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MINIMAL DETERMINING SETS OF LOCALLY FINITELY-DETERMINED FUNCTIONALS†

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Let N denote the set of natural numbers, N^N the set of all total functions from N into N. By functional we mean any function whose domain is N^N. If F is a functional and A is a partial function from N into N, we say that A is a *determining segment* (ds) of F if F has the same value on any two total extensions of A. A ds is called *minimal* (mds) if it does not properly contain another ds. For $\alpha \in N^N$, denote by $F^*(\alpha)$ the set of all mds's of F which are subsets of α.

A functional F is called *finitely-determined* (fd) if every $\alpha \in N^N$ contains a finite ds. F is *locally fd* (lfd) if there exists a set $\{F_i \mid i \in I\}$ of fd functionals such that $F(\alpha) = \{F_i(\alpha) \mid i \in I\}$ and $F_i(\alpha) \neq F_j(\beta)$ for $i \neq j$ and $\alpha, \beta \in N^N$. Total continuous operators (from N^N to N^N) are lfd.

Examples for fd F show that $F^*(\alpha)$ may contain (even $2^{\aleph_0}$) infinite mds's. The two main results for lfd functionals are that every ds contains a mds and that if $F^*(\alpha)$ consists only of finite sets then $F^*(\alpha)$ is itself finite. This follows from a combinatorial

Theorem. *If $A = \bigcup_{n=1}^{\infty} A_n$ where the A_n's are finite and $A_n \not\subset A_m$ for $n \neq m$, then $\exists B \subset A$ such that $\forall n\, (A_n \not\subset B)$ and for an infinite sequence $n_1, n_2, \ldots$, $(A_{n_i} - B) \cap (A_{n_j} - B) = \emptyset$ when $i \neq j$.*

A partial recursive functional F, if undefined on α, behaves differently when fd or non-fd on α. From any oracle-machine for F we can effectively construct another which makes finitely many queries about α when F is undefined and fd on α.

1. Introduction

Let N denote the set of nonnegative integers, N^N the set of all total functions from N into N, and P the set of all partial functions from N into N. We use small Greek letters $\alpha, \beta, \ldots$ to denote elements of N^N. These may be viewed either as infinite sequences of natural numbers or as sets of pairs:

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = \{(n, \alpha_n) \mid n \in N\}.$$

Capital letters $A, B, \ldots$ will be used to denote elements of P, which will also be regarded as sets of pairs.

By "functional" we mean any function defined on N^N and taking values in a set R. The nature of R plays little role in most of the following, and we shall mainly be interested in whether two elements of R are equal or unequal. For a given

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functional F and $\alpha \in N^N$, it may be that $F(\alpha)$ is determined only by a subset of α. For example, if F is a total recursive functional $(F: N^N \to N)$ then for every $\alpha \in N^N$, $F(\alpha)$ is determined by a finite subset of α. This leads to the following definition:

1.1. Definition. (a) If F is a functional and $A \in P$, we say that A *determines* F if for every $\alpha, \beta \in N^N$, if $A \subset \alpha$ and $A \subset \beta$, then $F(\alpha) = F(\beta)$. We call A a *determining segment* (ds) of F and denote $A \rhd F$. Denote also $A \rhd F(\alpha)$, $A \rhd (F = a)$ and $A \rhd (F(\alpha) = a)$ if $A \subset \alpha$ and $F(\alpha) = a$ $(a \in R)$.

(b) We say that A *determines F minimally* if $A \rhd F$ and no proper subset of A determines F. A is called a *minimal ds* (mds), and we denote $A \unrhd F$, $A \unrhd F(\alpha)$, $A \unrhd (F = a)$ and $A \unrhd (F(\alpha) = a)$.

The word "segment" is borrowed from [4, p. 259], where it is used only for partial functions with a recursive domain.

1.2. Definition. (a) A functional F is called *finitely-determined* (fd) on α if there exists $A \subset \alpha$ such that A is finite and $A \rhd F$.

(b) F is called *fd* if for every α, F is fd on α.

If we consider partial recursive functionals (range $N \cup \{\omega\}$), then total recursive functionals are fd while partial recursive functionals may or may not be fd. We regard ω as a value which a partial recursive functional may have, and we equate "$F(\alpha)$ is undefined" with "$F(\alpha) = \omega$". ω is considered as an element of the range for the purposes of Definitions 1.1 and 1.2.

If we take the discrete topology on R and the Baire topology on N^N, then the *fd* functionals are exactly the continuous ones. We use the term "fd" because in recursion theory the discrete topology is taken over N and not $N \cup \{\omega\}$, and in this sense, every partial recursive functional is continuous over its domain [4, p. 361].

In [2, Chapter 1] and [3] we consider the problem of computing partial recursive functionals by oracle-machines working on infinite sequences of integers. The problem considered there was to minimize the length of the initial segment (of such a sequence) for which we want answers from the oracle. This led to the definition of the "dependence functional" F' of a given functional F as

$$F'(\alpha) = \min\{n \mid \forall \beta\ (\beta|_n = \alpha|_n \to F(\alpha) = F(\beta))\}$$

where $\alpha|_n$ is the initial segment of length n of α. It is seen that the notion of a mds is closely related, but since there may be several mds's, we have to consider all of them. This leads to

1.3. Definition. for a functional F and $\alpha \in N^N$, denote

$$F^*(\alpha) = \{A \subset \alpha \mid A \unrhd F(\alpha)\}.$$

The main object of this work is to study the nature of $F^*(\alpha)$ for functionals which are fd and locally fd (defined below). This is done by first giving, in Section 2, various illustrative examples of fd functionals, among them a case where $F^*(\alpha)$ contains $2^{\aleph_0}$ infinite sets, the intersection of any two being finite.

We now introduce the following generalization of fd functionals:

1.4. Definition. A functional F is called *locally finitely-determined* (lfd) if there exists an index set I and a set of fd functionals $\{F_i \mid i \in I\}$ such that:

(1) for every $\alpha \in N^N$, $F(\alpha) = \{F_i(\alpha) \mid i \in I\}$,

(2) for every α, $\beta \in N^N$ and $i, j \in I$, if $i \neq j$, then $F_i(\alpha) \neq F_j(\beta)$.

