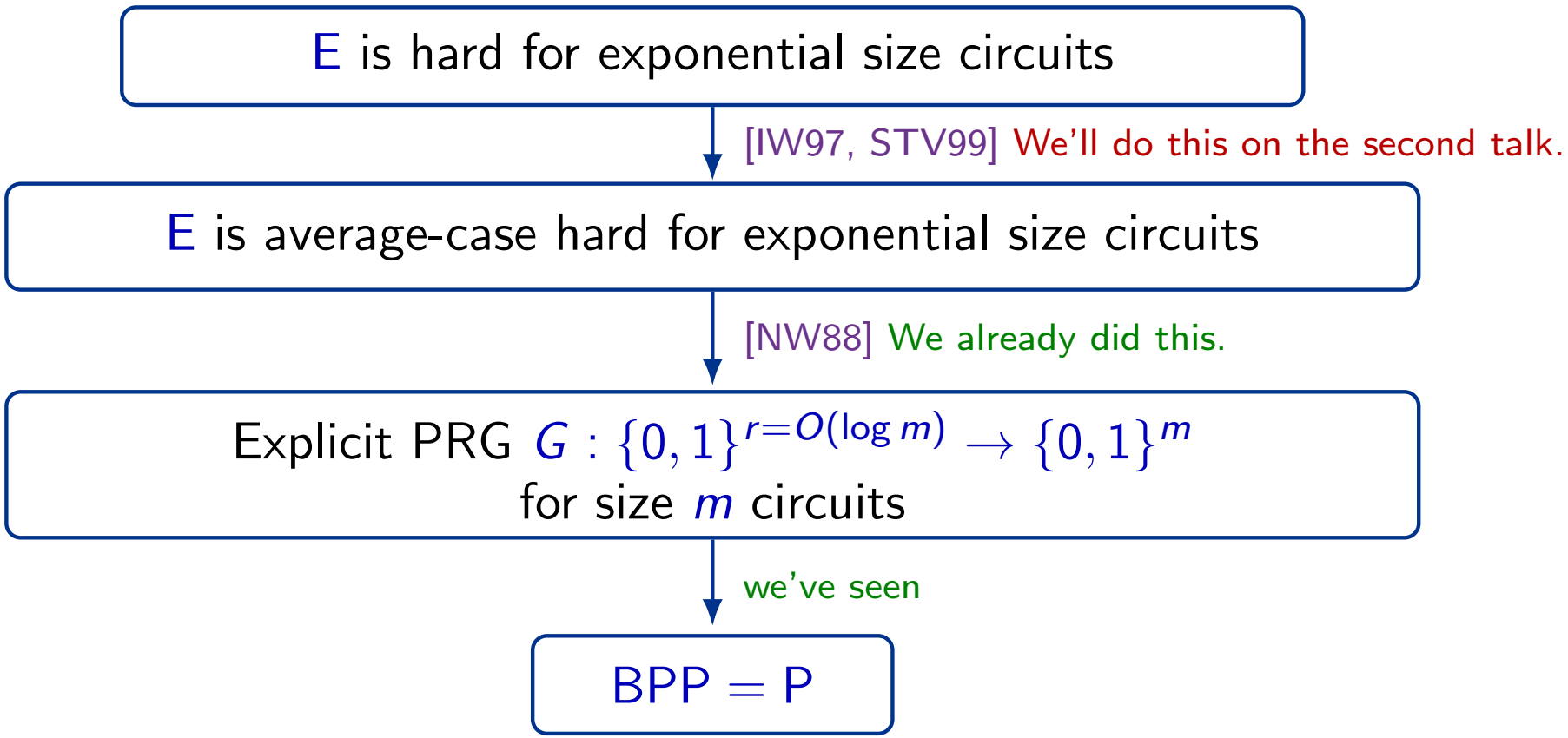
Summary (and where we are)



Hardness $\Rightarrow$ PRGs (Where we are)

Theorem (IW97)

E *is hard for exponential size circuits* $\Rightarrow$ BPP $=$ P.



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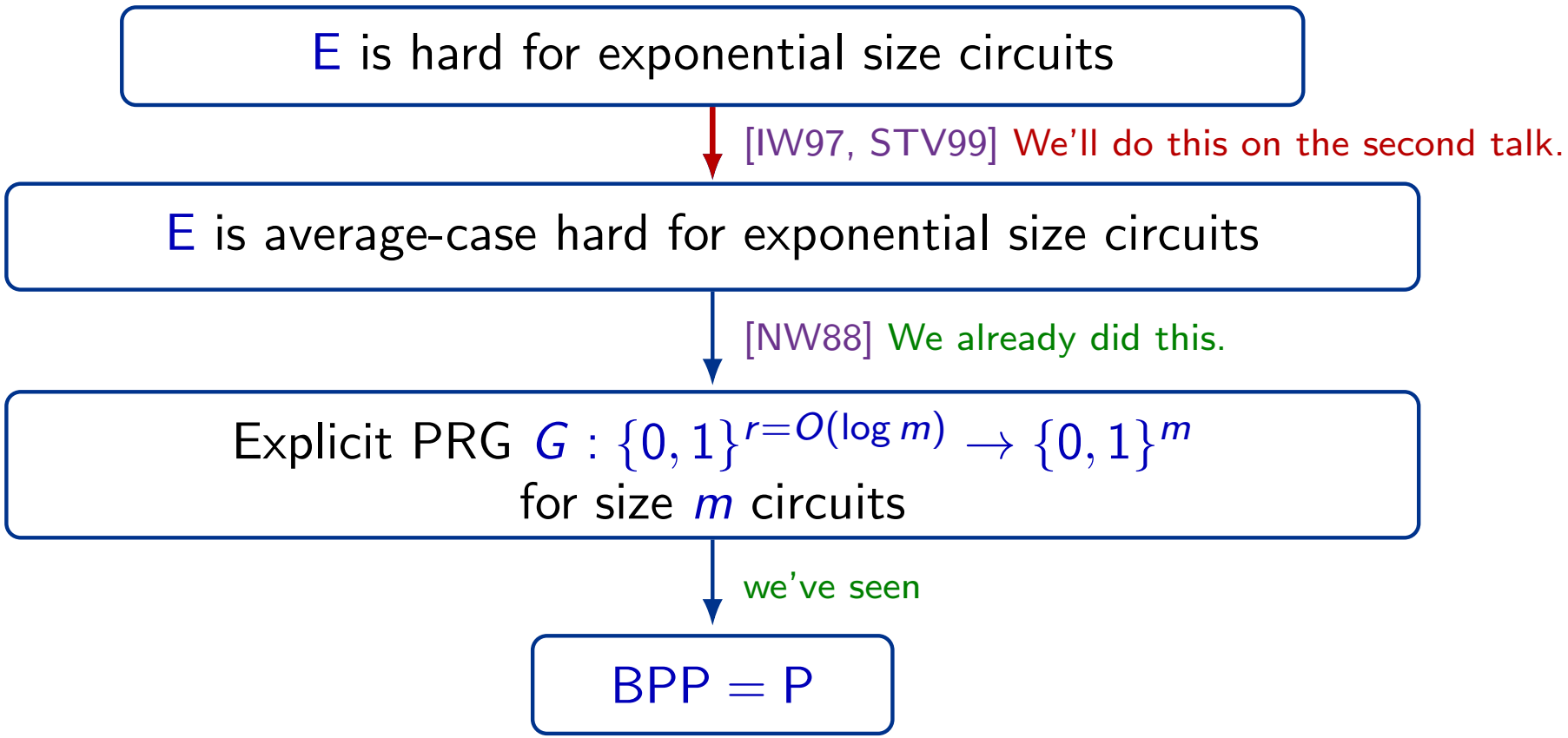

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Excerpt from a 2016 talk on hardness vs. randomness for Avi's birthday.

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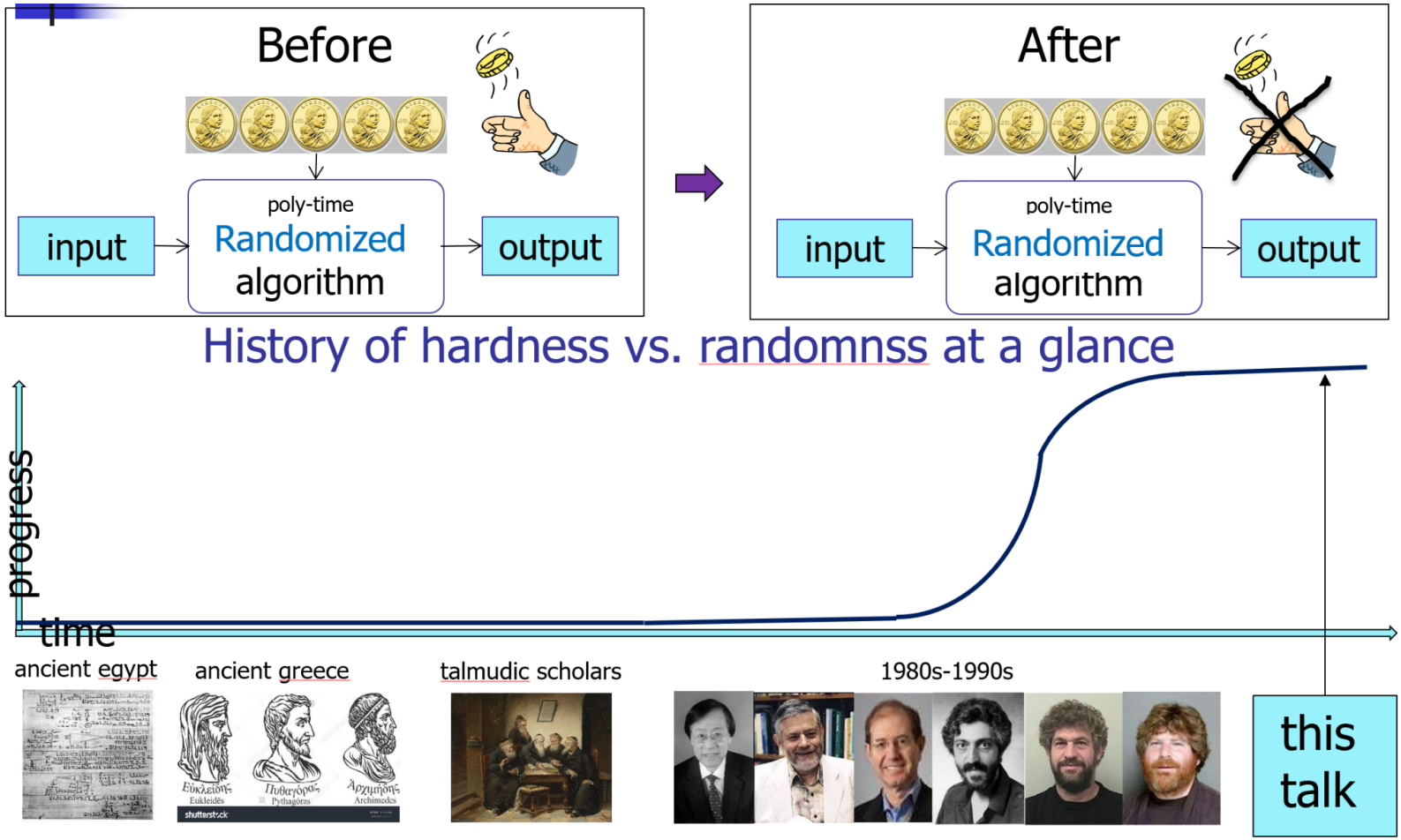

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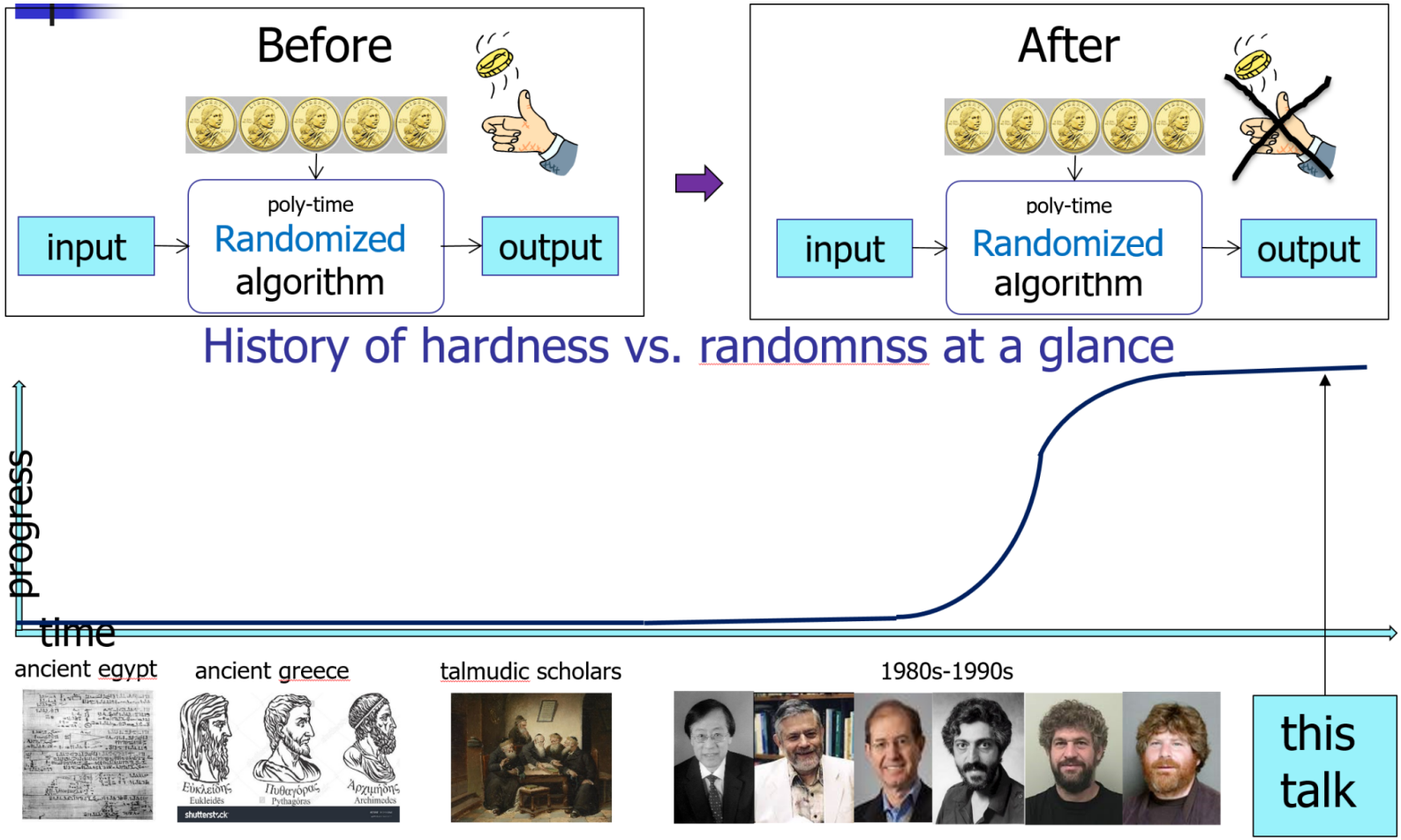


Amazing progress in recent years!

We'll hear about these developments in future talks and seminars.

End of 1st talk

Thank you!

