On the Computation of the Cross of Dispersion

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For several decades the cartographical presentation of variance of spatial data has been possible through the use of the 'cross of dispersion' or, as it has sometimes been called 'the standard deviational ellipse' (Lefever, 1926; Furfey, 1927; Bachi, 1963; Shimoni, 1966). The purpose of this note is to point to an error related to one of the computational procedures for the cross or the ellipse. With one recent exception, the cross of dispersion has not found its way into textbooks on quantitative methods in the spatial sciences (Ebdon, 1977), in spite of its usefulness for the analysis of spatial data. The most likely reason for this neglect is the considerable amount of cumbersome calculations and precise drawing effort required, through the results provide the researcher with much richer information on the variation when compared to the univariate standard deviation. The cross of dispersion, focused on the mean centre, lets the researcher know the mean location of the mapped variable, the principal direction of the series or the direction where most of the variation exists (σ_x') and its counterpart, the minor direction where the rest of the variation lies (σ_y').

In addition,

$$\sigma_x^{2\prime} + \sigma_y^{2\prime} = d^2 \quad (1)$$

so that d the Standard Distance (Bachi, 1963) may be calculated and an 'ellipse' may be drawn delimiting an area of

$$A = \tfrac{1}{2}\pi d^2 \quad (2)$$

as shown by Furfey (1927), who has also shown that an ellipse is not always achieved so that the term 'standard deviational ellipse' is less accurate than the term 'cross of dispersion'.

The recent introduction of mini-computers with graphic components has provided us with the facilities for instant calculation and graphical presentation, making it a relatively easy task to produce the mapping of data, the calculation of the mean centre and the angle of rotation of the original σ_x, σ_y, (angle α), computation of σ_x, σ_y, σ_x', σ_y' and draw them on the map.

This ease of computation and mapping made possible the comparison of two avenues of solution for the arms σ_x' σ_y' enabling us to show that one of them is incorrect.

Furfey's. (1927) original solution, adopted by Shimoni (1966) and Bachi (1966), was based on the transformation of each point so that each transformed point was equal to

$$x' = x \cos \alpha + y \sin \alpha \quad (3)$$

$$y' = y \cos \alpha - x \sin \alpha \quad (4)$$

where α is that angle by which the original σ_x and σ_y have to be rotated so that σ_x' and σ_y' will be $\sigma_{\max}$ and $\sigma_{\min}$ respectively. Shimoni's refined version of α, based on Linders (1931), was

$$\tan 2\alpha = \frac{2 \operatorname{cov}(x; y)}{\sigma_x^2 - \sigma_y^2} \quad (5)$$

The two new axes, σ_x' and σ_y' can be calculated so that

$$\sigma_x' = \sigma_x^2 \cos^2\alpha + \sigma_y^2 \sin^2\alpha + 2 \operatorname{cov}(x; y) \sin\alpha \cos\alpha \quad (6)$$

$$\sigma_y' = \sigma_y^2 \sin^2\alpha + \sigma_y^2 \cos^2\alpha + 2 \operatorname{cov}(x; y) \sin\alpha \cos\alpha \quad (7)$$

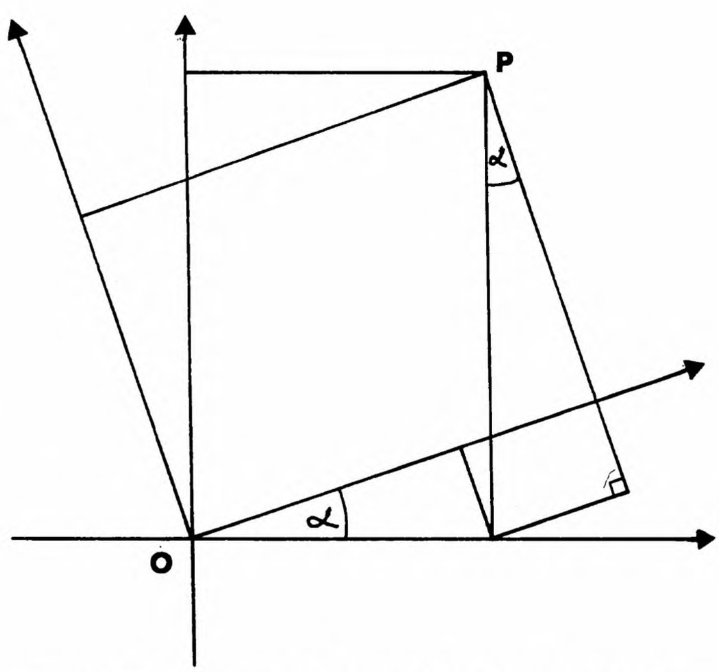

Figure 1. The transformation of data points and σ_x; σ_y, by angle α. (After Bachi [2, p. 177]).