One important class of lfd functionals are total continuous operators: If $F: N^N \to N^N$ is continuous (under the Baire topology), then F is lfd. We shall also consider fd functionals as lfd, without bothering to go into unnecessary detail concerning the structure of the elements of R. One main result is the following:

1.5. Theorem. *If F is a* lfd *functional, then every* ds *contains a* mds.

This theorem is of interest in view of the fact that even a fd functional can have an infinite mds. The proof, given in Section 3, is "constructive" and the mds is obtained by deleting elements from the ds. Another main result, which uses Theorem 1.5 is:

1.6. Theorem. *If F is* lfd, *$\alpha \in N^N$ and $F^*(\alpha)$ contains infinitely many finite sets, then $F^*(\alpha)$ also contains an infinite set.*

1.7. Corollary. *If F is* lfd *and all elements of $F^*(\alpha)$ are finite, then $F^*(\alpha)$ is itself finite.*

An example for Theorem 1.6 with a fd functional is given in Section 2. Both Theorems 1.5 and 1.6 were originally proved in [2] for fd functionals, but the extension to lfd functionals is quite simple.

The proof of Theorem 1.6 requires a combinatorial theorem which is proved separately in Section 4 together with another combinatorial result due to Paul Erdös [1]. The two results are grouped separately because both are examples of constructions of certain sets from a denumerable family of finite sets.

We mentioned that partial recursive functionals may or may not be fd. An example:

1.8. Example

$$f(\alpha) = \begin{cases} 0 & \text{if for some } n,\ \alpha_n \neq 0, \\ \omega & \text{otherwise.} \end{cases}$$

F is partial recursive but is not fd on the sequence $(0, 0, 0, \ldots)$. □

In Section 5 we study the difference between partial recursive functionals being fd (but undefined) and non-fd on their arguments, by considering the behavior of algorithms (oracle-machines) computing them. We show that if F is not fd on α then F is, in a certain sense, "searching" α for "something" which is finite and which can be arbitrarily far on α (the above is a good example). Any algorithm computing F is unbounded on α, meaning that it requires infinitely many answers from an oracle for α.

We also prove that if F is partial recursive then there exists an algorithm computing F which is bounded on any α on which F is fd (even if undefined). Such an algorithm can be effectively constructed from any given algorithm for F.

2. Examples

We now bring some examples of $F^*(\alpha)$ for fd F. All the examples are, in fact, of $\{0, 1\}$-valued recursive functionals. We shall sometimes use the notation $A = (\_, \alpha_1, \alpha_2, \_, \alpha_4, \ldots)$ to denote a partial function (element of P) which is undefined at certain values (in the above example, A is undefined at 0 and at 3).

2.1. Example

$$F(\alpha) = \begin{cases} 0 & \text{if } \alpha_0 = 0 \quad \text{or} \quad \alpha_1 = 1, \\ 1 & \text{otherwise.} \end{cases}$$

This functional is clearly fd. For $\alpha = (0, 1, \ldots)$ it is obvious that the mds's are $\{(0, 0)\}$ and $\{(1, 1)\}$. For $\alpha = (2, 2, \ldots)$, the only mds is $\{(0, 2), (1, 2)\}$. □

The following example shows that an mds may be infinite:

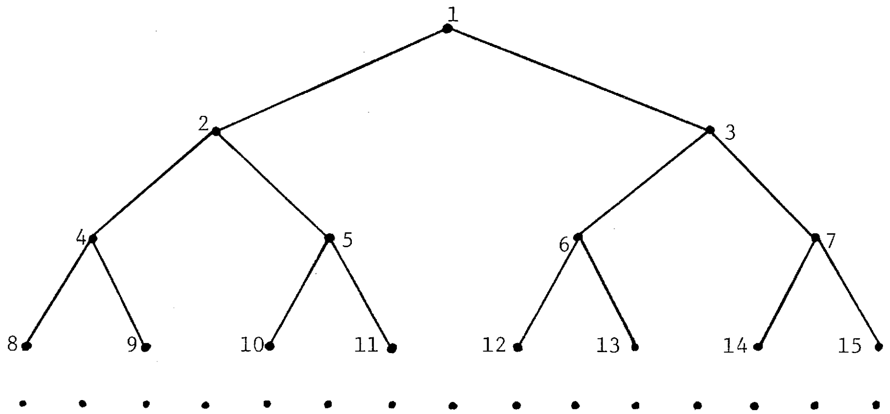
2.2. Example

$$F(\alpha) = \begin{cases} 0 & \text{if } \alpha_{\alpha_0+1} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

F is fd. For any α, it is obvious that $\{(0, \alpha_0), (\alpha_0 + 1, \alpha_{\alpha_0+1})\}$ is a mds. If we take $\alpha = (\alpha_0, 0, 0, 0, \ldots)$, where α_0 is any integer, then $F(\alpha) = 0$ and the segment $A = (\_, 0, 0, 0, \ldots)$ determines F. But clearly, $A \not\trianglerighteq F$. □

In the following example, it is shown that $F^*(\alpha)$ may contain $2^{\aleph_0}$ infinite sets, the intersection of any two being finite.

2.3. Example. *Enumerate the nodes of an infinite binary tree, starting with* 1:



We make free use of terms like "branch of length n" and "infinite branch", where a branch always starts with 1. Thus, $\{1, 3, 6, 13\}$ is a branch of length 4 and $\{2^n \mid n = 0, 1, 2, \ldots\}$ is an infinite branch.