An Introduction to the Hardness vs. Randomness Paradigm

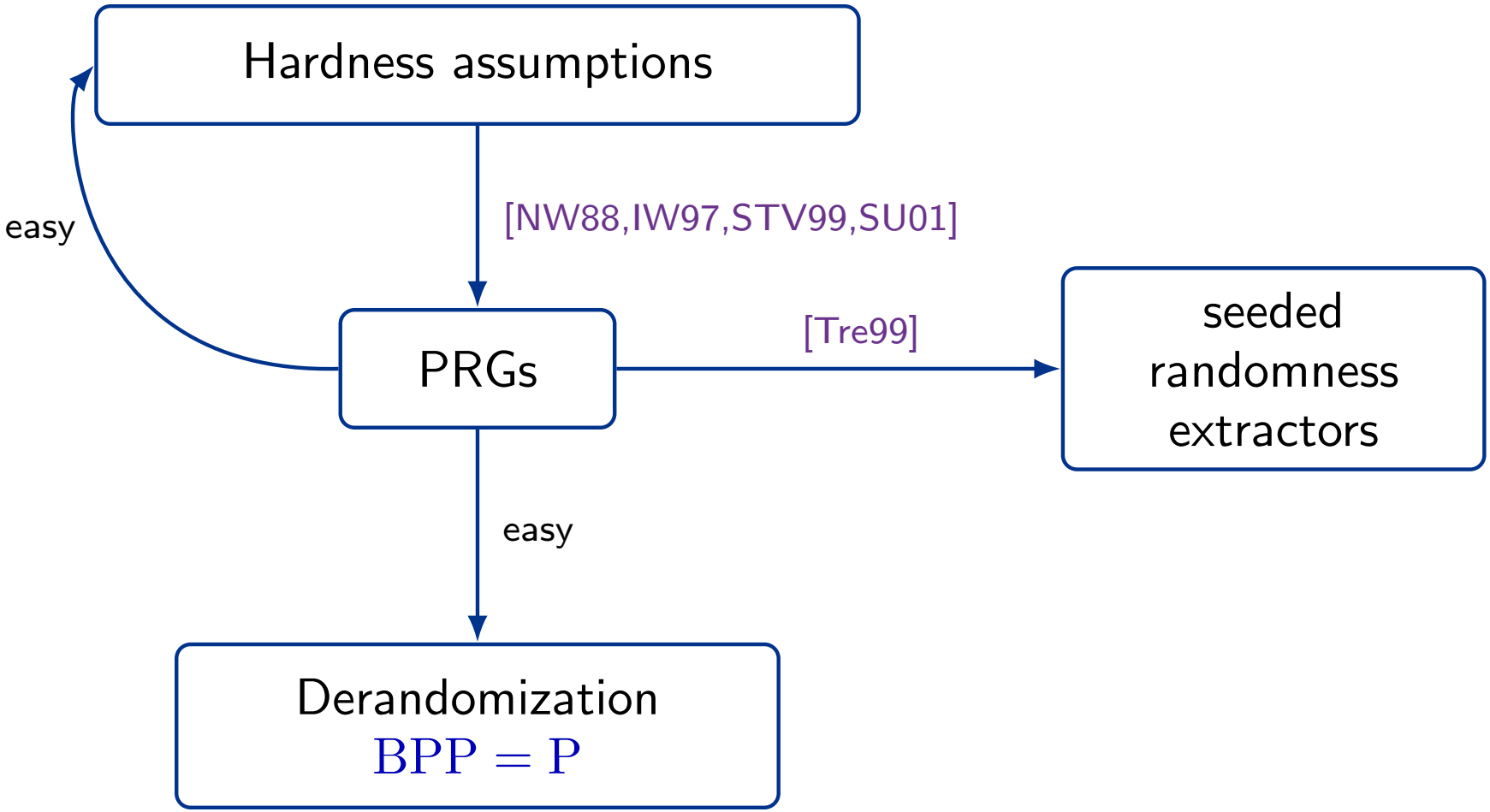
Tutorial for the Simons HDX and Pseudorandomness Workshop

Ronen Shaltiel

University of Haifa

August 2026

Overall summary for the two talks



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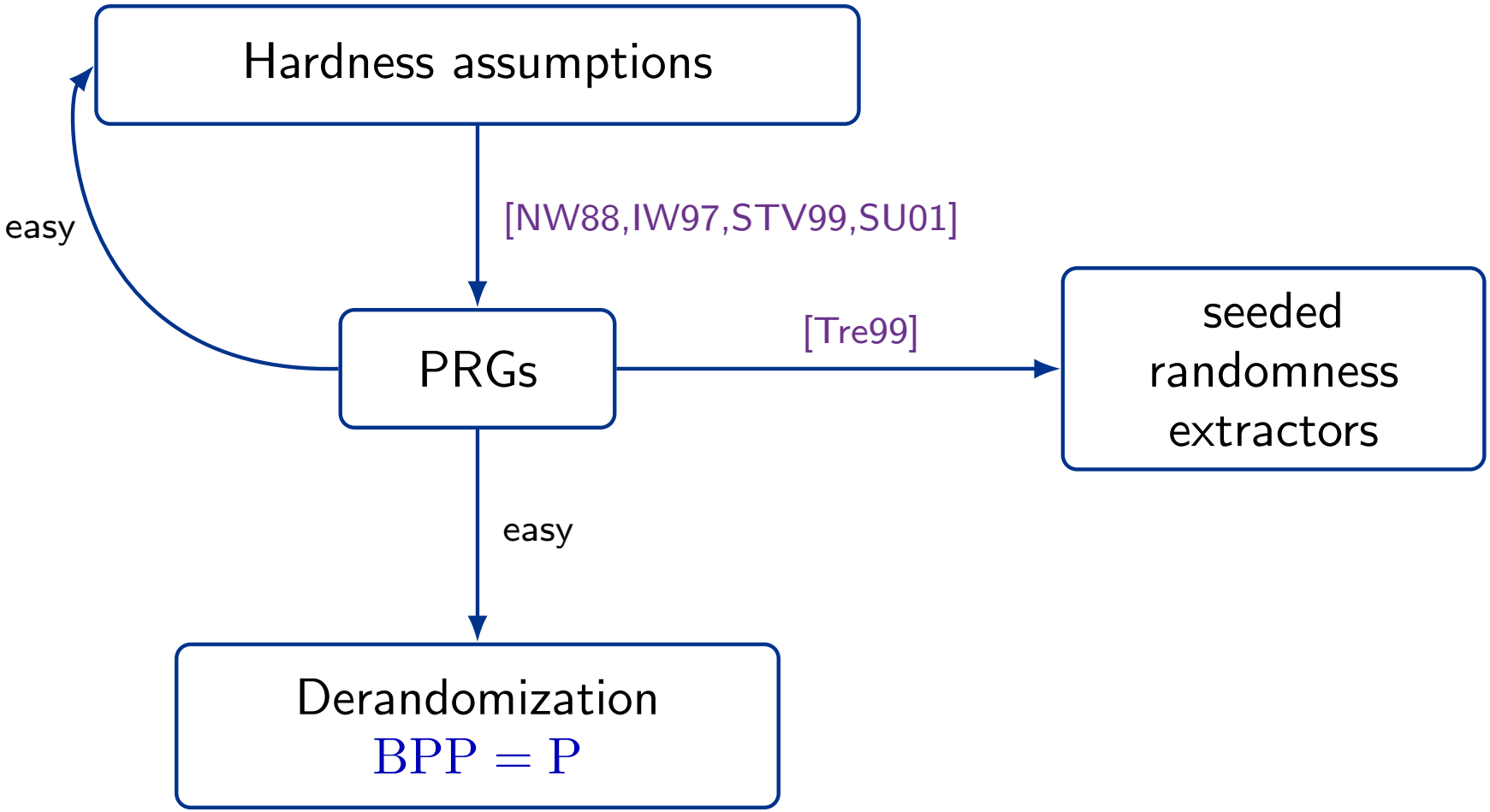

Then: extensions, generalizations, applications,
and some other research directions.

Reminder of key definitions from first talk

Definition (Pseudorandom generator (PRG))

$G : \{0,1\}^r \to \{0,1\}^m$ is an ϵ-PRG for $D : \{0,1\}^m \to \{0,1\}$ if

$$|\Pr[D(G(U_r)) = 1] - \Pr[D(U_m) = 1]| \leq \epsilon.$$

Throughout this talk, $\epsilon = 1/10$.

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Definition (Hard functions)

$f : \{0,1\}^n \to \{0,1\}$ is

- s-hard if $\forall$size s circuit C, $\exists x \in \{0,1\}^n$ s.t. $C(x) \neq f(x)$.
- s-average-case-hard if $\forall$size s circuit C, $\Pr_{x \leftarrow U_n}[C(x) = f(x)] \leq \frac{1}{2} + \frac{1}{s}$.

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- E is hard for exponential size circuits if $\exists \beta > 0$, and $\exists L \in \mathrm{E} = \mathrm{DTIME}(2^{O(n)})$ s.t. $\forall n$, 1_L is $2^{\beta n}$-hard (on inputs of length n).
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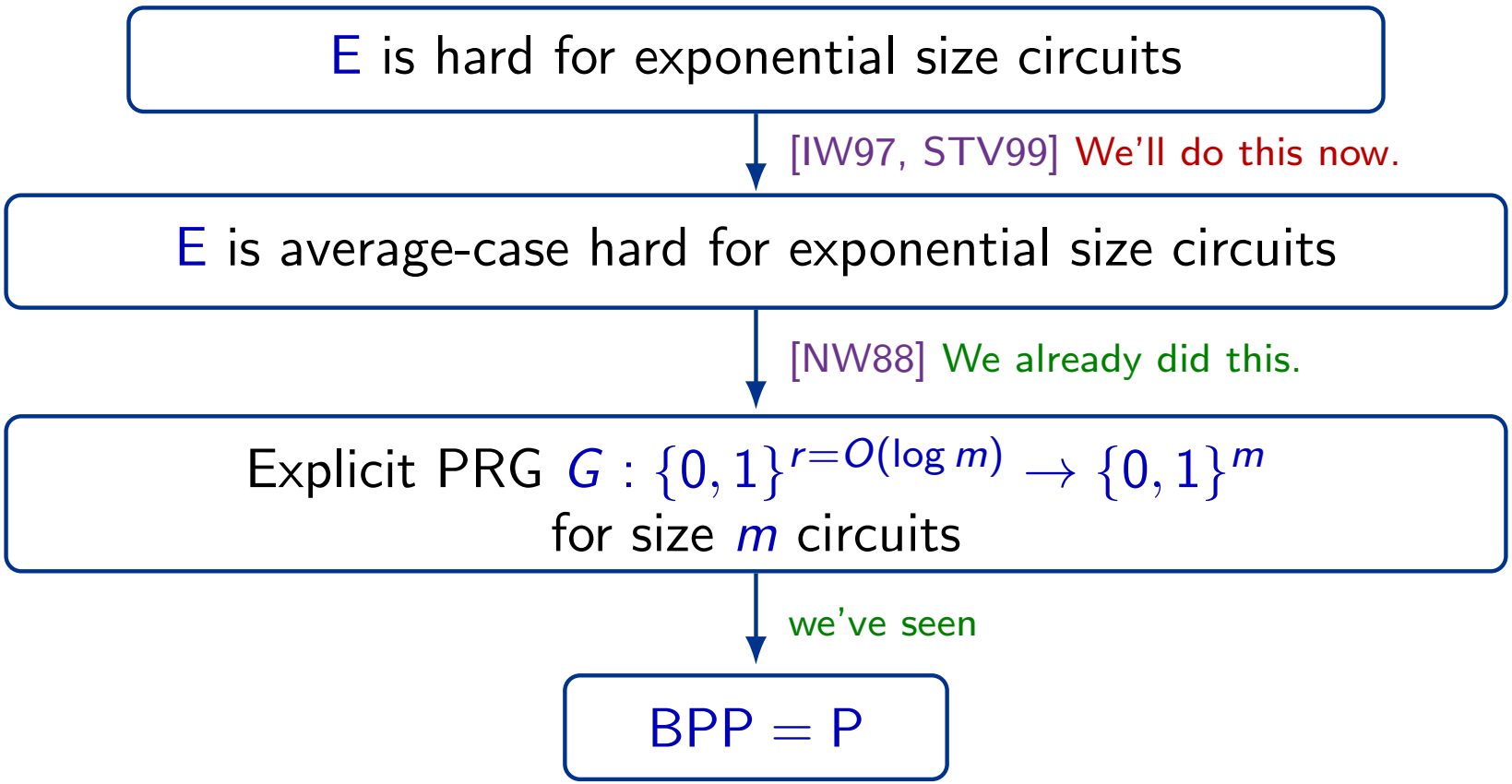

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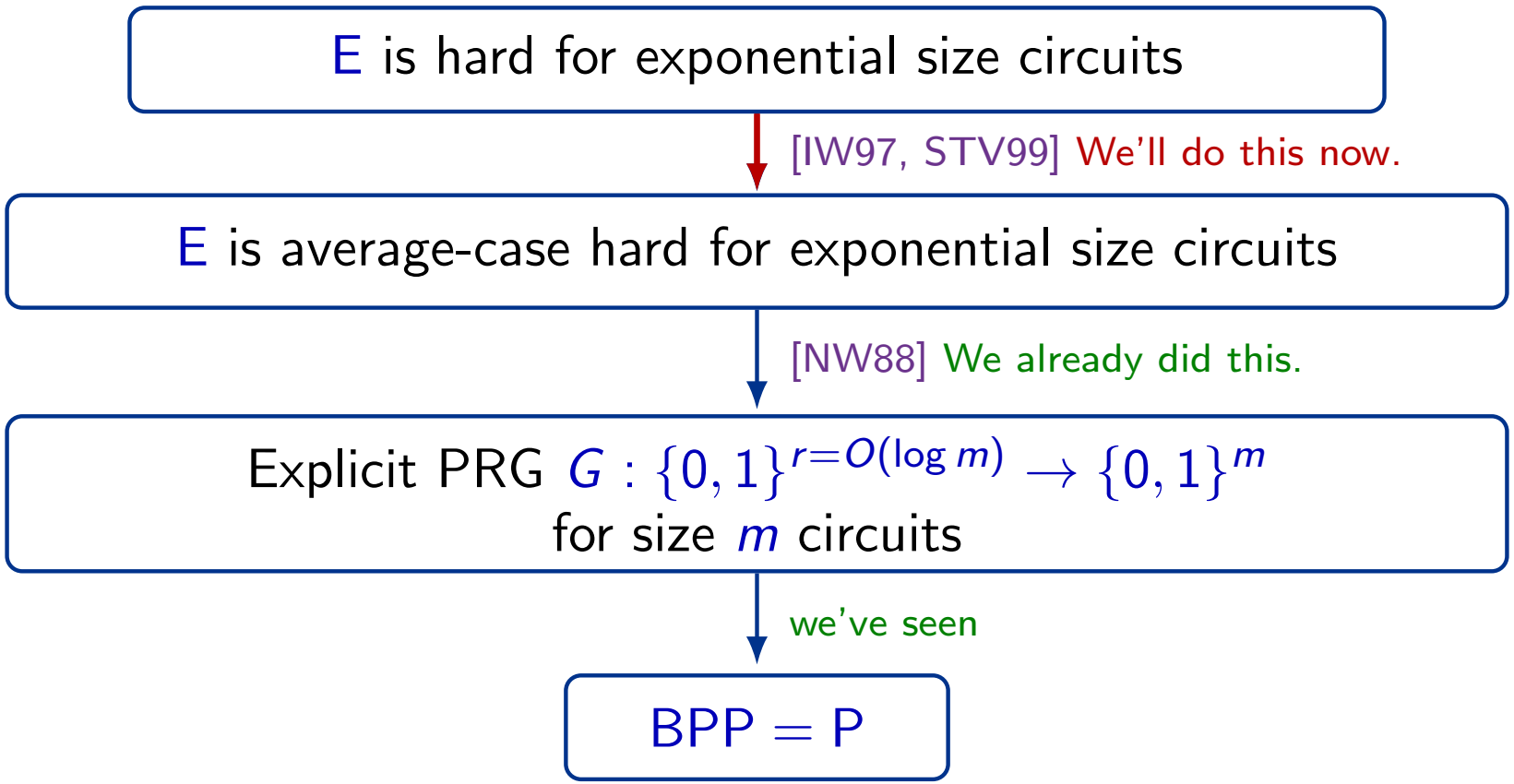

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Consequence: Worst-case to average-case hardness amplification.

Hardness amplification and list decoding [STV99]

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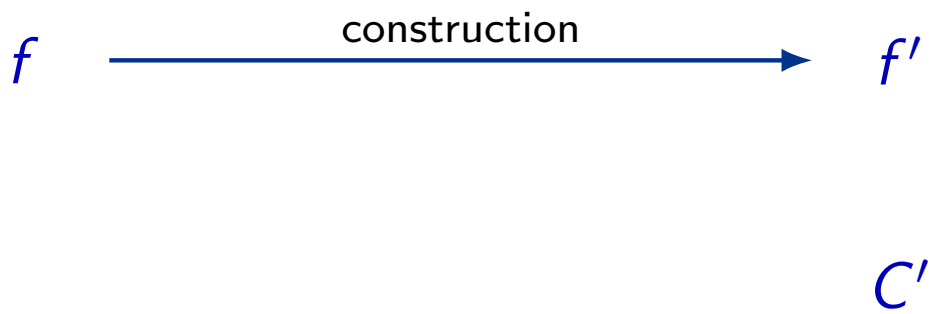

Hardness amplification and list decoding [STV99]

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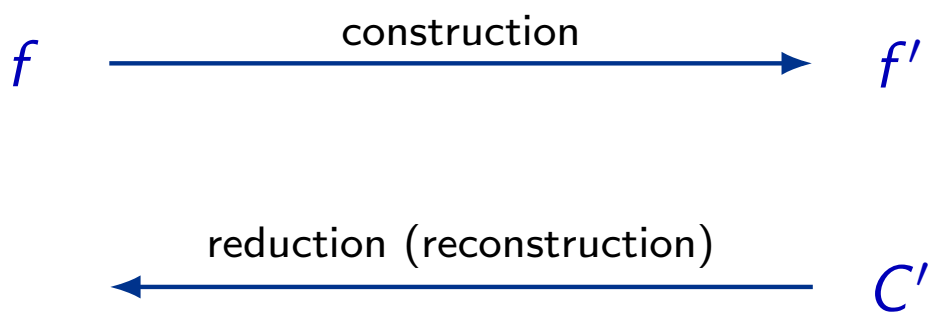

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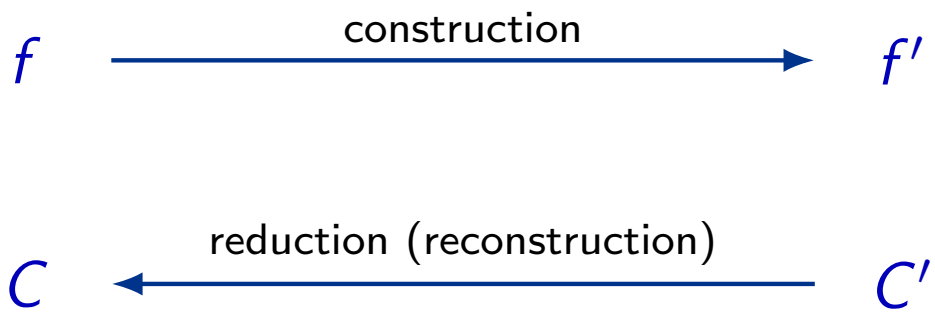