The only source in which this procedure has been demonstrated graphically is Bachi (1966) (Figure 1). In this figure, the two rectangles created by σ_x; σ_y and σ_x'; σ_y' respectively have two common corner points O and P. As will be demonstrated, this is misleading, as it is not generally true. This figure served as the basis, however, for Kadmon's solution (1968; 1971) (Figure 2). Since the rectangles are supposed to coincide, d thus serves as a common diagonal. This quality permits the definition of $\tan \beta$, as

$$\tan \beta = \frac{\sigma_y}{\sigma_x} \tag{8}$$

and, therefore,

$$\sigma_x' = d \cdot \cos (\beta - \alpha) \tag{9}$$

$$\sigma_y' = d \cdot \sin (\beta - \alpha) \tag{10}$$

Though this solution is much easier to calculate than the former, empirical testing shows that it is simply incorrect (Figure 3). Only O is a common point and both rectangles have a diagonal of the same length, but this does not imply that the rectangles have another common corner point.

Bachi's mathematical solution is correct, but Kadmon's mathematical solution relied on Bachi's faulty figure and is thus incorrect.

It is necessary, therefore, to use the more cumbersome solution. However, as has already been noted, this lends itself readily to programming and computerised plotting, thereby removing many of the objections to its use as an anlytical tool in spatial studies.

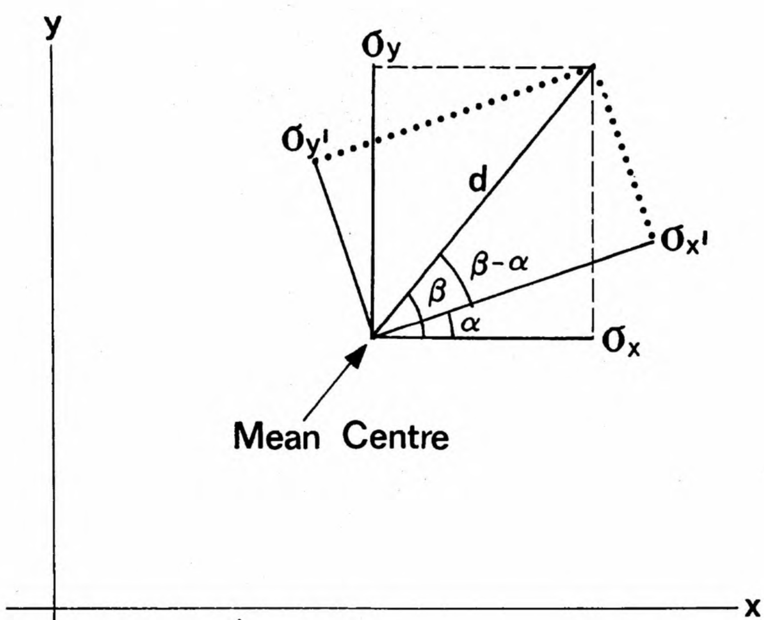

Figure 2. The derivation of σ_x'; σ_y' by d. (After Kadmon [5, p. 68]).

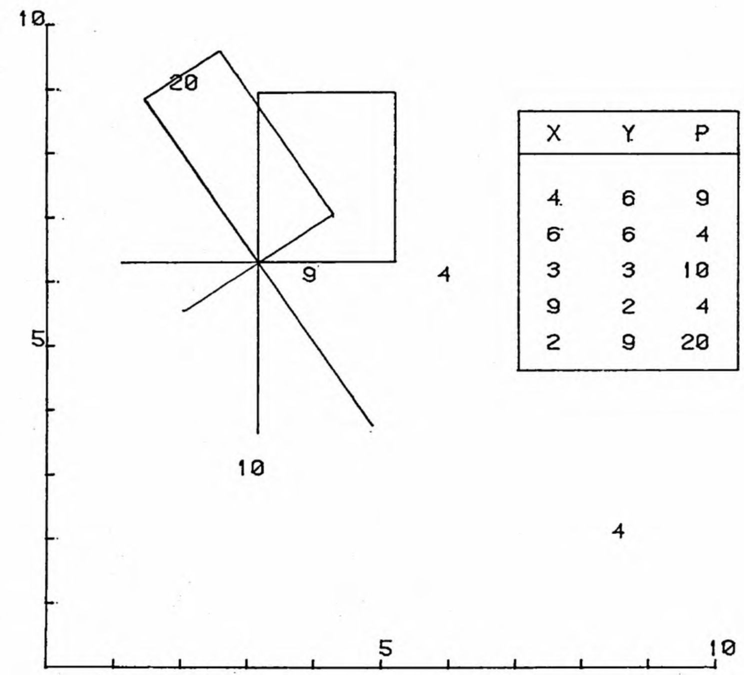

Figure 3. A numerical example of the Lefever—Shimoni solution.

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Editorial note.

Readers may care to note that the cross of dispersion is more widely known as eigenvalue/vector analysis.