Define:

$$F(\alpha) = \begin{cases} 1 & \text{if there exists a branch } B \text{ of length} \\ & \alpha_0 + 1 \text{ such that, for every } i \in B,\ \alpha_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

F is obviously total and recursive, because for any α we only have to check $2^{\alpha_0+1} - 1$ elements to determine the value of F. Consider the segment $A = (\_, 1, 1, \ldots)$. If $A \subset \alpha$, then $F(\alpha) = 1$. For every infinite branch B, let $A_B = \{(i, 1) \mid i \in B\}$. Then $A_B \subset \alpha$. Clearly, $A_B \rhd F(\alpha)$ because if $A_B \subset \beta$, then β has arbitrarily long finite branches of 1's to suit any choice of β_0. Also, the A_B's are infinite, there are $2^{\aleph_0}$ of them and the intersection of any two is finite. It remains to show that $A_B \mathrel{\|\!\rhd} F$. Let $n \in B$. We can construct β such that $(A_B - \{(n, \alpha_n)\}) \subset \beta$ and $f(\beta) = 0$ by setting $\beta_i = 1$ if $i \in B - \{n\}$, $\beta_0 \geqslant n$ and $\beta_i = 0$ everywhere else. This ensures that β has no branch of 1's of length $\beta_0 + 1$, and therefore $F(\beta) = 0$. Therefore $A_B \mathrel{\|\!\rhd} F(\alpha)$. □

The following is an example of the occurrence of Theorem 1.6 with a fd functional.

2.4. Example

$$F(\alpha) = \begin{cases} 0 & \text{if } \alpha_1 = \alpha_3 = \cdots = \alpha_{2\alpha_0+1} = \alpha_{2\alpha_0+2} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

I.e., $F(\alpha) = 0 \Leftrightarrow \alpha_{2i+1} = 0$ for $0 \leqslant i \leqslant \alpha_0$ and $\alpha_{2\alpha_0+2} = 0$. F is obviously total recursive and hence fd. For $n = 1, 2, \ldots$ define:

$$A_n = \{(2, 1), (4, 1), \ldots, (2n, 1), (2n+1, 1)\}.$$

Claim. $A_n \trianglerighteq (F = 1)$. Explanation: for every n and every possible choice of α_0, A_n is inconsistent at exactly one point with the set that permits $F(\alpha) = 0$, and therefore $A_n \rhd (F = 1)$. But, if we delete any element from A_n we leave an "opening" so that for a suitable choice of $\alpha_0, \alpha_1, \alpha_3, \ldots, \alpha_{2\alpha_0+1}, \alpha_{2\alpha_0+2}$ we can get $F(\alpha) = 0$. The reader can easily work this example out in detail. Let $\alpha = (\alpha_0, \alpha_1, 1, 1, \ldots)$. Then for every n, $A_n \in F^*(\alpha)$ and so $F^*(\alpha)$ has infinitely many finite sets. Consider $B = \{2, 1), (4, 1), \ldots, (2n, 1), \ldots\}$. $B \subset \alpha$ and it is easily seen that $B \trianglerighteq (F = 1)$. Therefore B is an infinite element of $F^*(\alpha)$, the existence of which follows from Theorem 1.6. □

The following is an example of a non-fd functional and a ds of which no proper subset is a mds.

2.5. Example

$$F(\alpha) = \begin{cases} 1 & \text{if } \exists m \geqslant 1 \text{ such that for every } i, \\ & m \leqslant i \leqslant m + \alpha_0, \quad a_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (\_, 1, 1, \ldots, 1, \ldots)$. Obviously, $A \rhd (F = 1)$. It is easily seen that for every $B \subset A$, $B \rhd F$ iff B contains arbitrarily long strings of consecutive 1's (to suit any possible choice of α_0). Now, if we take a set with that property and delete a finite number of elements from it, it would still contain arbitrarily long strings of consecutive 1's. Therefore, any determining subsegment of A cannot be minimal.

Note that if we replace 0 by ω in the definition of F, we get a partial recursive functional which is not fd on any sequence which does not contain a string of length $\alpha_0 + 1$ of consecutive 1's. The only way to compute $F(\alpha)$ is to "search" α for such a string. □

3. Main results

In this section we bring the proofs of Theorems 1.5 and 1.6, and a few more facts about fd functionals and ds's.

3.1. Lemma. *Let F be a lfd functional, $F(\alpha) = \{F_i(\alpha) \mid i \in I\}$. If $A \rhd F$, then for every $i \in I$, $A \rhd F_i$.*

Proof. Immediate from the definitions. □

It is this lemma which allows us to extend the proofs of Theorems 1.5 and 1.6 to lfd functionals.

Notation. π is the projection function on the first coordinate of a pair: $\pi(x, y) = x$, and if A is a set of pairs, then $\pi(A) = \{n \mid \exists m \text{ such that } (n, m) \in A\}$.

The following lemma is extremely useful. It states that two ds's which determine different values are inconsistent.

3.2. Lemma. *If F is any functional, $A \rhd F(\alpha)$, $B \rhd F(\beta)$ and $F(\alpha) \neq F(\beta)$, then $\exists i \in \pi(A) \cap \pi(B)$ such that $\alpha_i \neq \beta_i$.*

Proof. Assume otherwise. Define $\gamma \in N^N$ as follows:

$$\gamma_i = \begin{cases} \alpha_i & \text{if } i \in \pi(A), \\ \beta_i & \text{otherwise.} \end{cases}$$

Obviously, $A \subset \gamma \Rightarrow F(\gamma) = F(\alpha)$. If $i \in \pi(B)$ we get: *Case 1.* $i \in \pi(A) \Rightarrow i \in \pi(A) \cap \pi(B) \Rightarrow \alpha_i = \beta_i \Rightarrow \gamma_i = \beta_i$. *Case 2.* $i \notin \pi(A) \Rightarrow \gamma_i = \beta_i$. Therefore, $B \subset \gamma \Rightarrow F(\gamma) = F(\beta)$. Contradiction. □

Proof of Theorem 1.5. Let F be a lfd functional and $A \rhd F$. Let $\alpha \in N^N$ be any sequence containing A. We define a descending sequence of sets A_n as follows:

$$A_0 = A,$$

$$A_{n+1} = \begin{cases} A_n & \text{if } A_n \mathrel{\|\!\!\rhd} F, \\ A_n - \{(k, \alpha_k)\} & \text{if } k \in \pi(A_n) \text{ is smallest such that } \\ & (A_n - \{(k, \alpha_k)\}) \rhd F. \end{cases}$$

It is easily seen that for every n, $A_n \rhd F$.