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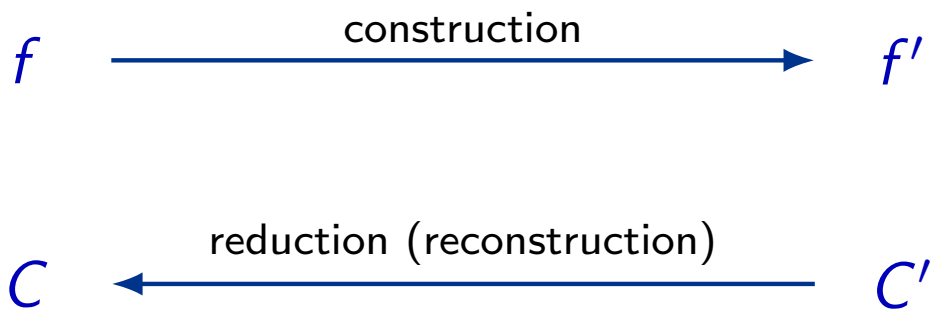
Hardness amplification

f, f' are functions



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Hardness amplification **(list)-decoding**

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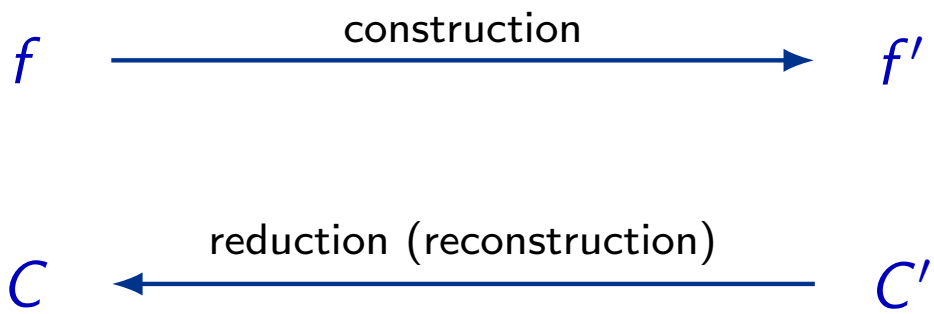

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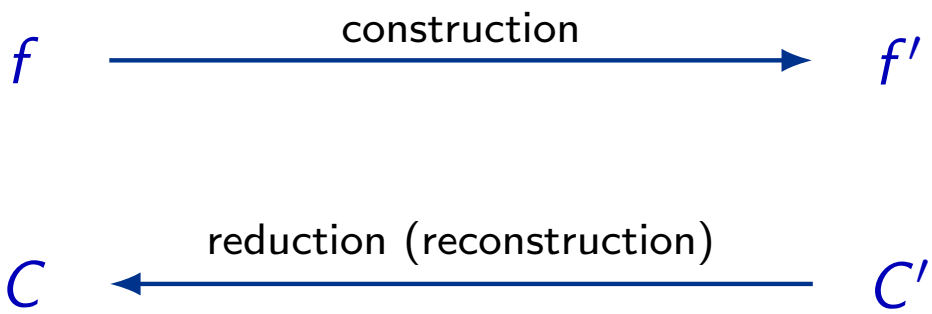

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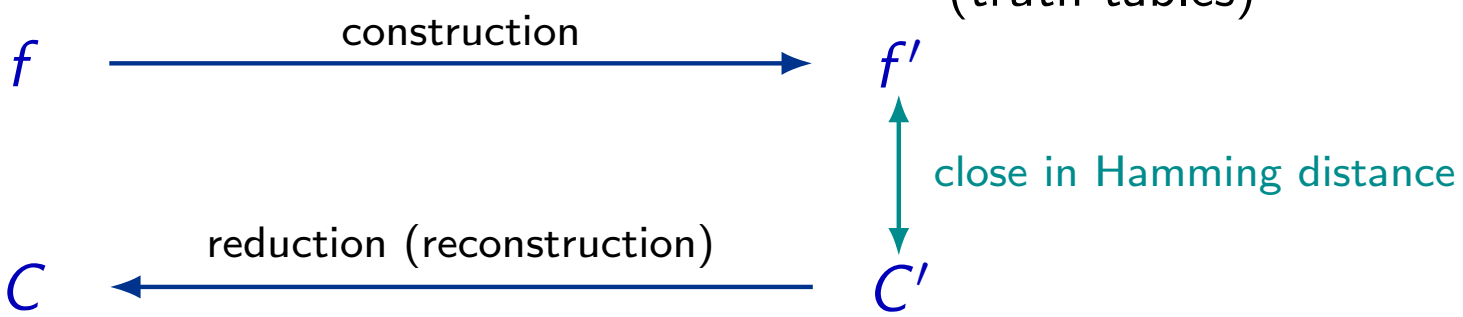

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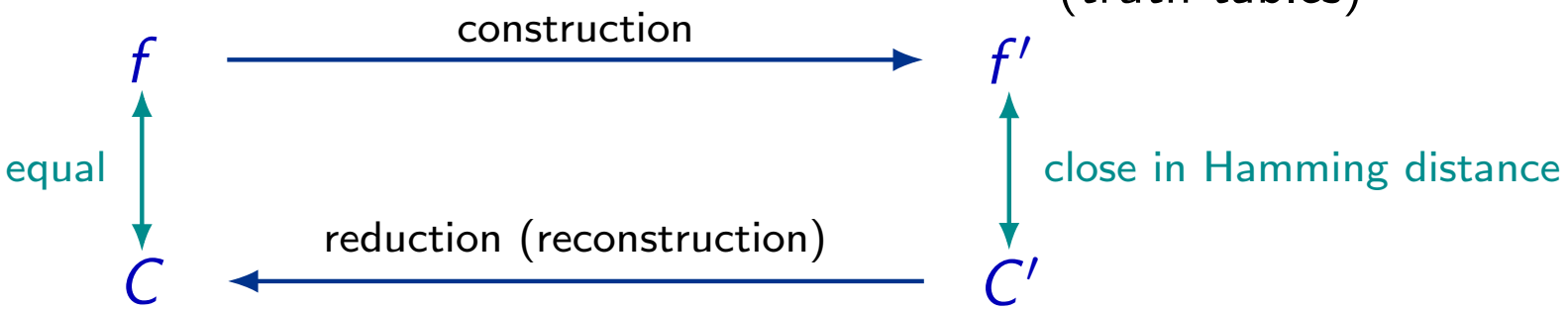

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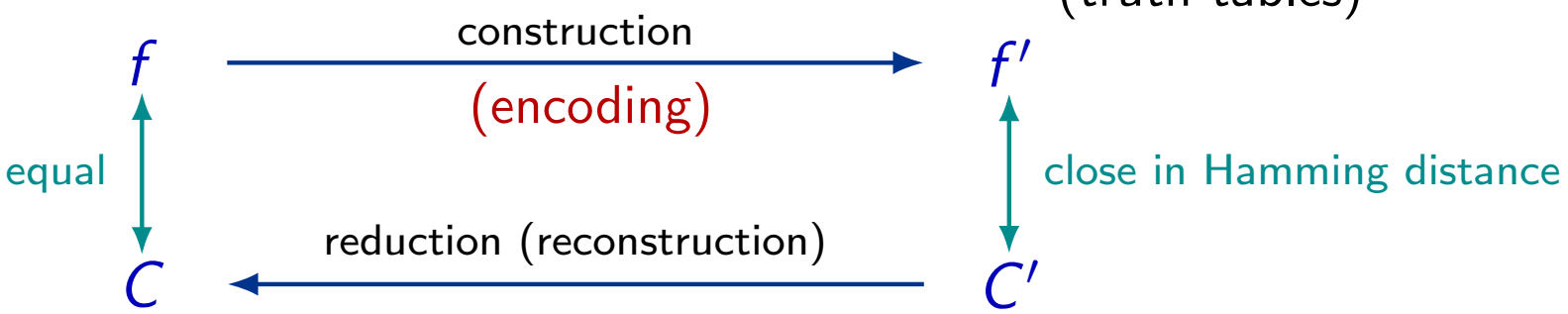

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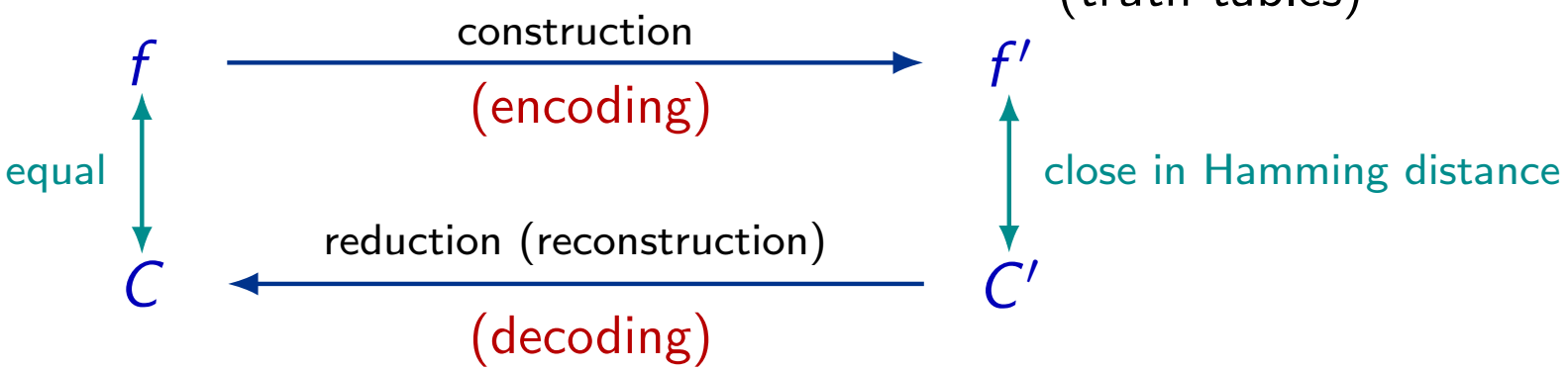

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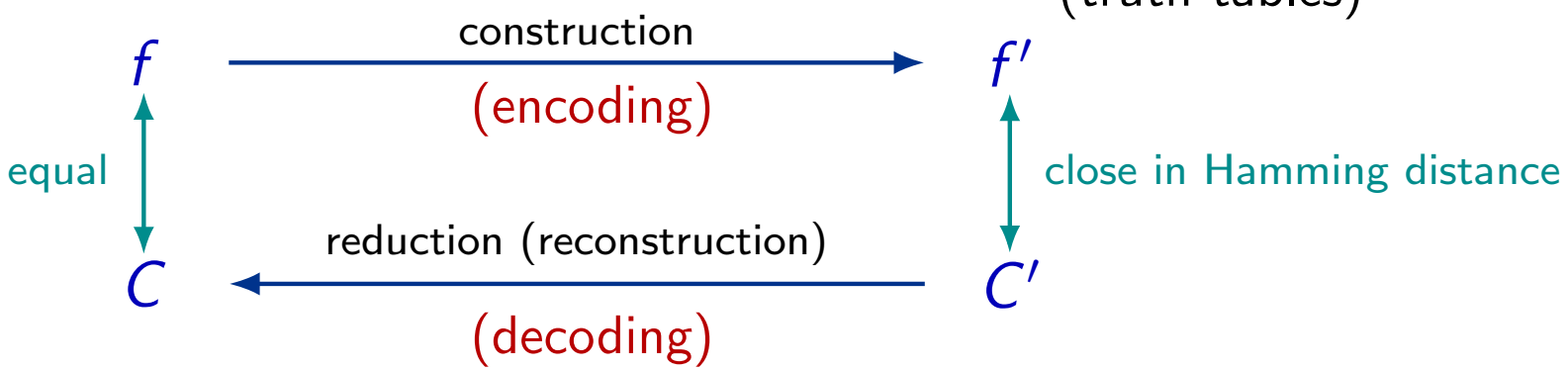

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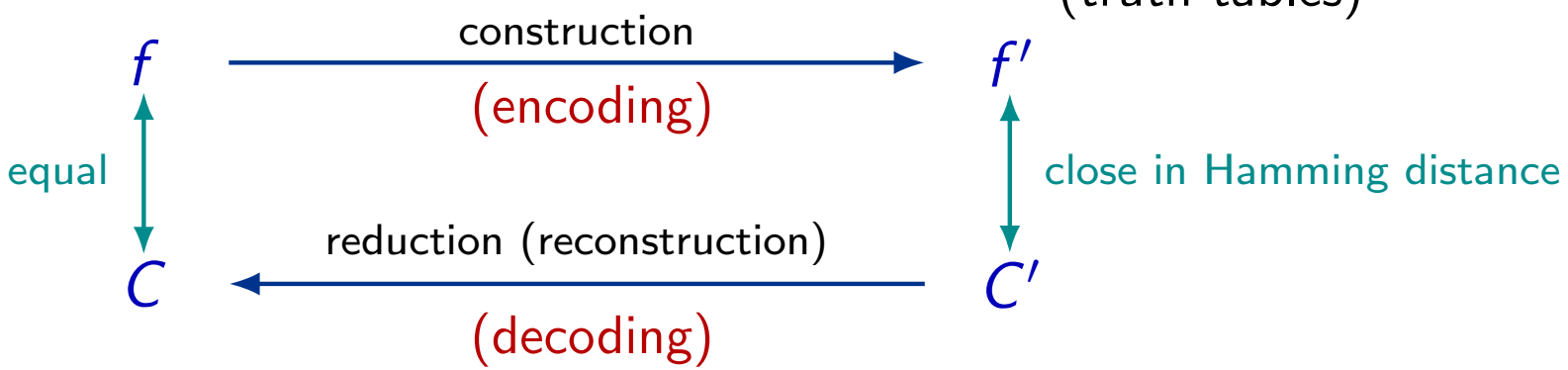

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"Computational Construction" = "info-theory object" + "efficient reduction".

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Theorem (STV99: hardness amplification - non-boolean version)

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$h = 2^{\ell/d}$

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$|H^d| = h^d = 2^\ell$

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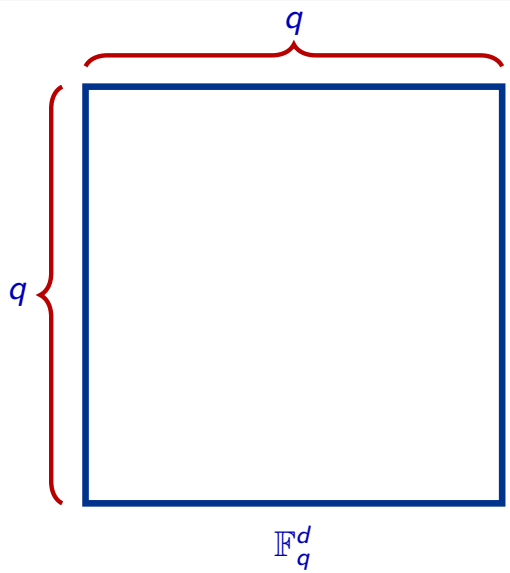

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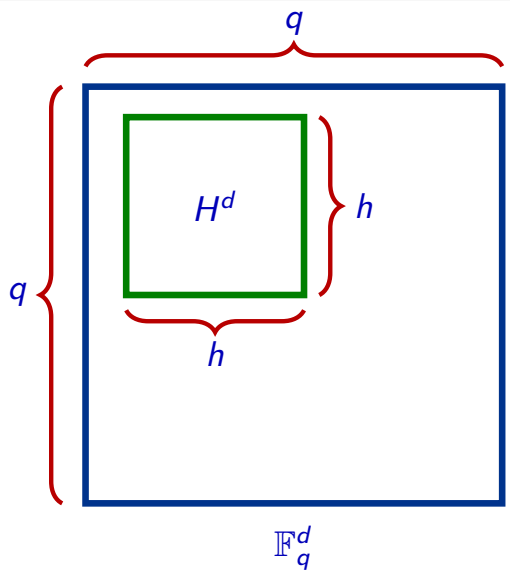

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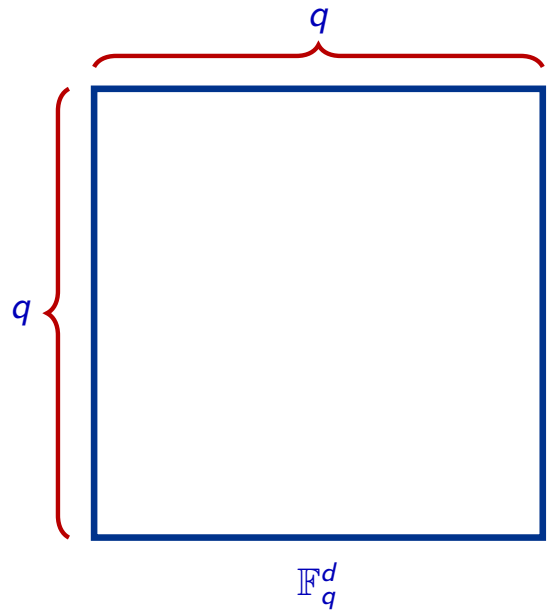
Reduction (decoding): $\hat{C} \to C$

Starting point: A circuit $\hat{C}$ of size $\hat{s}$ s.t. $\Pr_{x \leftarrow \mathbb{F}_q^d}[\hat{C}(x) = \hat{f}(x)] > 1/\hat{s}$.



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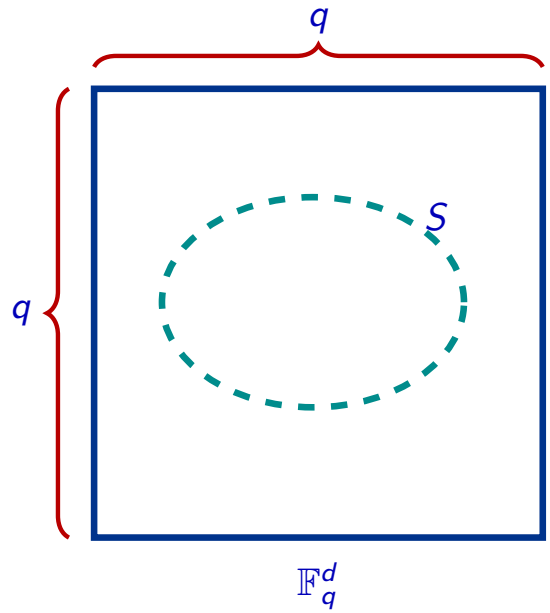

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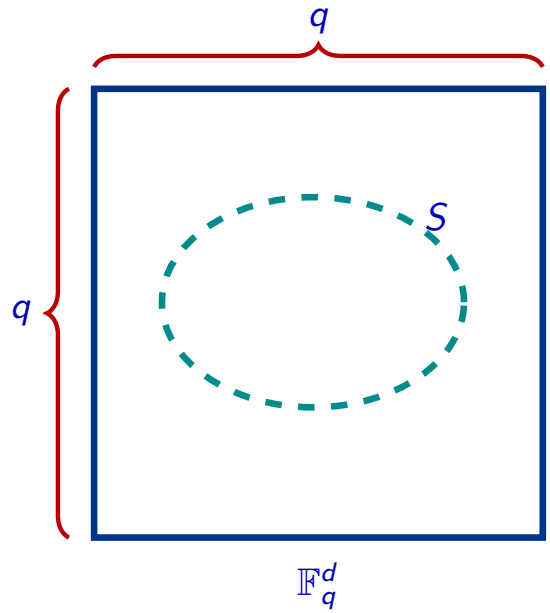

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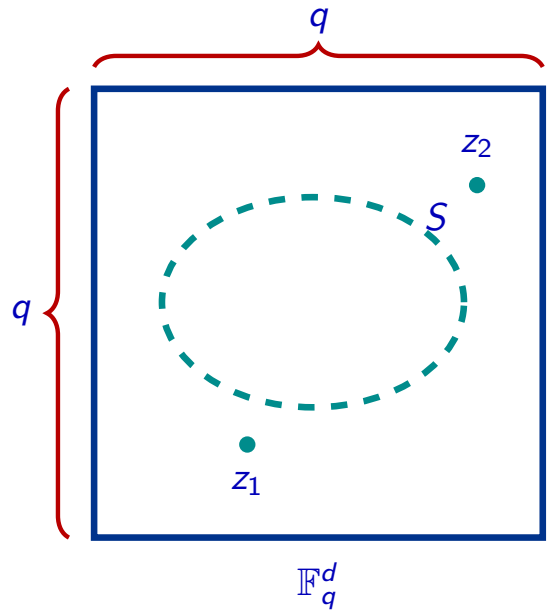

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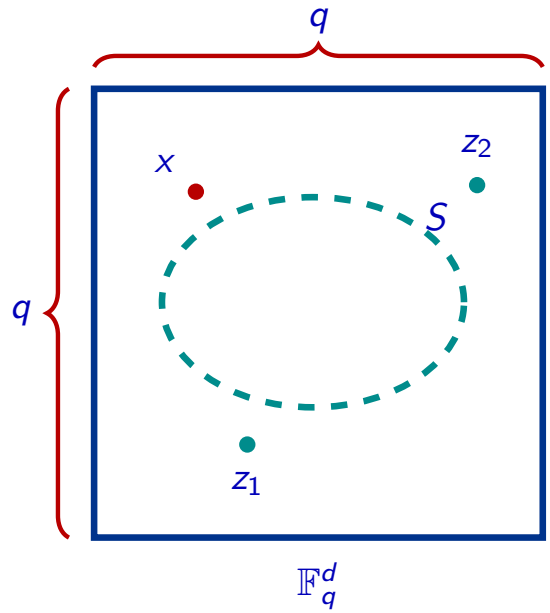

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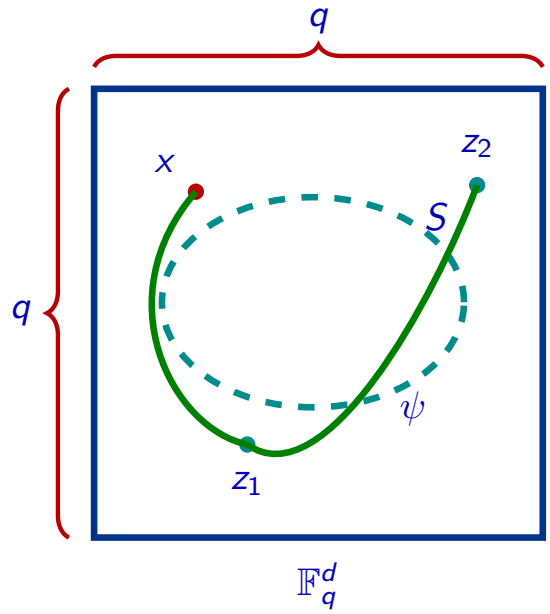

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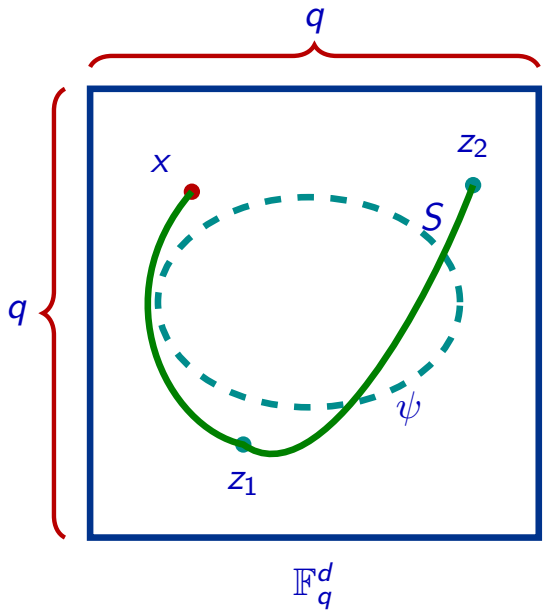

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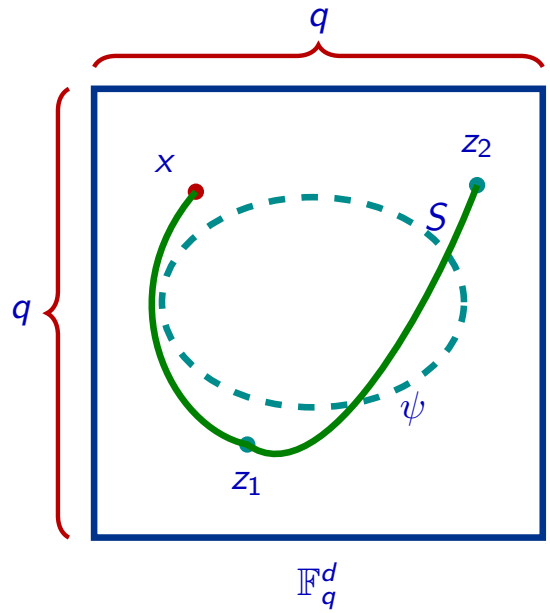

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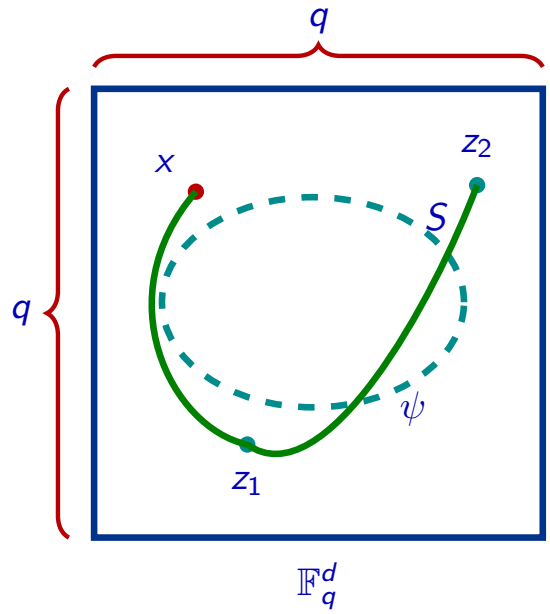

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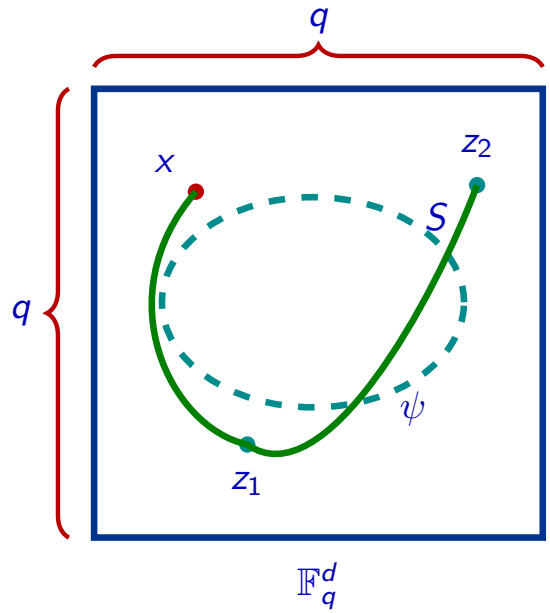

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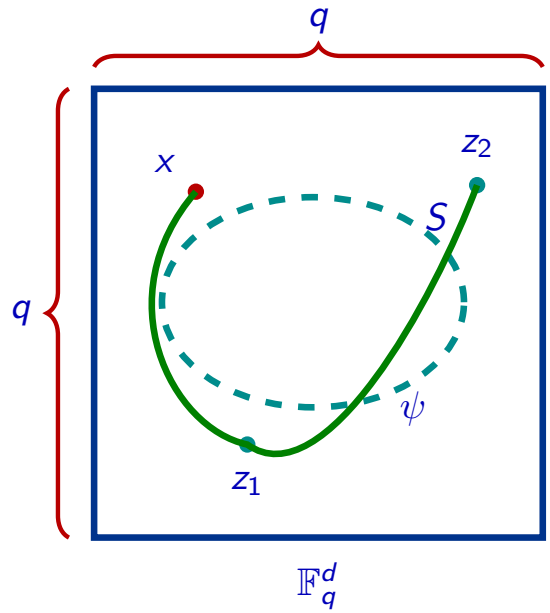

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Obtaining p (which gives us $p(0) = f(x)$) is a Reed-Solomon decoding problem!

Using Reed-Solomon (list)-decoding

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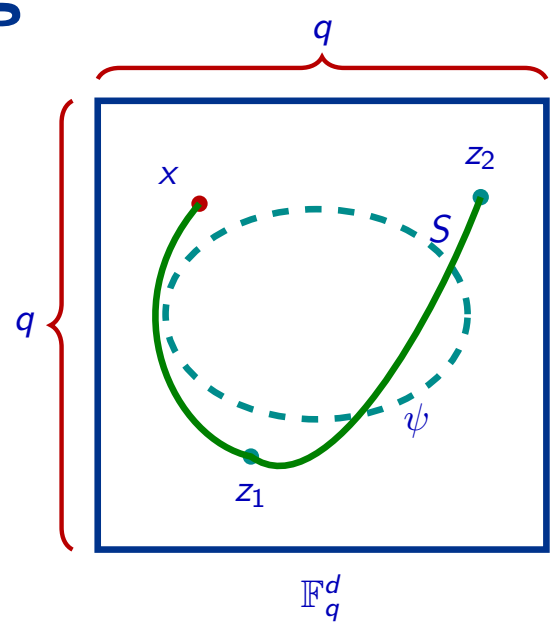

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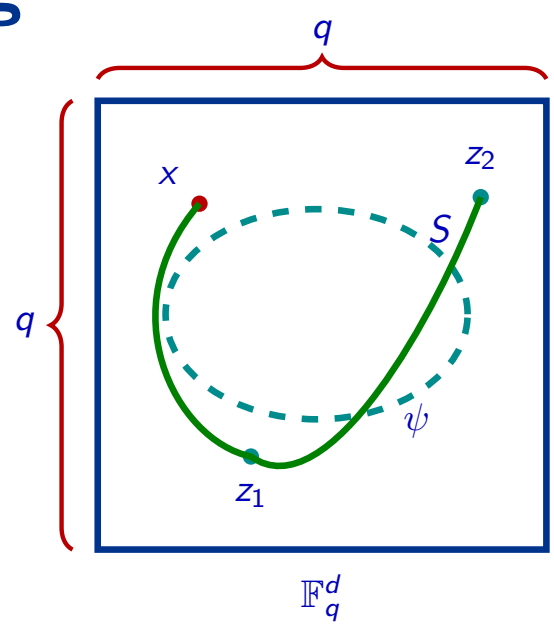

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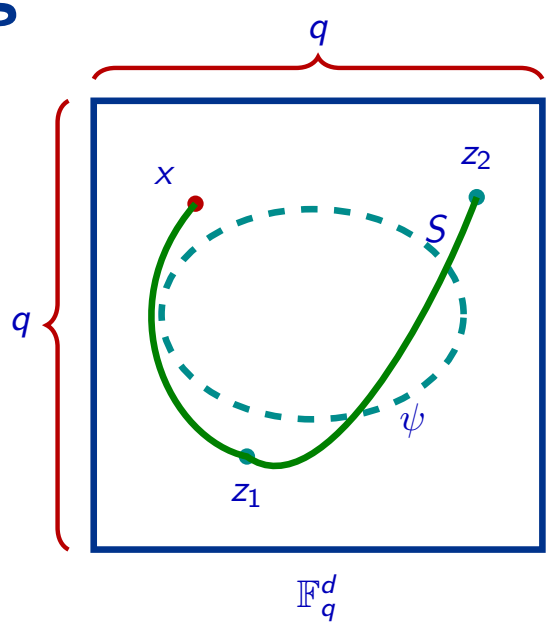

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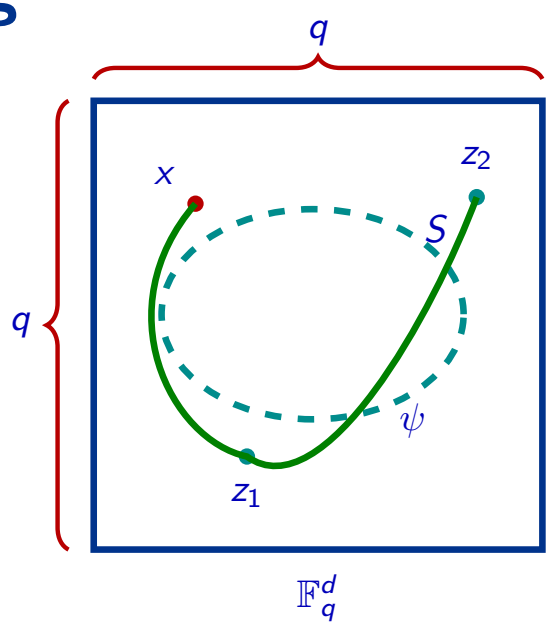

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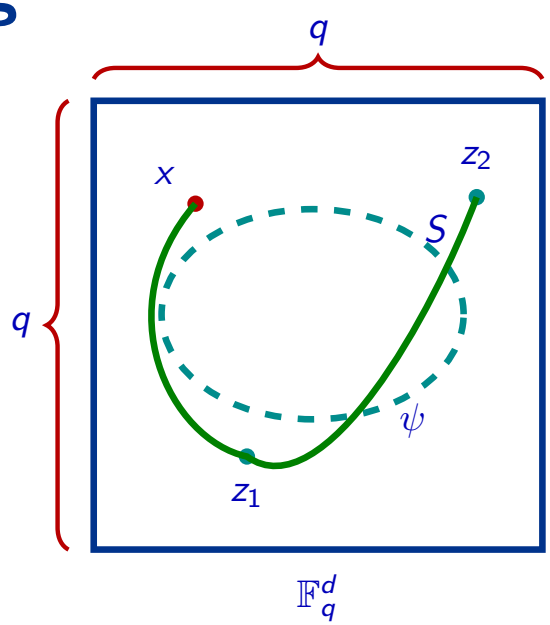

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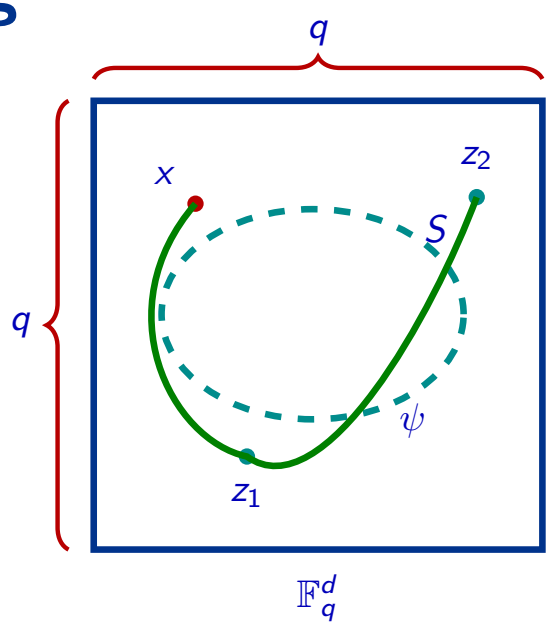