Let $B = \bigcap_{n=0}^{\infty} A_n$. We shall show that $B \mathrel{\|\!\!\rhd} F$. It is sufficient to show $B \rhd F$, and this implies $B \mathrel{\|\!\!\rhd} F$, because if B contains any redundant pair (k, α_k), we would have deleted it at some stage.

Let $\beta \in N^N$ such that $F(\beta) \neq F(\alpha)$. We shall show that this implies $B \not\subset \beta$. F is locally fd and so there exists a set of fd functionals $\{F_i \mid i \in I\}$ such that $F(\alpha) = \{F_i(\alpha) \mid i \in I\}$. therefore, for some $i \in I$, $F_i(\alpha) \neq F_i(\beta)$.

Claim. There exists j such that $\alpha_j \neq \beta_j$ and for every n, $j \in \pi(A_n)$.

Proof of claim. Assume no such j exists. F_i is fd and therefore there exists $Y \subset \beta$, Y finite, such that $Y \rhd F_i(\beta)$. Let

$$\{j_1, \ldots, j_k\} = \{j \in \pi(Y) \mid \alpha_j \neq \beta_j\}.$$

It follows from Lemma 3.2 that the above set is not empty, because $Y \rhd F_i(\beta)$,

$A \rhd F_i(\alpha)$ and $F_i(\alpha) \neq F_i(\beta)$. We are assuming that the claim is untrue, and so there exist $n_1, \ldots, n_k$ such that $j_1 \notin \pi(A_{n_1}), \ldots, j_k \notin \pi(A_{n_k})$. We can assume without loss of generality that $A_{n_1} \subset A_{n_2} \subset \cdots \subset A_{n_k}$. Therefore $j_1, \ldots, j_k \notin \pi(A_{n_1})$. This is a contradiction to Lemma 3.2, because we have $A_{n_1} \rhd F_i(\alpha)$ (by Lemma 3.1), $Y \rhd F_i(\beta)$, $F_i(\alpha) \neq F_i(\beta)$ and there is no $j \in \pi(A_{n_1}) \cap \pi(Y)$ such that $\alpha_j \neq \beta_j$. This proves our claim.

Therefore, $j \in \bigcap_{n=0}^{\infty} \pi(A_n)$. Since every A_n is a subset of A, $\bigcap_{n=0}^{\infty} \pi(A_n) = \pi(\bigcap_{n=0}^{\infty} A_n) = \pi(B)$. Therefore $j \in \pi(B)$ and $\alpha_j \neq \beta_j$, and this implies that β cannot include B.

We showed that $F(\beta) \neq F(\alpha) \Rightarrow B \not\subset \beta$. therefore $B \rhd F(\alpha)$. □

The proof given in [2] used Zorn's Lemma and considered the set of all subsets of A which determine F, partially ordered by set-inclusion. The proof that $\bigcap_{n=0}^{\infty} A_n \rhd F$ is the same as the proof that the intersection of all elements of a chain is a ds. Note that in Example 1.8, every ds contains a mds, but the functional is not lfd, so the converse of Theorem 1.5 is not true.

The following combinatorial theorem, which is proved in the next section, is used in Theorem 1.6:

3.3. Theorem. *Let $A_1, A_2, \ldots, A_n, \ldots$ be a denumerable sequence of finite sets such that if $n \neq m$, then $A_n \not\subset A_m$. Then there exists a set $B \subset \bigcup_{n=1}^{\infty} A_n$ such that:*

(1) *for every n, $A_n \not\subset B$,*

(2) *there exists an infinite sequence of integers $n_1, n_2, n_3, \ldots$ such that if $i \neq j$, then*

$$(A_{n_i} - B) \cap (A_{n_j} - B) = \emptyset.$$

Proof of Theorem 1.6. Let F be a lfd functional and $\alpha \in N^N$ such that $F^*(\alpha)$ contains infinitely many finite sets, which we shall denote $A_1, A_2, A_3, \ldots$. There are at most a denumerable number of finite sets because there are a denumerable number of finite subsets of α. We assume that every finite set of $F^*(\alpha)$ appears once and only once in the above enumeration. Since the elements of $F^*(\alpha)$ are minimal ds's, no set is contained in another. Therefore there exists a set B as in Theorem 3.3.

Claim. $B \rhd F(\alpha)$.

Proof of claim. Assume $B \subset \beta$ and $F(\beta) \neq F(\alpha)$. There exists a set of fd functionals as in Definition 1.4, therefore for some $i \in I$, $F_i(\beta) \neq F_i(\alpha)$. F_i is fd and so there exists $Y \subset \beta$ such that Y is finite and $Y \rhd F_i(\beta)$. For every k, $A_{n_k} \rhd F_i(\alpha)$ and therefore by Lemma 3.2, there exists $j_k \in \pi(Y) \cap \pi(A_{n_k})$ such that $\alpha_{j_k} \neq \beta_{j_k}$. But $B \subset \beta$ and therefore $j_k \in \pi(A_{n_k} - B)$. The sets $A_{n_k} - B$ are pairwise disjoint and they are all consistent (subsets of α). Therefore the sets $\pi(A_{n_k} - B)$ are also pairwise disjoint. Therefore the set $\{j_1, j_2, \ldots, j_k, \ldots\}$ is infinite. But this is impossible because this set is contained in $\pi(Y)$ which is finite. This proves the claim.

Since $B \rhd F(\alpha)$, it follows from Theorem 1.5 that there exists $B' \subset B$ such that $B' \unrhd F(\alpha)$, i.e. $B' \in F^*(\alpha)$. B' must be infinite because B does not contain any A_n and the A_n's are all the finite sets of $F^*(\alpha)$. □

Example 1.8 serves to show that the property "every ds contains a mds" is not sufficient for the conclusion of Theorem 1.6 to hold: Consider $\alpha = (1, 1, 1, \ldots)$. $F^*(\alpha)$ contains infinitely many finite sets (all sets of the form $\{(n, 1)\}$), but no infinite set.