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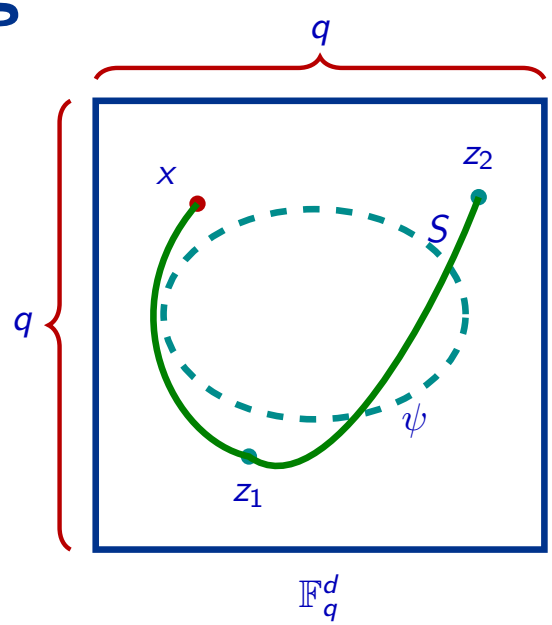

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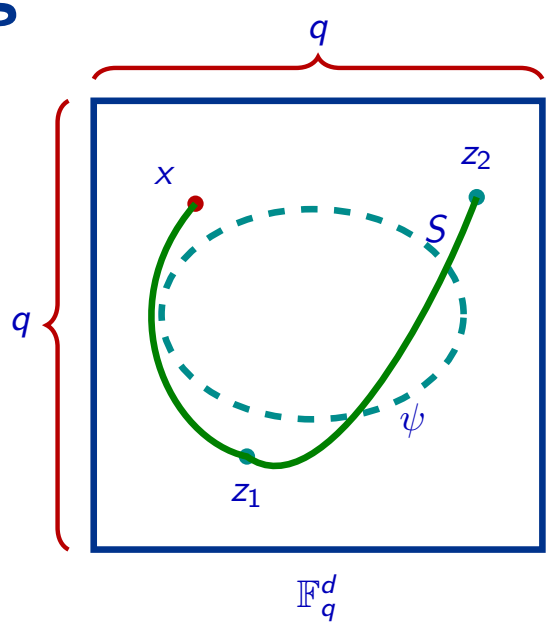

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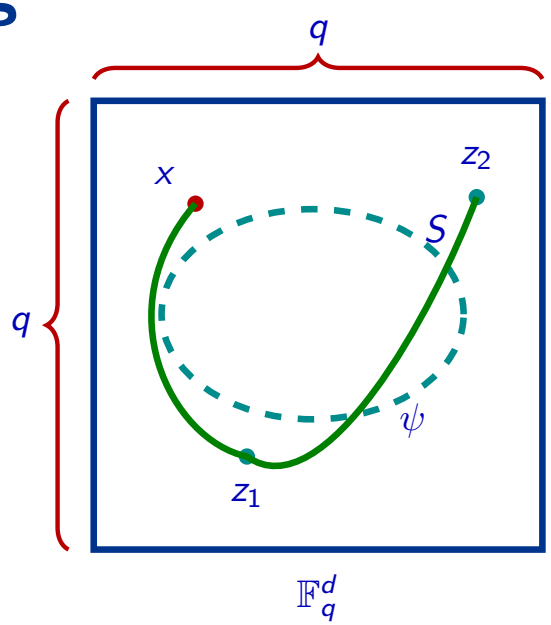

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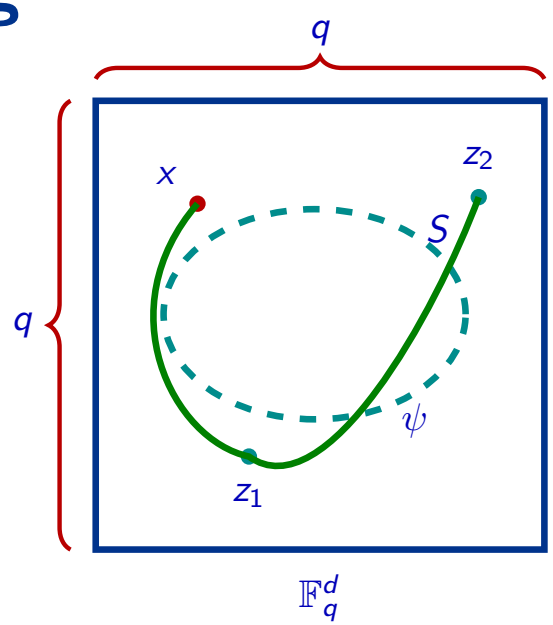

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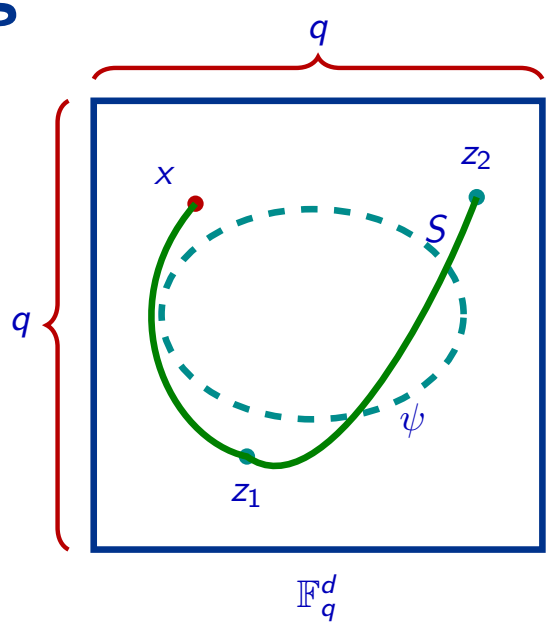

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(Important point is that time is $\text{poly}(q) \ll 2^{\ell}$ and not $\text{poly}(q^d) > 2^{\ell}$).

Choosing a sufficiently large d, can make time as small as required.

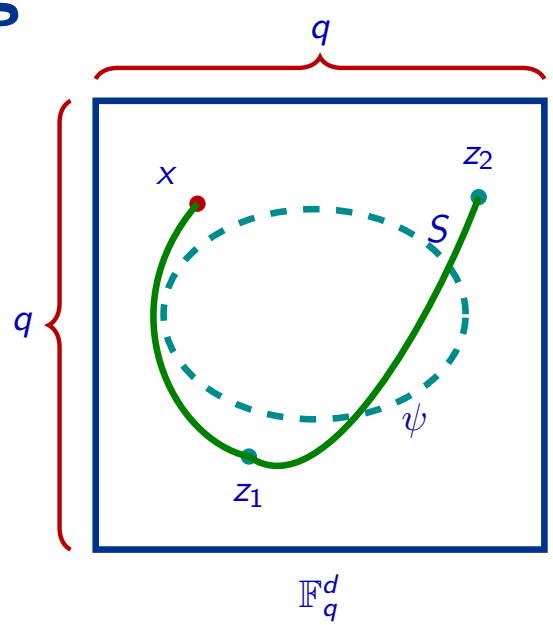
Using Reed-Solomon (list)-decoding

We only know that $a > 1/\hat{s}$ fraction of evaluations agree with p, where $a < 1/2$.

In this range unique-decoding is impossible.

List-decoding algorithm [Sud97] returns a list of $L = \text{poly}(\hat{s})$ candidate polynomials $p_1, \ldots, p_L$, such that one of them is the correct polynomial p.

How do we find the correct polynomial p?



The circuit C that we will construct will also be hardwired with $\hat{f}(z_1)$.

Claim: w.h.p. p is the only polynomial in the list that agrees with $\hat{f}$ on z_1.

Proof: Two different low degree polynomials differ w.h.p. on a random point.

This claim means that we can identify the correct polynomial p.

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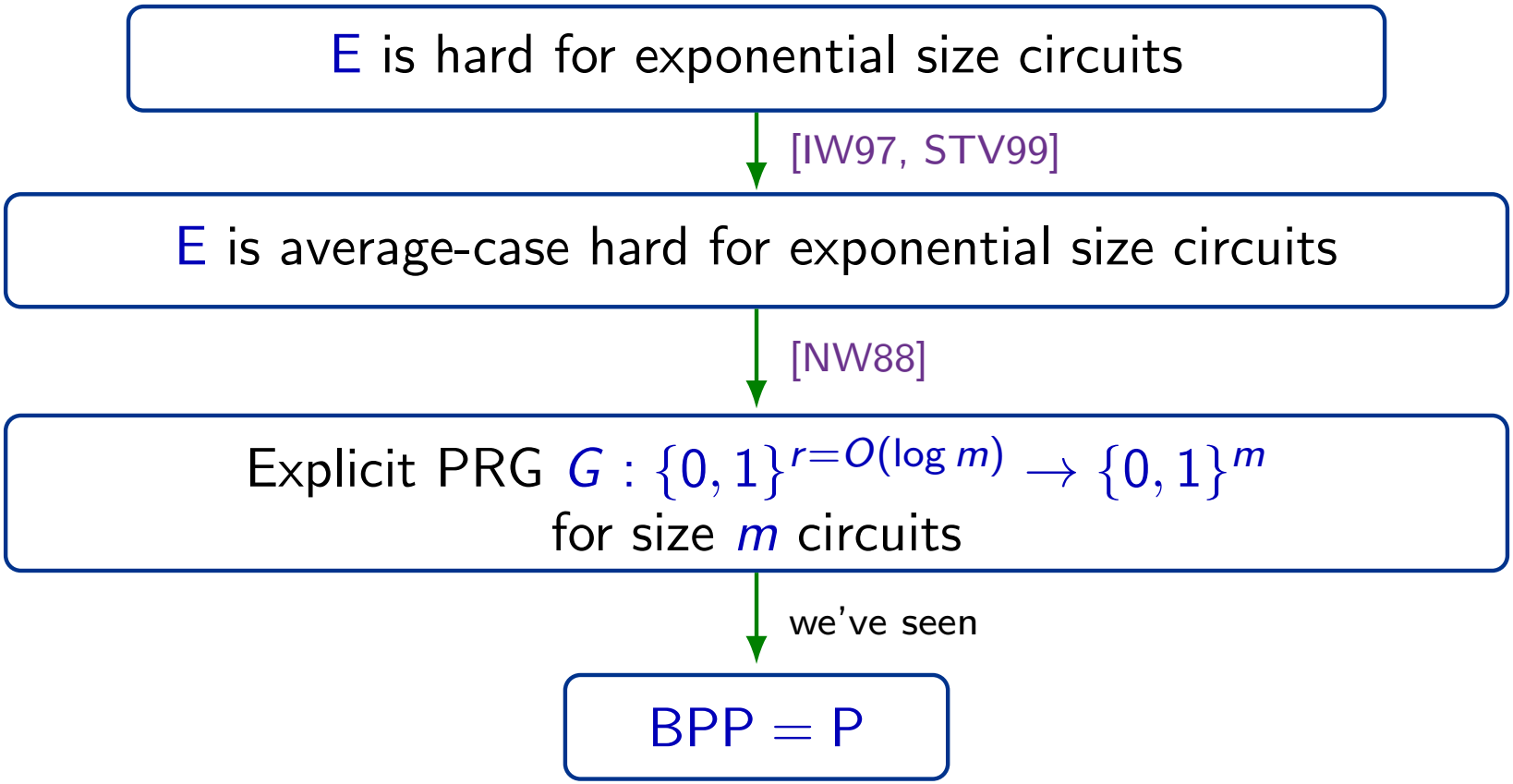
Comment: This presentation is different from [STV99] which uses lines (and not curves) and a more delicate argument.

Hardness assumptions $\Rightarrow$ derandomization: recap

Theorem (IW97)

E *is hard for exponential size circuits* $\Rightarrow$ BPP $=$ P.



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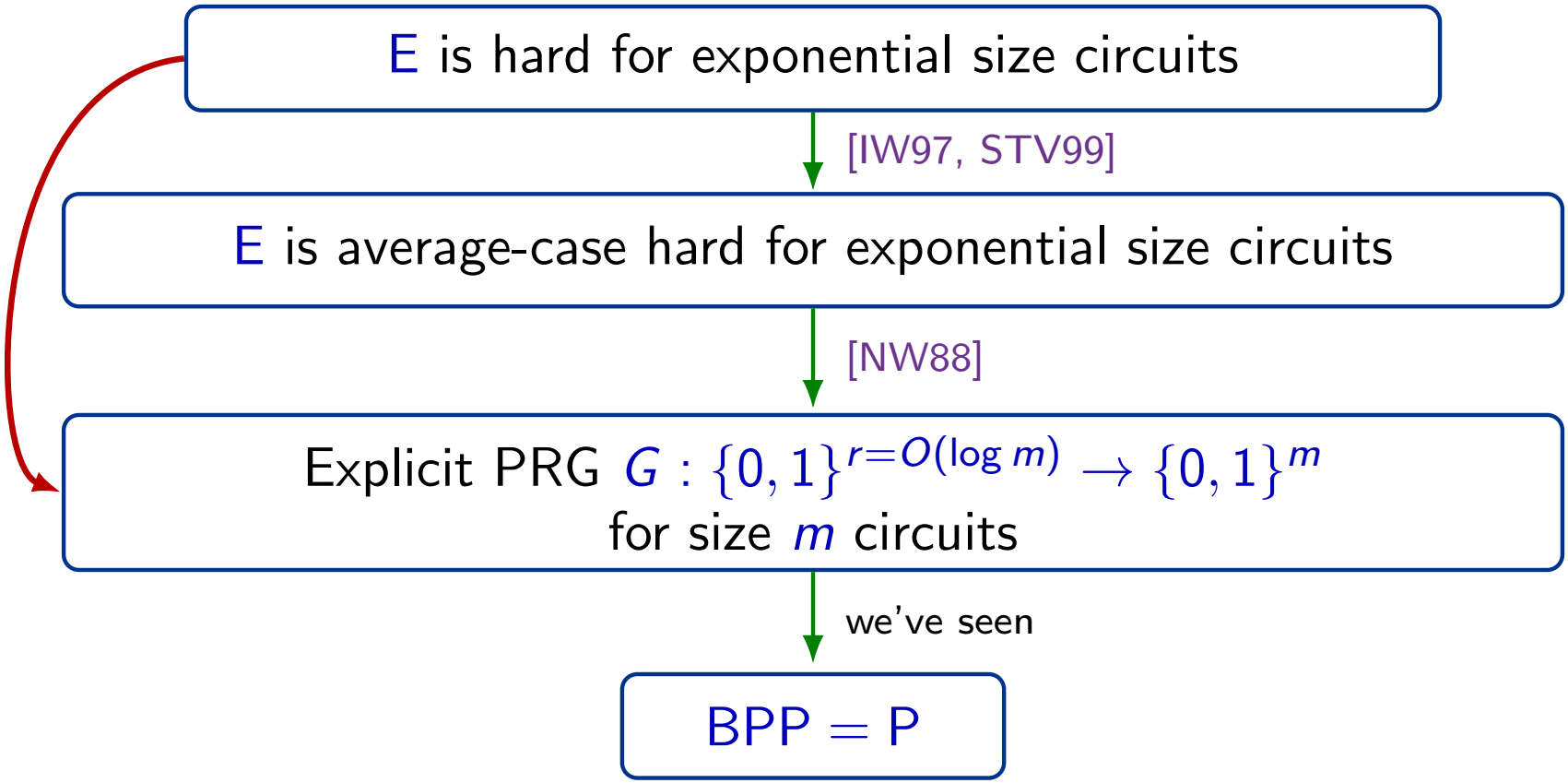

We'll now sketch a different PRG construction [SU01,Uma02].

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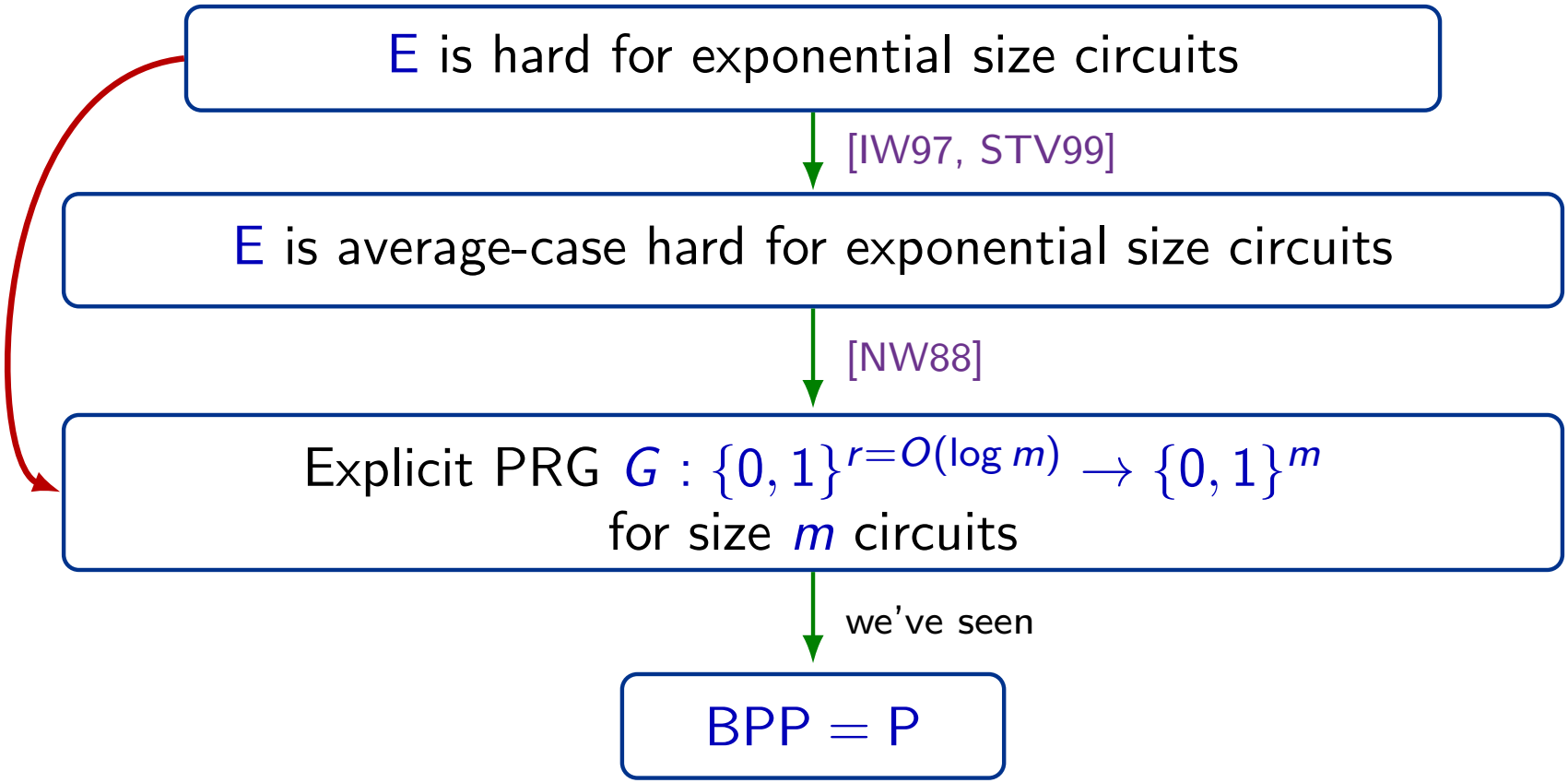

We'll now sketch a different PRG construction [SU01,Uma02].

Does not use designs, and has shorter seed length in some scenarios.

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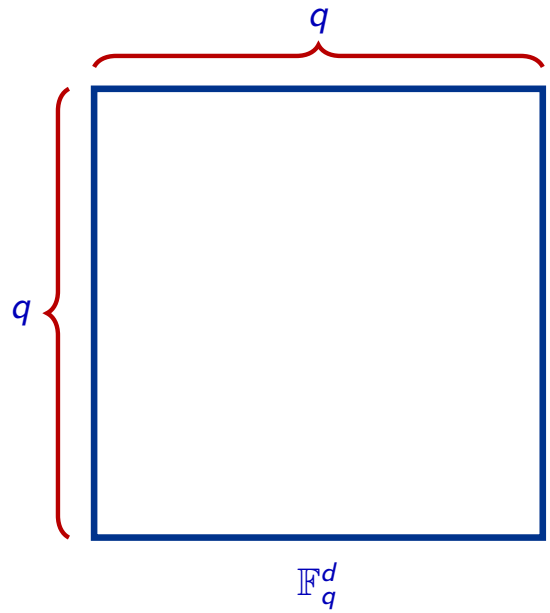

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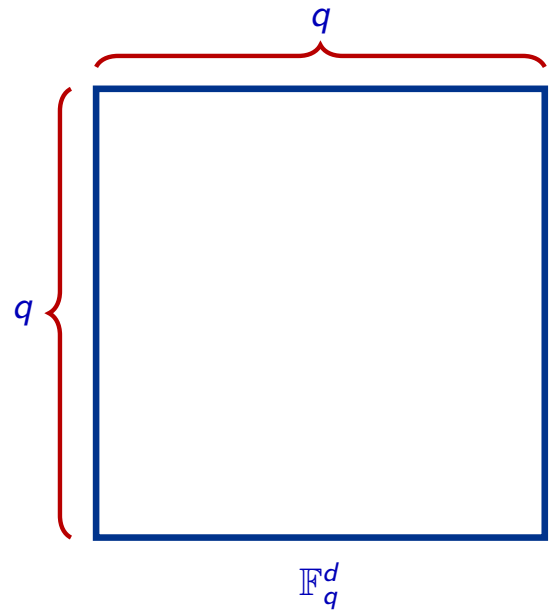

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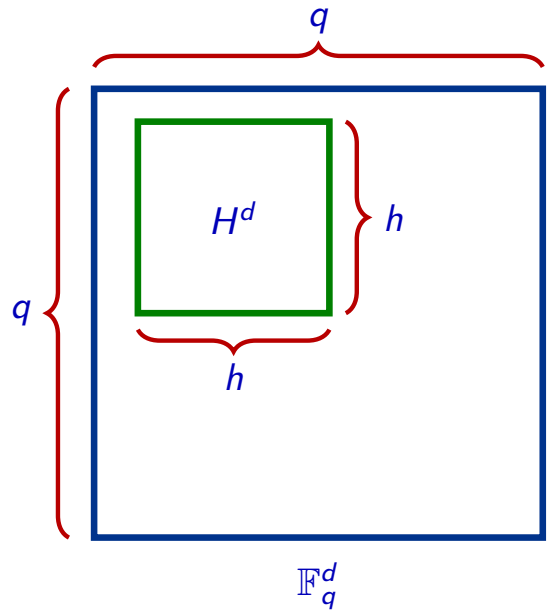

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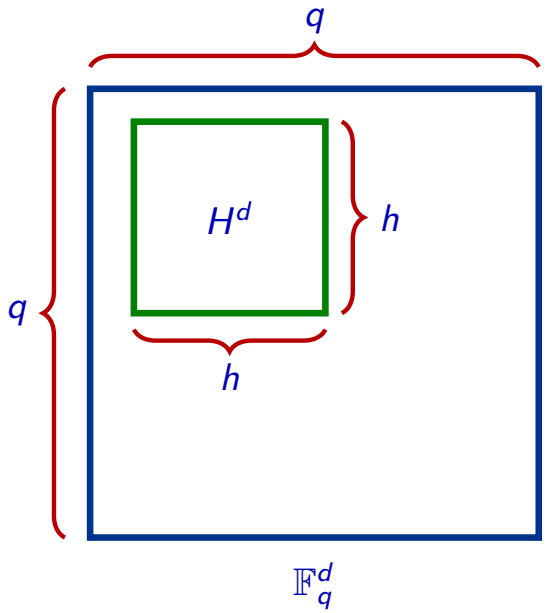

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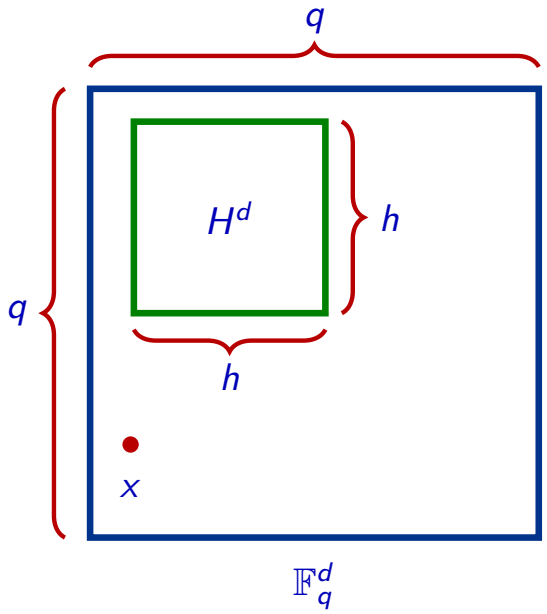

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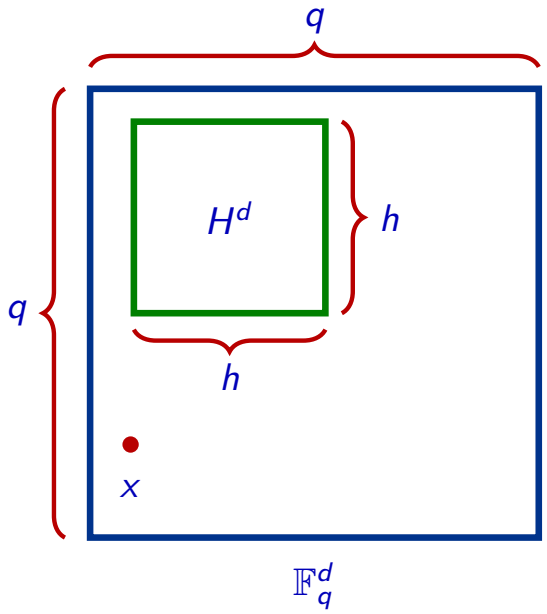

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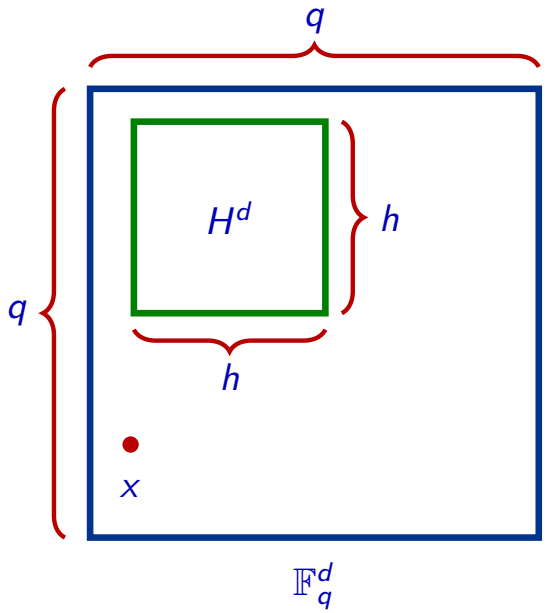

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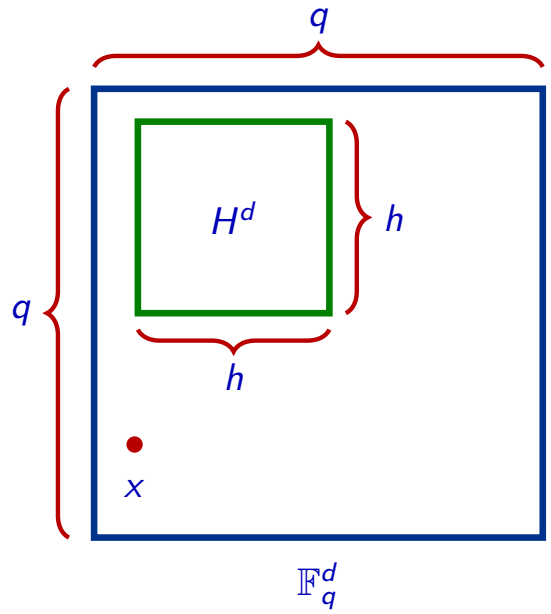

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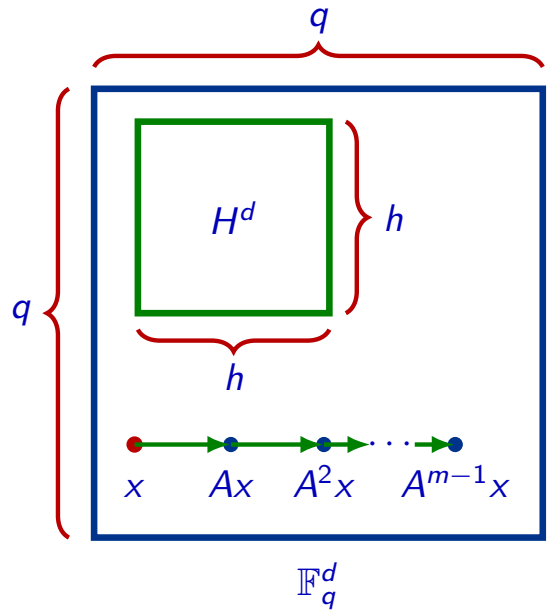

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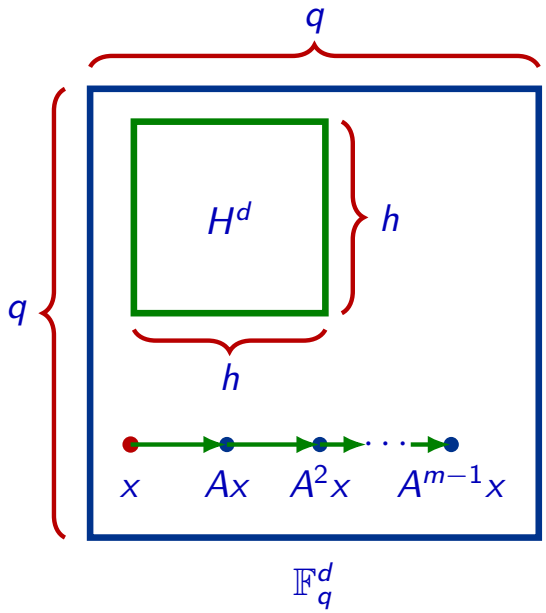

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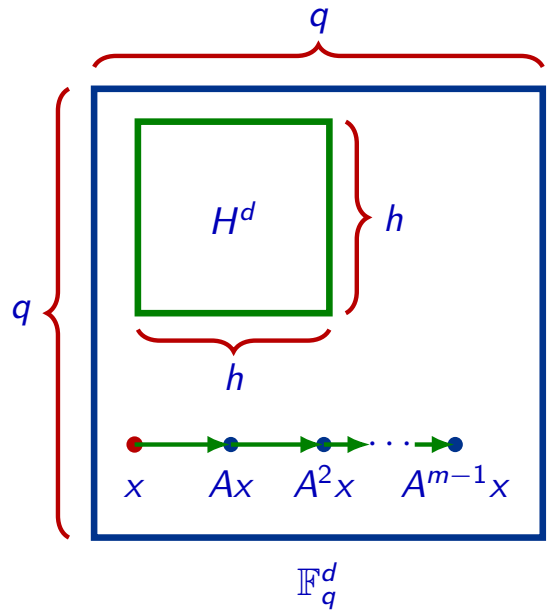

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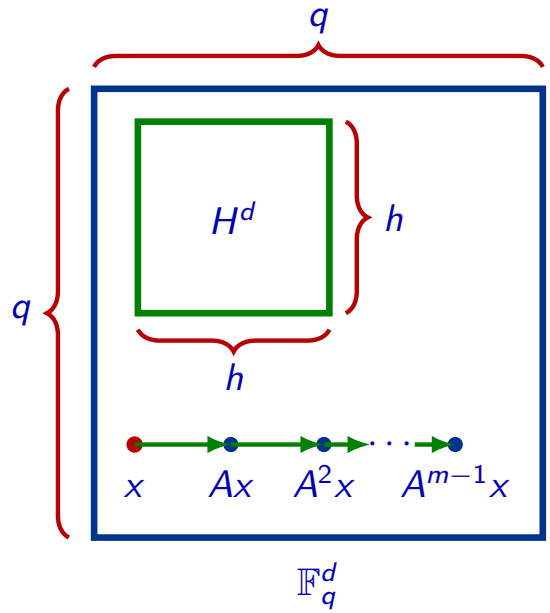

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Idea: PRG = STV99 + predictor (from previous elements).