We proceed with a few more properties of fd and lfd functionals. As seen from the previous paragraph, $F^*(\alpha)$ may contain infinitely many pairwise disjoint sets. The following theorem and corollary show that this is impossible for lfd functionals.

3.4. Theorem. *Let F be a nonconstant* lfd *functional. Then*

(a) *For every $\alpha \in N^N$, there exists a finite set of integers $J \subset N$ such that $J \cap \pi(A) \neq \emptyset$ for every* ds *$A \subset \alpha$.*

(b) *If F is also* fd, *then there exists a finite set $J \subset N$ such that for every $\alpha \in N^N$, $J \cap \pi(A) \neq \emptyset$ for every* ds *$A \subset \alpha$, i.e., the same J is suitable for all α.*

Proof. Note that $\emptyset$ is a ds for any constant-valued functional.

(a) Let $\alpha \in N^N$. Since F is not constant, there exists $\beta \in N^N$ such that $F(\alpha) \neq F(\beta)$. Assuming the usual representation for lfd functionals, there exists $i \in I$ such that $F_i(\alpha) \neq F_i(\beta)$. F_i is fd and so there exists $Y \subset \beta$, Y finite and $Y \rhd F_i(\beta)$. Now for every ds $A \subset \alpha$, $A \rhd F_i(\alpha)$ and so by Lemma 3.2, $\exists j \in \pi(Y) \cap \pi(A)$ such that $\alpha_j \neq \beta_j$. We take $J = \{j \in \pi(Y) \mid \alpha_j \neq \beta_j\}$, and this of course is the required set.

(b) Since F is not constant, there exist $\beta, \gamma \in N^N$ such that $F(\beta) \neq F(\gamma)$. F is fd and so there exist $B \subset \beta$, $C \subset \gamma$, both finite such that $B \rhd F(\beta)$ and $C \rhd F(\gamma)$. Take $J = \pi(B) \cup \pi(C)$. Now, for any $\alpha \in N^N$, $F(\alpha) \neq F(\beta)$ or $F(\alpha) \neq F(\gamma)$. Assume $F(\alpha) \neq F(\beta)$ and let $A \subset \alpha$ be a ds. By Lemma 3.2, $\exists j \in \pi(A) \cap \pi(B)$ such that $a_j \neq \beta_j$, and so $j \in J \cap \pi(A) \Rightarrow J \cap \pi(A) \neq \emptyset$. Similarly if $F(\alpha) \neq F(\gamma)$. □

3.5. Corollary. *If F is a nonconstant* lfd *functional, then for every α, $F^*(\alpha)$ does not contain an infinite subset of pairwise disjoint sets.*

The following theorem tells us that an (infinite) mds of a fd functional is the union of its intersections with finite ds's.

3.6. Theorem. *Let F be a* fd *functional, A a* mds *of F. For every $\beta \in N^N$, let $A_\beta \subset \beta$ be a finite* ds *of F. Then*

$$A = \bigcup_{\beta \supset A} (A_\beta \cap A).$$

Proof. Denote $B = \bigcup_{\beta \supset A} (A_\beta \cap A)$. The union is taken over all β which contain A. Clearly, $B \subset A$, because B is the union of subsets of A. We shall prove that $B \rhd F$. Assume $B \subset \gamma$. We have to prove that $F(\gamma) = F(\alpha)$.

$$B \subset \gamma \;\Rightarrow\; \text{for all } \beta, \text{ if } A \subset \beta, \text{ then } (A_\beta \cap A) \subset \gamma. \qquad (*)$$

Define β as follows:

$$\beta_j = \begin{cases} \alpha_j & \text{if } j \in \pi(A), \\ \gamma_j & \text{if } j \notin \pi(A). \end{cases}$$

$A \subset \beta$ and so $F(\beta) = F(\alpha)$. We shall now show that $A_\beta \subset \gamma$: $A_\beta = [A_\beta \cap A] \cup [A_\beta \cap (\beta - A)]$. This equality splits A_β into two disjoint subsets. The first of them, $A_\beta \cap A$, is a subset of γ because of $(*)$ and $A \subset \beta$. $[A_\beta \cap (\beta - A)] \subset \gamma$ because $\gamma_j = \beta_j$ for $j \notin \pi(A)$. Therefore $A_\beta \subset \gamma$. $A_\beta \rhd F(\beta)$ and so $F(\gamma) = F(\beta) = F(\alpha)$. This proves that $B \rhd F$. Since $B \subset A$ and A is a mds, it follows that $B = A$. □

Notation. If M is a set of natural numbers, $\alpha \in N^N$, denote $\alpha|_M = \{(i, \alpha_i) \mid i \in M\}$. Also, for natural numbers k, denote $\alpha|_k = \{(i, \alpha_i) \mid i < k\} = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$. The two notations are actually the same if we adopt the convention that $k = \{0, 1, \ldots, k-1\}$.

The next theorem tells us that if there is always a "fixed part" of α that determines $F(\alpha)$, then every mds is necessarily contained in that fixed part.

3.7. Theorem. *Let F be a functional for which there exists a set of natural numbers M such that for every α, $\alpha|_M \rhd F(\alpha)$. Then for every* mds *A, $\pi(A) \subset M$, i.e., if $A \in F^*(\alpha)$, then $A \subset \alpha|_M$.*

Proof. Let $\alpha \in N^N$ be given, $A \subset \alpha$ such that $A \rhd f(\alpha)$. Let $B = A \cap (\alpha|_M)$. We shall show that $B \rhd f(\alpha)$: Assume $B \subset \beta$. Define γ as follows:

$$\gamma_i = \begin{cases} \alpha_i & \text{if } i \in \pi(A), \\ \beta_i & \text{if } i \notin \pi(A). \end{cases}$$

$A \subset \gamma$ and so $F(\gamma) = F(\alpha)$. We also have $\gamma|_M = \beta|_M$ and therefore $F(\gamma) = F(\beta) \Rightarrow F(\beta) = F(\alpha)$.