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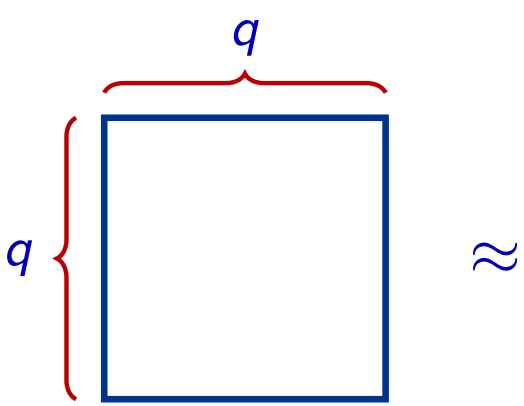

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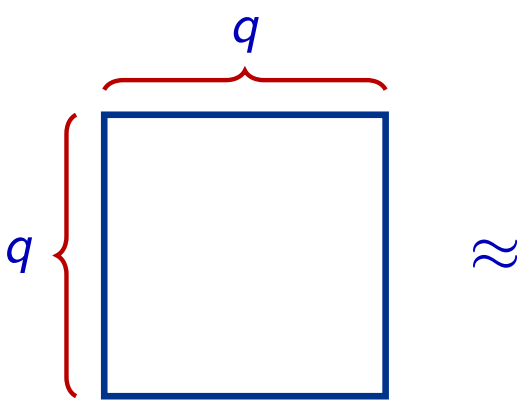

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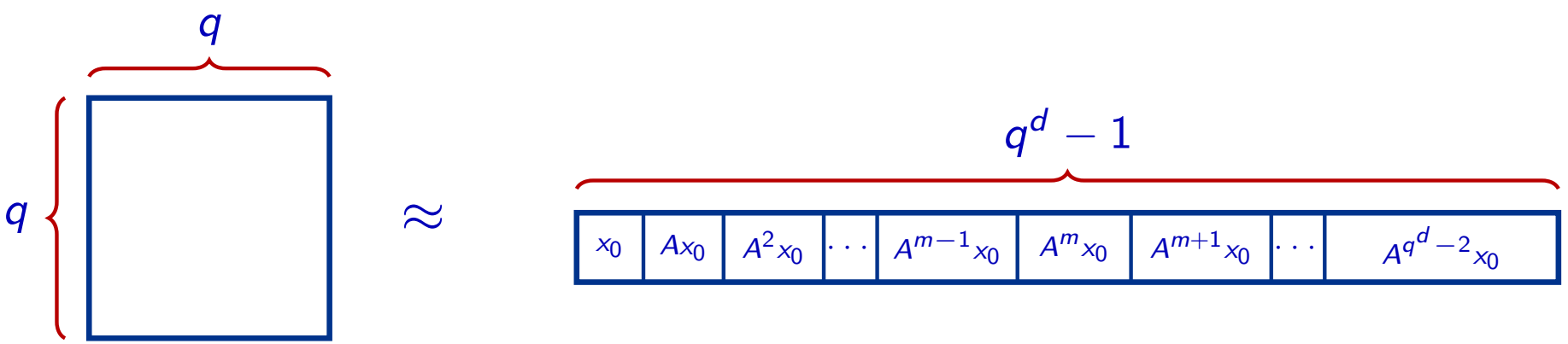

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x_0	Ax_0	A^2x_0	$\cdots$	$A^{m-1}x_0$	$A^m x_0$	$A^{m+1}x_0$	$\cdots$	$A^{q^d-2}x_0$

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We assume that we have an "error-corrected" predictor $\hat{P}$ s.t. for every $x \in \mathbb{F}_q^d$,

$$\hat{P}\Big(x, \hat{f}(A^{-m}x), \hat{f}(A^{-(m-1)}x), \ldots, \hat{f}(A^{-1}x)\Big) = \hat{f}(x).$$

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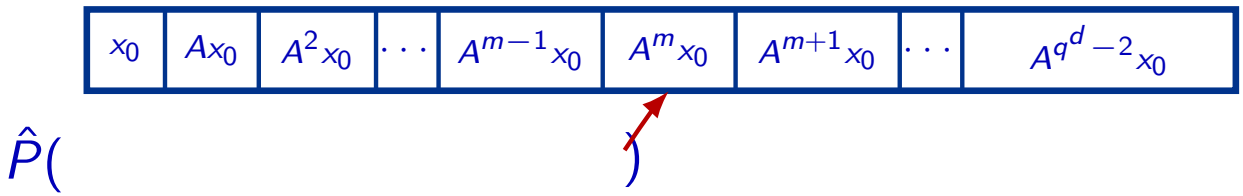

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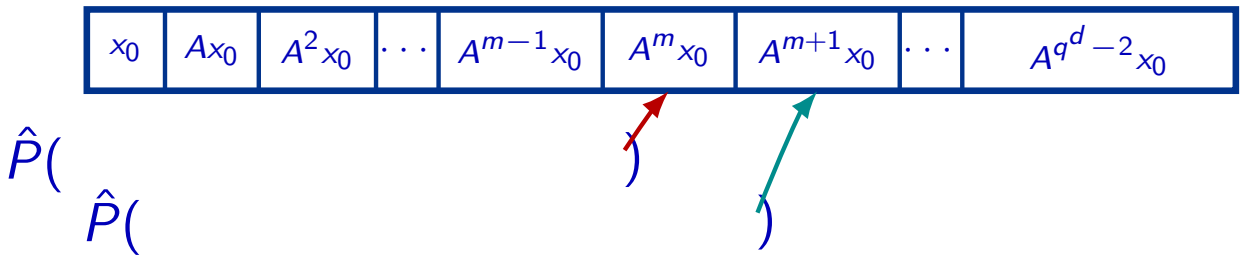

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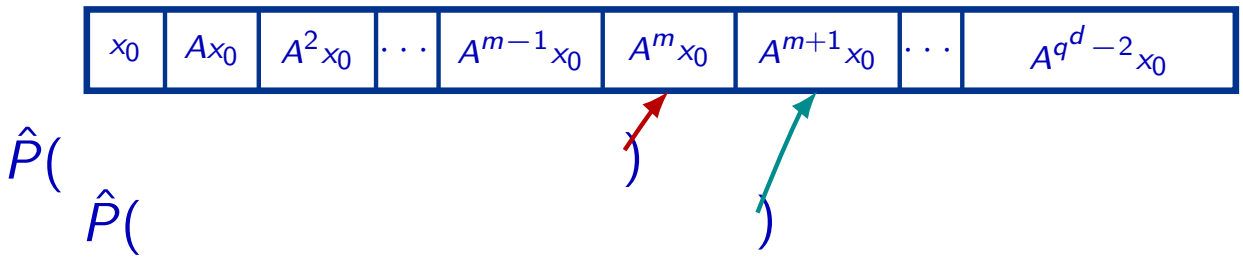

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$\Rightarrow$ Eventually we reconstruct $\hat{f}$ correctly on every point of $\mathbb{F}_q^d$, and hence reconstruct f correctly on every point of $\{0,1\}^\ell$.

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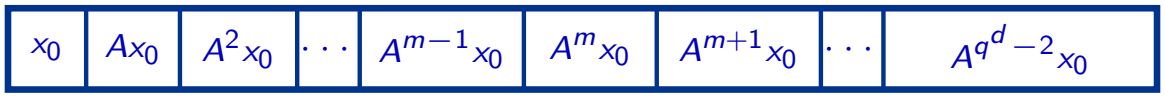

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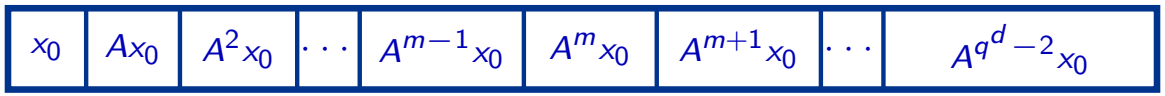

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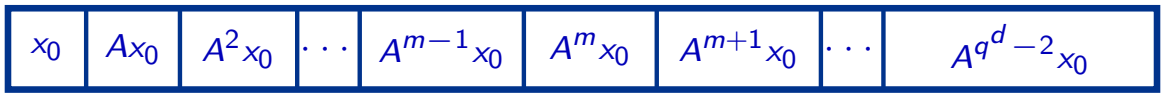

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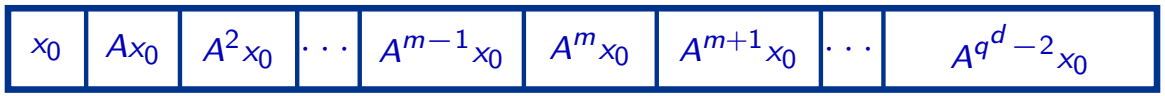

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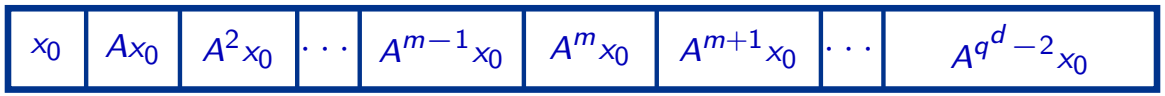

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- We will now explain one clean part: how to find a traversing matrix.

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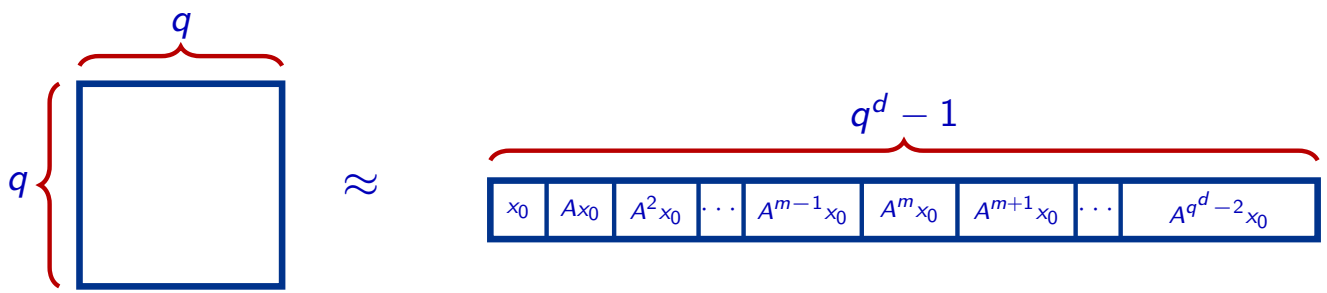

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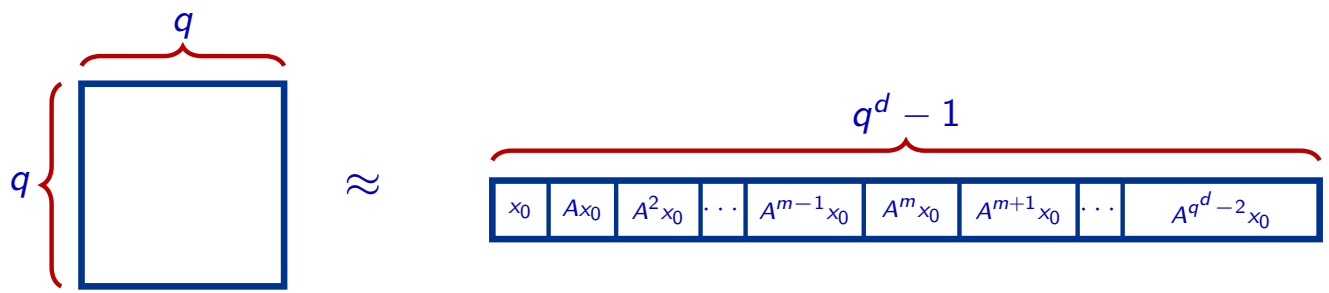

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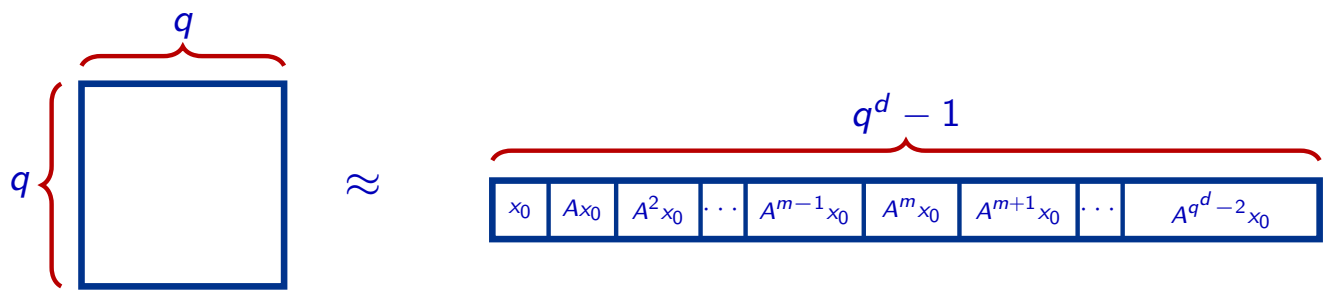

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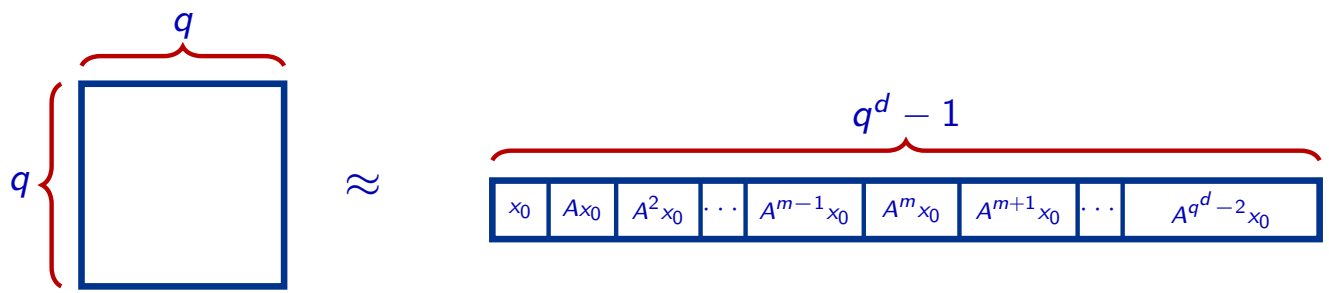

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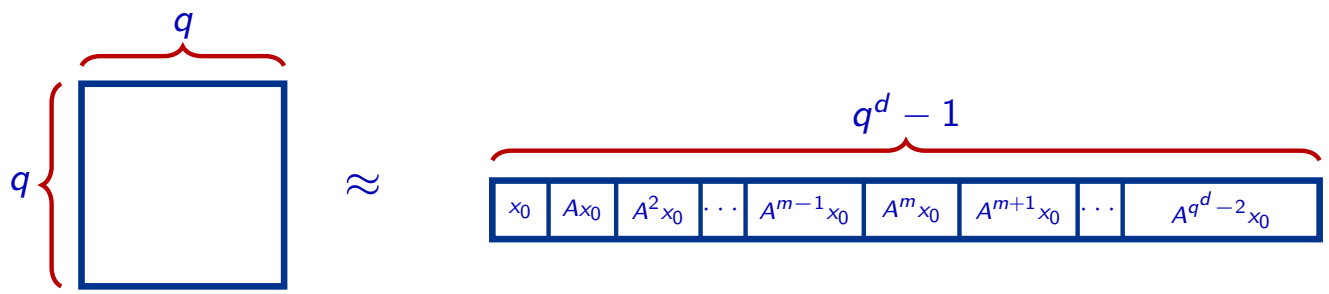

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Hardness $\Rightarrow$ PRGs (review of argument):

Theorem (IW97)

E *is hard for exponential size circuits* $\Rightarrow$ *there is an explicit PRG*

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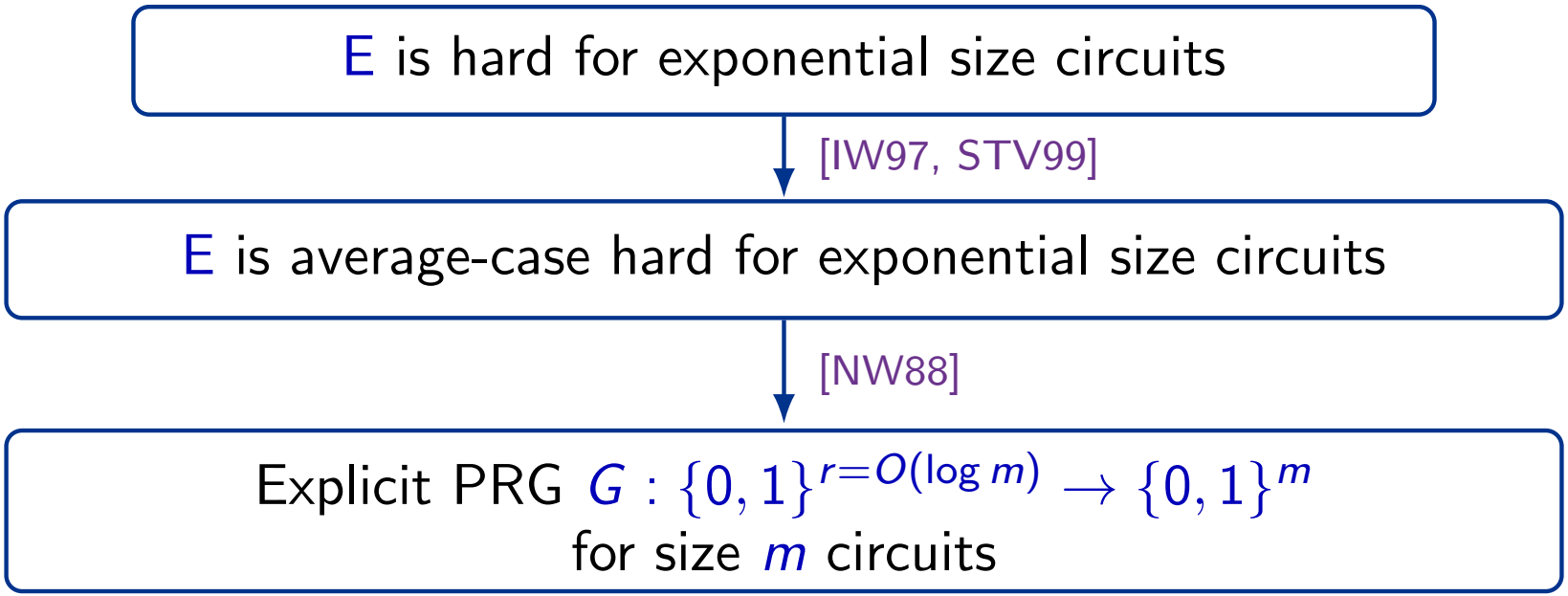

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Theorem (hardness $\Rightarrow$ PRGs - fully-black-box view)

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  $G^f : \{0,1\}^{r=O(\log m)} \to \{0,1\}^m$ *is computable in* E^f.
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- $\forall D : \{0,1\}^m \to \{0,1\}$ *(not necessarily a small circuit), if*

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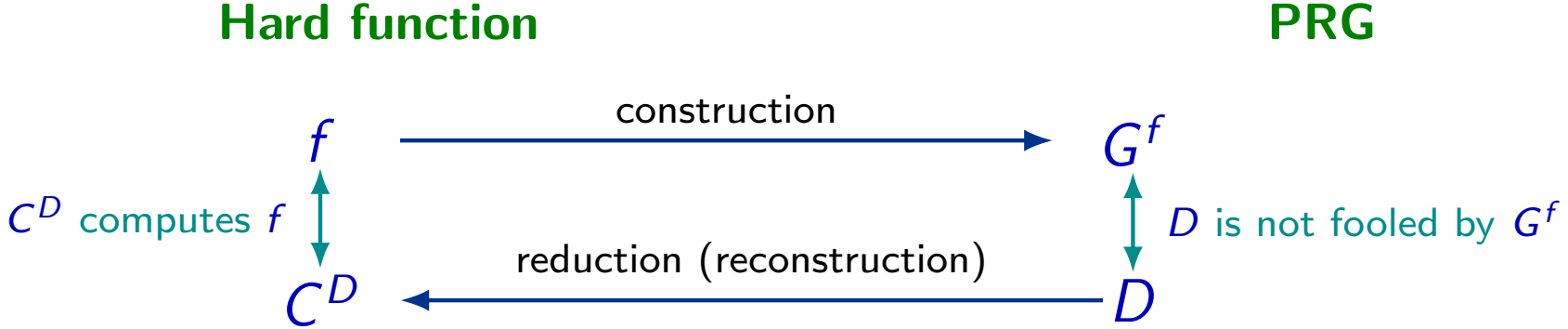

Key point: We do not assume that f or D are efficient (except for the consequently clauses).