We have shown that the restriction to M of any ds is a ds, and therefore if A is a mds, $\pi(A) \subset M$. □

A special case of the above is when M is finite, i.e., there exists a natural number k such that for every α, $\alpha|_k \rhd F(\alpha)$. It follows from Theorem 3.7 that every set in $F^*(\alpha)$ is a subset of $\alpha|_k$ and so all elements of $F^*(\alpha)$ are finite. The following example shows that the existence of such a k does not characterize functionals F for which every mds is finite.

3.8. Example. $F(\alpha)=$ first n for which $\alpha_{n+1}\geqslant\alpha_n$. For any α, the sequence of inequalities $\alpha_0>\alpha_1>\cdots>\alpha_n>\cdots$ is finite because every α_n is a natural number. Therefore, there is a first n for which $\alpha_{n+1}\geqslant\alpha_n$. Every mds of F is finite, but there are arbitrarily large mds's. □

4. Two combinatorial theorems

In this section we bring two combinatorial theorems which are of interest in their own right.

3.3. Theorem. *Let $A_1, A_2, \ldots$ be a denumerable sequence of finite sets such that $n\neq m \Rightarrow A_n\not\subset A_m$. Then there exists a set $B\subset\bigcup_{n=1}^{\infty} A_n$ such that:*

(1) *For every n, $A_n\not\subset B$;*

(2) *There is an infinite sequence $n_1, n_2, \ldots$ such that if $i\neq j$, then*

$$(A_{n_i}-B)\cap(A_{n_j}-B)=\phi.$$

Proof. Denote $A=\bigcup_{n=1}^{\infty} A_n$.

Assertion. There exists a set $B\subset A$ and an infinite set of indices M such that if we denote $\mathfrak{A}=\{A_n \mid n\in M\}$, then:

(a) For every n, $A_n\not\subset B$.

(b) Every element of $A-B$ appears in only a finite number of the sets of $\mathfrak{A}$.

We shall show first that the assertion implies the result. Part (a) of the assertion is part (1) of the result. To show part (2): Let $n_1\in M$. $A_{n_1}-B$ is finite and by (b), every element in $A_{n_1}-B$ appears in only a finite number of the sets of $\mathfrak{A}$. And so for $n_2\in M$ sufficiently large, $(A_{n_1}-B)\cap A_{n_2}=\emptyset$. The set

$$(A_{n_1}-B)\cup(A_{n_2}-B)=(A_{n_1}\cup A_{n_2})-B$$

is finite and by (b), every element in that set appears in only a finite number of the sets of $\mathfrak{A}$. So for $n_3\in M$ sufficiently large,

$$[(A_{n_1}-B)\cup(A_{n_2}-B)]\cap A_{n_3}=\emptyset.$$

M is infinite and so we can continue and get a sequence $n_1, n_2, n_3, \ldots$ as required.

It remains to prove the assertion.

Case 1. Every element of A appears in only a finite number of sets. We then take $B=\emptyset$ and $M=\{1,2,3,\ldots\}$. Since no set is contained in another, $A_n\neq\emptyset$ for every n, and therefore $A_n\not\subset B$. Obviously part (b) of the assertion holds too.

Case 2. Some elements of A appear in infinitely many sets. A is denumerable, and we can assume that the elements of A are natural numbers (or, at least, ordered like the natural numbers). Let $x_1\in A$ be the first to appear in infinitely many sets. So there exists an infinite set N_1 such that $\forall n\in N_1$, $x_1\in A_n$. Consider $A-\{x_1\}$: If this set has elements which appear in infinitely many sets out of

$\{A_n \mid n \in N_1\}$, let x_2 be the first of them. So there is an infinite set $N_2 \subset N_1$ such that $\forall n \in N_2$, $x_2 \in A_n$ (and also $x_1 \in A_n$ and $x_1 \neq x_2$). We continue in this fashion: $x_3 \in (A - \{x_1, x_2\})$ is the first element which appears in infinitely many sets out of $\{A_n \mid n \in N_2\}$ (if such an element exists). There are two possibilities:

Possibility 1. The process stops after a finite number of steps. We then get elements $x_1, x_2, \ldots, x_k$ and infinite sets $N_1 \supset N_2 \supset \cdots \supset N_k$ such that for every $n \in N_k$, $\{x_1, \ldots, x_k\} \subset A_n$. We take $B = \{x_1, \ldots, x_k\}$ and $M = N_k$, and we now show that the assertion holds: (a) For every m, $A_m \not\subset B$ because for $n \in N_k$, $B \subset A_n$ and no set is contained in another. (b) Since the process stopped after a finite number of steps, it follows that, apart from $x_1, \ldots, x_k$, no other element of A appears in infinitely many sets out of $\{A_n \mid n \in N_k\} = \mathfrak{A}$. Therefore (b) is also true.

Possibility 2. The process does not stop. We then get an infinite sequence $x_1, x_2, \ldots$ and an infinite sequence of infinite sets $N_1 \supset N_2 \supset \cdots$ such that for every k and every $n \in N_k$, $\{x_1, \ldots, x_k\} \subset A_n$. Take $B = \{x_1, x_2, \ldots\}$. We first show that part (a) of the assertion holds: Assume $A_m \subset B$ for some m. A_m is finite and so for some k, $A_m \subset \{x_1, \ldots, x_k\}$. But for $n \in N_k$, $\{x_1, \ldots, x_k\} \subset A_n$ and therefore $A_m \subset A_n$. Contradiction.

We now choose M as follows: Let $m_1 \in N_1$, $m_2 \in (N_2 - \{m_1\})$, and for every k, $m_k \in (N_k - \{m_1, \ldots, m_{k-1}\})$. This choice is possible because every N_k is infinite. We now take $M = \{m_1, m_2, \ldots\}$. This set is infinite, and it remains to prove part (b): Assume that there exists an element $x \in (A - B)$ such that x appears in infinitely many sets out of $\mathfrak{A}$. A has the ordering of the natural numbers, B is infinite, and $x \notin B$. Therefore, for some k, $x_{k-1} < x < x_k$. M is not quite a subset of N_{k-1}, but it is clear that $M - N_{k-1}$ is finite. Since x appears in infinitely many sets out $\mathfrak{A} = \{A_n \mid n \in M\}$, it also appears in infinitely many sets out of $\{A_n \mid n \in N_{k-1}\}$. Therefore, at stage k, we would have chosen x and not x_k. Contradiction. (If $x < x_1$, we would have chosen x as the first element.) □

The following theorem is due to Paul Erdös. Its proof follows the original outline as given to the author in [1].

4.1. Theorem. *Let $A_1, A_2, \ldots$ be a denumerable sequence of finite sets such that if $n \neq m$, then $A_n \not\subset A_m$. Then there exists an infinite set $B \subset \bigcup_{n=1}^{\infty} A_n$ such that:*

(1) *For every n, $A_n \not\subset B$;*

(2) *For an infinite sequence $n_1, n_2, \ldots$, $A_{n_i} \cap B \cap A_{n_j} = \emptyset$ if $i \neq j$ and $A_{n_i} \cap B \neq \emptyset$.*

Proof. If there is an infinite subclass $A_{n_1}, A_{n_2}, \ldots$ of pairwise disjoint sets, we pick $x_i \in A_{n_i}$ and $B = \{x_1, x_2, \ldots\}$ and we are finished.

Assume that there is no such infinite subclass. We can assume without loss of generality that $A_1, \ldots A_l$ is a maximal subclass of pairwise disjoint sets, i.e., if $n > l$, then $A_n \cap (\bigcup_{i=1}^{l} A_i) \neq \emptyset$. Denote $C = \bigcup_{i=1}^{l} A_i$. C is finite and so at most a finite number of sets are contained in C, and we can assume that they are $A_{l+1}, \ldots, A_m$. Therefore, for $n > m$, we have $A_n \cap C \neq \emptyset$ and $A_n - C \neq \emptyset$. B will be chosen out of $\bigcup_{n=1}^{\infty} A_n - C$ and this will ensure that $A_n \not\subset B$ for all n.

C is finite and has a nonempty intersection with infinitely many sets, and so some subset C' of C is equal to infinitely many of these intersections, i.e., $C' \subset C$ and for some infinite set of indices M, $A_n \cap C = C'$ for every $n \in M$.

Let $n_1 > m$, $n_1 \in M$.

Claim. $\exists x_1 \in (A_{n_1} - C)$ such that for infinitely many values of n, $x_1 \notin A_n$.

Proof of claim. Assume otherwise. Then for every $x \in (A_{n_1} - C)$ there is an index n_x such that if $n > n_x$ then $x \in A_n$. Let $n_0 = \max\{n_x \mid x \in (A_{n_1} - C)\}$ and then for $n > n_0$, every element of $A_{n_1} - C$ is in A_n, i.e., $(A_{n_1} - C) \subset A_n$. But $\forall n \in M$, $A_n \cap C = C'$ and $C' = A_{n_1} \cap C$ (because $n_1 \in M$). Therefore $\forall n \in M$, $A_n \cap C = A_{n_1} \cap C$. If we now take any $n > n_0$ and $n \in M$, we get both $(A_{n_1} - C) \subset A_n$ and $A_n \cap C = A_{n_1} \cap C$ and therefore $A_{n_1} \subset A_n$. Contradiction.

This proves our claim. x_1 will be the first element of B and n_1 the first in the infinite sequence. Denote $N_1 = \{n \mid x_1 \notin A_n\}$. We have proved that N_1 is infinite. $n_2, n_3, \ldots$ will be chosen out of N_1. Let $C_1 = (C \cup A_{n_1}) - \{x_1\}$. C_1 is finite and has a nonempty intersection with infinitely many A_n's, $n \in N_1$ (because C has that property and $x_1 \notin C$). Therefore there exists $C_1' \subset C$ such that for infinitely many $n \in N_1$, $A_n \cap C_1 = C_1'$. Let $n_2 \in N_1$ such that $A_{n_2} \cap C_1 = C_1'$. The rest is similar: We can prove that for some $x_2 \in (A_{n_2} - C_1)$, $x_2 \notin A_n$ for infinitely many $n \in N_1$. From the choice of n_1, x_1, n_2, x_2 we get $x_1 \notin A_{n_2}$ and $x_2 \notin A_{n_1}$.

$n_3, n_4, \ldots$ are chosen out of $\{n \in N_1 \mid x_2 \notin A_n\}$. We continue in this fashion and we get elements $x_1\, x_2, \ldots$ and sets $A_{n_1}, A_{n_2}, \ldots$ such that $x_i \in A_{n_i}$ and for $i \neq j$, $x_i \notin A_{n_j}$. If we now take $B = \{x_1, x_2, \ldots\}$, then for every i, $A_{n_i} \cap B = \{x_i\}$ and so $A_{n_i} \cap B \cap A_{n_j} = \emptyset$ for $i \neq j$. As mentioned before, $x_i \notin C$ and so for every n, $A_n \not\subset B$. $\square$

5. Partial recursive functionals and finite-determinedness

In this section we study partial recursive functionals considering the notion of fd. If F is partial recursive and $F(\alpha)$ is undefined, it may be that F is fd on α or that F is not fd on α. Intuitively, the difference is the following: Imagine an oracle-machine trying to compute $F(\alpha)$ when F is not fd on α. As time goes on, the algorithm demands more and more elements from α. Theorem 5.1 tells us in effect that we can, at any stage, stop the process, review what the algorithm has read and effectively determine what changes in α (in the part that hasn't been read) would cause the computation to halt. In other words, the oracle-machine is "searching" α and we can effectively determine what it is searching for.

When $F(\alpha)$ is undefined and F is fd on α, then F is undefined in the same sense that an ordinary Turning Machine may be undefined on some of its input. If we recall the Kleene Normal Form Theorem, we see that it is now searching the natural numbers for a number to fulfill a certain recursive predicate, but it is not searching α. However, we may have been given a "wasteful" algorithm for F that even in this case demands infinitely many answers from α. Theorem 5.2 tells us that we can effectively construct another algorithm (from the given one) which will always remain bounded in such cases.