PRGs against other classes of circuits

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E is hard for exponential size circuits of type X if $\exists \beta > 0$, and $\exists L \in \mathrm{E} = \mathrm{DTIME}(2^{O(n)})$ s.t. $\forall n$, 1_L is $2^{\beta n}$-hard for circuits of type X on inputs of length n.

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Theorem (KvM99,MV99,SU01 Derandomization of AM)

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Reflection: PRGs for nondeterministic circuits

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Hardness assumptions against nondeterministic circuits have many applications: [KvM99,AK99,MV99,SU01,BOV02,GW02,GST03,SU05,SU07,Dru13,AASY15,BV17, AIKS16,HNY17,DMOZ20,BDL22,CT22,BDGM23,BSS24,SS24,BSS25,Sha25].

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Hardness assumptions against nondeterministic circuits have many applications: [KvM99,AK99,MV99,SU01,BOV02,GW02,GST03,SU05,SU07,Dru13,AASY15,BV17, AIKS16,HNY17,DMOZ20,BDL22,CT22,BDGM23,BSS24,SS24,BSS25,Sha25].

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E *is hard for exponential size nondeterministic circuits* $\Rightarrow$ *there is an explicit PRG*

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Codes for computationally bounded channels [GS10,SS16,BDL22,SS24,BSS25].

Perspective: limitations of black-box constructions

Definition (Fully black-box PRG)

oracle TM $G^{(\cdot)}$ s.t. $\forall f : \{0,1\}^{\ell} \to \{0,1\}$, setting $m = 2^{\Omega(\ell)}$:
$G^f : \{0,1\}^{r=O(\log m)} \to \{0,1\}^m$ is computable in E^f.
s.t. $\forall D : \{0,1\}^m \to \{0,1\}$ (not necessarily a small circuit), if

$$\left|\Pr[D(G^f(U_r)) = 1] - \Pr[D(U_m) = 1]\right| > \epsilon,$$

then $\exists$size $2^{o(\ell)}$ oracle-circuit $C^{(\cdot)}$ s.t. $\forall x \in \{0,1\}^{\ell}$, $C^D(x) = f(x)$.

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Do we have constructions that aren't fully black box?

Perspective: beyond black-box techniques I

Using the *efficiency* of f in the reduction proving pseudorandomness.

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Theorem (IW98 uniform hardness vs. randomness tradeoffs)

$\mathrm{EXP} \neq \mathrm{BPP} \Rightarrow \mathrm{BPP} \subseteq i.o. - \mathit{Heuristic} - \mathrm{SUBEXP}$.

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Extensions and generalizations [CRT22,SvM23,BCT25,...].

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Theorem (DPTW25 Hardness vs. randomness for space)

$\forall \epsilon > 0$, $\mathsf{NSPACE}(n) \not\subseteq i.o.TISP(2^{n^{2-\epsilon}}, n^{O(1)}) \Rightarrow \mathsf{BPSPACE}(n) \subseteq \mathsf{SPACE}(n^{1+\epsilon})$.

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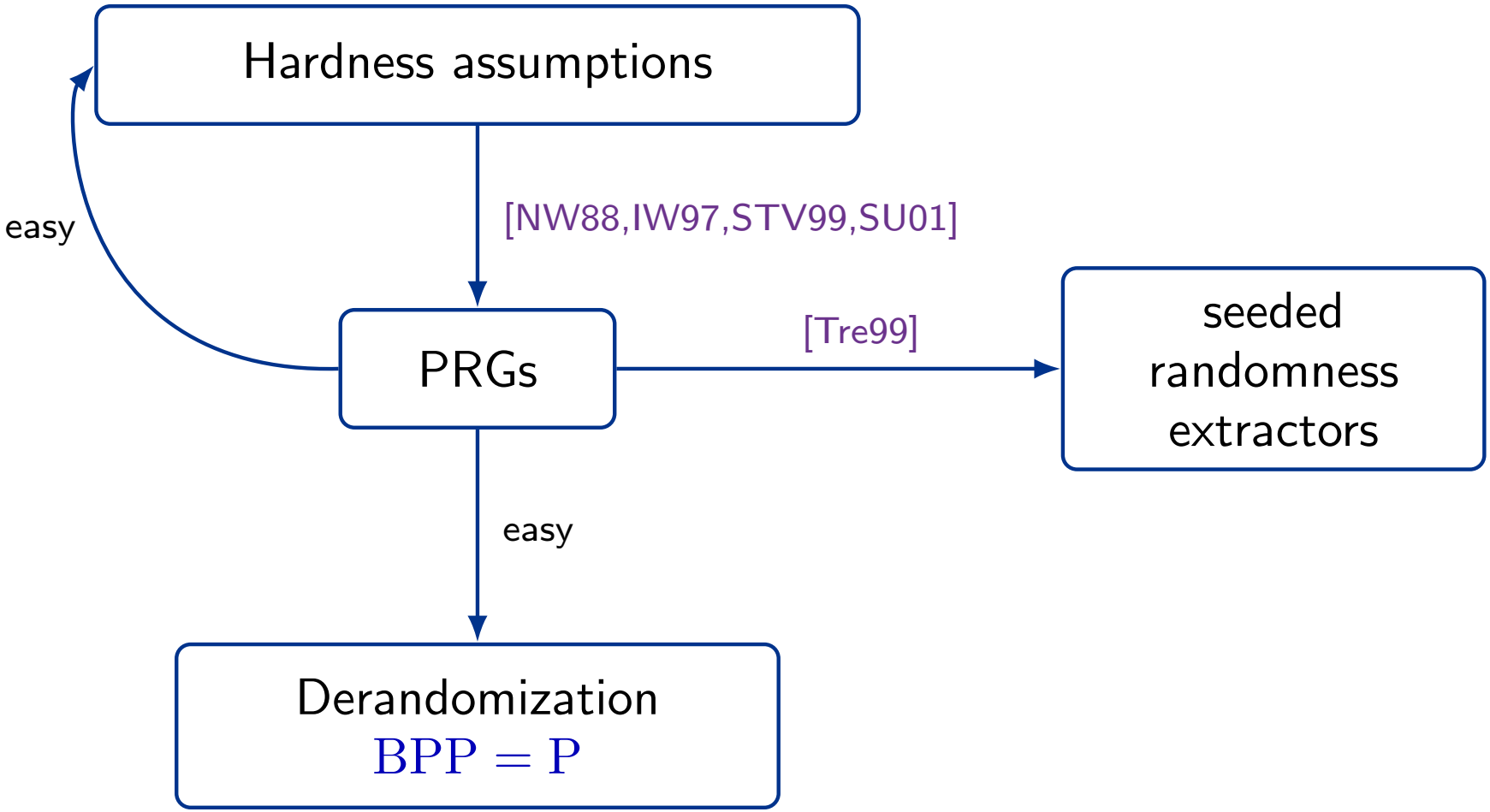
Other non-black-box uses of D: [LP22,LPT24].

Overall summary for the two talks



Then: extensions, generalizations, applications,
and some other research directions.

Current situation in hardness vs. randomness

Excerpt from a 2016 talk on hardness vs. randomness for Avi's birthday.

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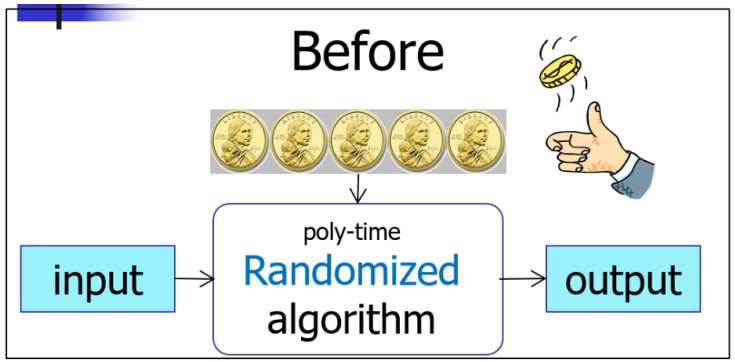
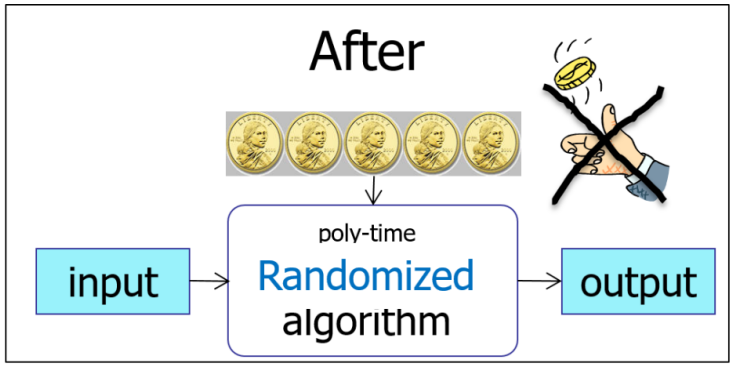
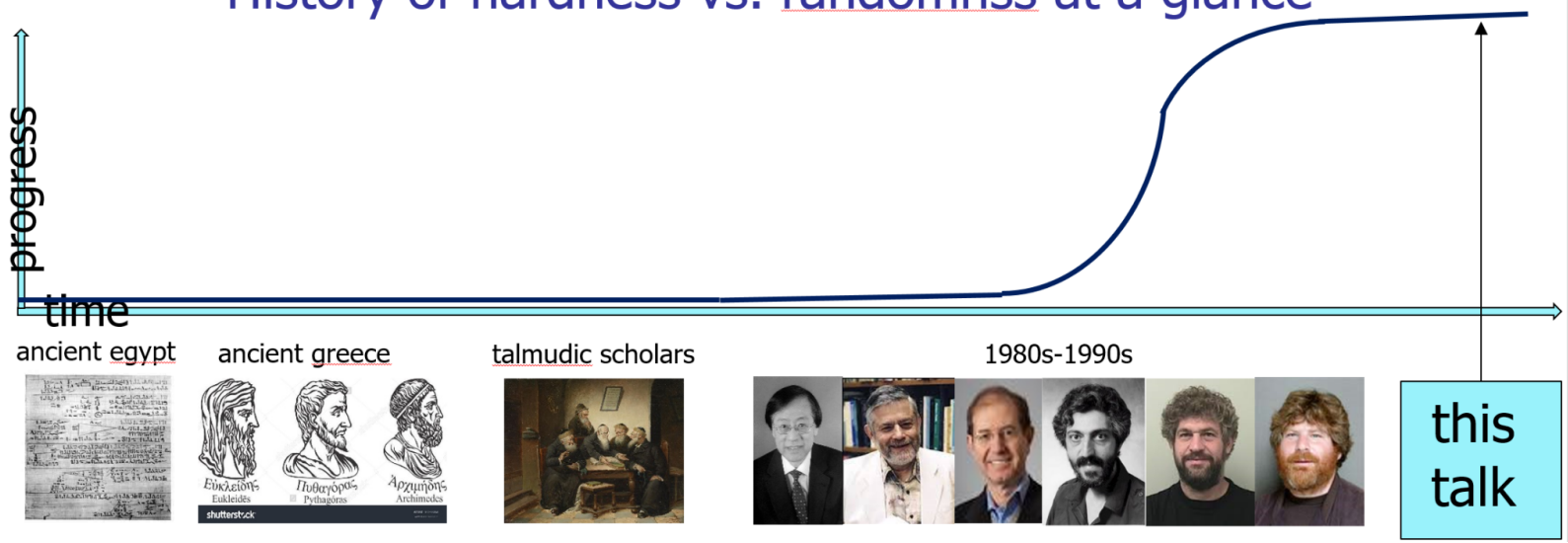




History of hardness vs. randomnss at a glance



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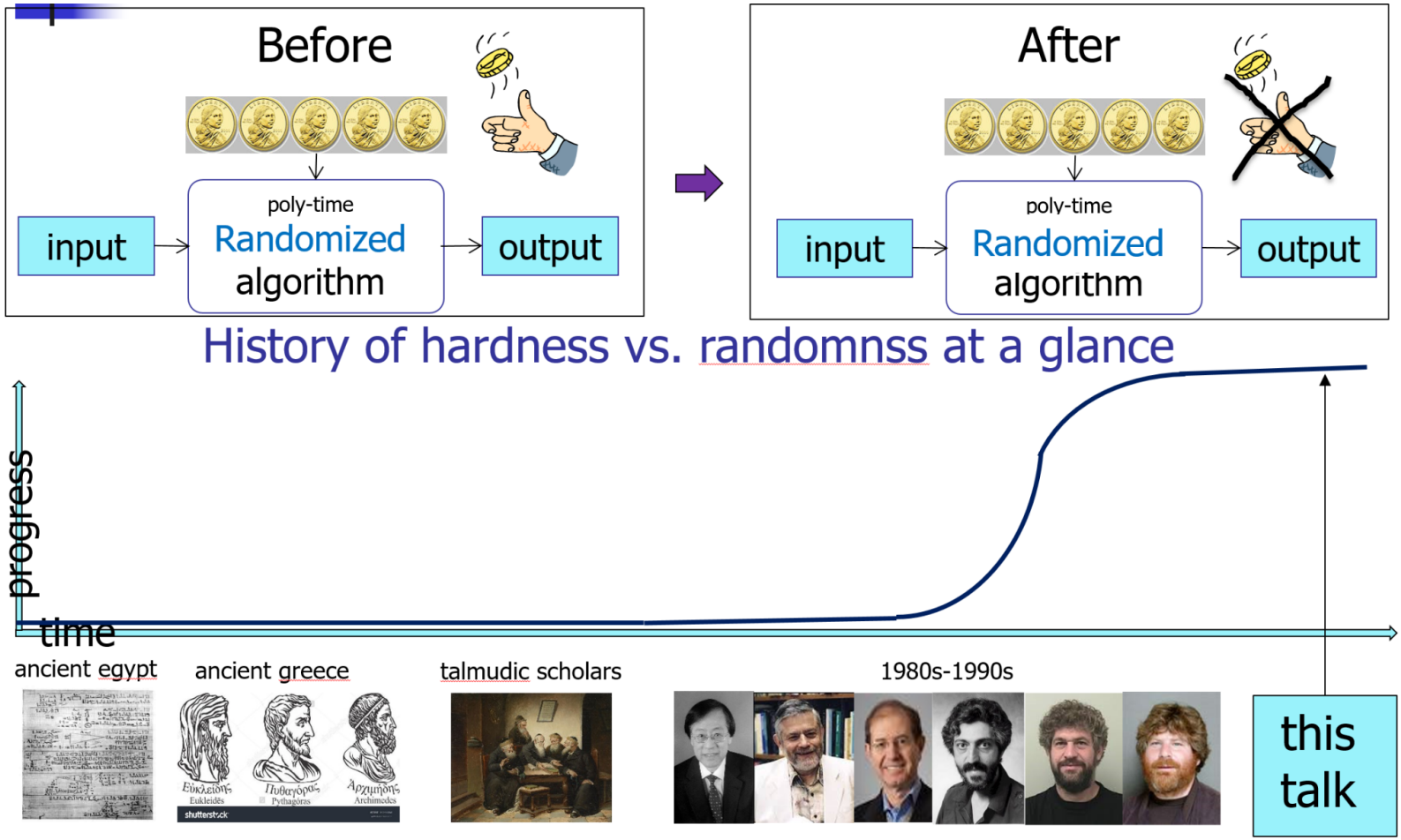

Amazing progress in recent years!

We'll hear about these developments in future talks and seminars.