For the sake of simplicity, let us assume that we are dealing with the following type of oracle-machines: The oracle for a sequence α is a ω-type tape on which the numbers $\alpha_0, \alpha_1, \ldots$ are written. The machine has a "read" state, and a reading head which at the beginning is in front of α_0. Every time the machine reaches the "read" state, the next number from the tape is copied onto one of the work-tapes and the computation continues. We assume that whatever is copied from the oracle is not erased, so that the machine does not have to read the same element more than once.

5.1. Theorem. *If F is a partial recursive functional and F is not* fd *on α, then for every $n \geqslant 0$, there exists $m \geqslant n$ and numbers $k_n, \ldots, k_m$ such that $F(\beta)$ is defined, where*

$$\beta_i = \begin{cases} k_i & \text{if } n \leqslant i \leqslant m, \\ \alpha_i & \text{otherwise.} \end{cases}$$

Furthermore, m and $k_n, \ldots, k_m$ can be effectively found from any algorithm for F and $\alpha|_n$.

Proof. The idea is quite simple. Given an algorithm for F and $\alpha|_n$, we set the algorithm to work, dovetail fashion, on all possible finite extensions of $\alpha|_n$. (If the algorithm requires more elements than any such extension has, we simply pass on to the next computation.) The algorithm must halt on at least one such extension, because otherwise we would have $\alpha|_n \rhd (F=\omega)$-contradicting the assumption that F is not fd on α. □

5.2. Theorem. *For any partial recursive functional F, there exists an algorithm which computes F and is bounded on any α on which F is* fd. *Furthermore, such an algorithm can be effectively constructed from any given algorithm for F.*

Proof. Let $\alpha \in N^N$ be given. Start the given algorithm working, dovetail fashion, on all possible finite sequences (if it requires information not given in a particular finite sequence, pass on to the next computation). If it never halts on any finite sequence then F is everywhere undefined and this process certainly "computes" $F(\alpha)$. Let s_1 be the first finite sequence on which the algorithm halts. We now compare s_1 and α and determine whether s_1 is an initial segment of α or not. If it is, we print the result and halt. If it is not, let n_1 be the number of elements we have read off α in order to determine that $s_1 \not\subset \alpha$ (we can assume n_1 to be minimal). We now proceed with all possible finite extensions of $\alpha|_{n_1}$ (including $\alpha|_{n_1}$ itself) in the same manner. If the algorithm does not halt on any such extension, then $F(\alpha)$ is undefined. If it does halt, let s_2 be the first extension of $\alpha|_{n_1}$ on which it halts. Compare s_2 to α. If $s_2 \subset \alpha$, print the result and halt. If $s_2 \not\subset \alpha$, let n_2 be the (minimal) number of elements of α which we need to determine that $s_2 \not\subset \alpha$. We now proceed with $\alpha|_{n_2}$, and so on.

Note that in the above process, the sequence $n_1, n_2, \ldots$ is strongly increasing, because whenever we have determined that $s_k \not\subset \alpha$, we need *more* than n_{k-1} elements from α, because s_k is an extension of $\alpha|_{n_{k-1}}$.

Let us examine what the above process yields in the following 3 cases:

(1) $F(\alpha)$ is defined,

(2) F is not fd on α,

(3) $F(\alpha)$ is undefined but F is fd on α.

In case (1), the algorithm will eventually halt on a sequence which will be found to be an initial segment of α, and the process will print $F(\alpha)$ and halt.

In case (2), the process will continue indefinitely and never halt, requiring always more and more elements from α.

In case (3), there exists a number k such that $\alpha|_k \rhd (F(\alpha) = \omega)$. The process will never read beyond $\alpha|_k$ because the (original) algorithm will not halt on any extension of $\alpha|_k$. It follows that the above process will read only a finite number of elements from α.

Note that if we make n_k minimal in the above process, then in case (3) we will not read beyond $\alpha|_k$, where $k = \min\{n \mid \alpha|_n \rhd (F = \omega)\}$. □

In view of Theorem 5.2, we now raise the following question: Is it possible to recursively bound the number of elements read from the oracle when the functional is fd? By "recursively bound" we mean: given a partial recursive functional F which is fd, does there exist a total recursive functional G and an algorithm computing F such that for every α, the algorithm does not read more than $G(\alpha)$ elements off α?

Paul Young [5] has shown the answer to be no by giving an example of which the following is a simplified version:

5.3. Example. Let A be a r.e., nonrecursive set of natural numbers, and $f: N \to N$ a recursive function such that range $f = A$. Define:

$$F(\alpha) = \begin{cases} 0 & \text{if } \exists m \text{ such that} \\ & f(m) < f(\alpha_0) \quad \text{and} \quad \alpha_m = 0, \\ \omega & \text{otherwise.} \end{cases}$$

F is clearly partial recursive. It is also fd because for any α, there are only a finite number of m's such that $f(m) < f(\alpha_0)$. (We are assuming that f enumerates each element of A only once.)

If F could be recursively bound, it means that there exists a recursive function $g: N \to N$ such that for any n, $m: f(m) < f(n) \Rightarrow m \leq g(n)$ ($g(\alpha_0)$ gives us the bound). We now use f and g to give a decision method for A: Let k be given. We calculate $f(0), f(1), \ldots$ until we find l such that $k < f(l)$ (such an l exists because A is nonrecursive and hence infinite). $k \in A \Leftrightarrow \exists m$ such that $f(m) = k < f(l)$. If $f(m) < f(l)$, then $m \leq g(l)$, therefore we can restrict the search for such an m to the set $\{0, 1, 2, \ldots, g(l)\}$. Therefore A is recursive. Contradiction. □

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References

[1] P. Erdös, Private communication, Jan. 1976.
[2] D. Gordon, Computability, determining sets and dependence degrees of functionals, D.Sc. Thesis, The Technion, Israel Institute of Technology, March 1976.
[3] D. Gordon and E. Shamir, Computation of recursive functionals using minimal initial segments (note), to appear in Theor. Comput. Sci.
[4] H. Rogers, Jr., Theory of Recursive Functions and Effective Computability (McGraw-Hill, New York, 1967).
[5] P.R. Young, Private communication, Oct. 1976.