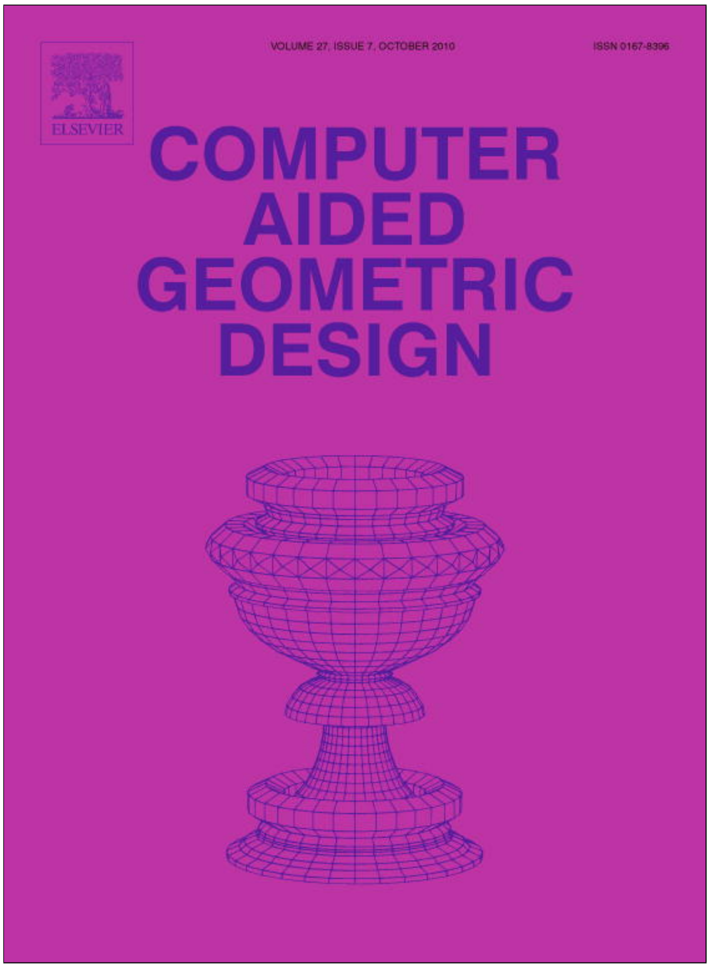
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Corner cutting and augmentation: An area-preserving method for smoothing polygons and polylines

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A B S T R A C T

A new subdivision method is presented for smoothing polygons and polylines while preserving the enclosed area. The new technique, called "corner cutting and augmentation" (CCA), operates by cutting corners with line segments and adding the cut area of each corner to two augmenting structures constructed on the two incident edges; this operation can be iterated as needed. Area is preserved in a local sense, meaning that when a corner is cut, the cut area is added to the other side of the line in immediate proximity to the cut corner. Thus, CCA is also applicable to self-intersecting polygons and polylines, and it enables local control. The concept of a "bounding hull" of a convex polygon is defined and used to show that certain implementation restrictions guarantee the existence and the G^1-continuity of the limit curve, and also the preservation of convexity when the original polygon is convex. CCA allows the definition of various profiles which determine how closely each iteration follows its previous polygon. Potential applications include computer aided geometric design, an alternative to spline approximation, as an aid to artistic design, and for some applications, as a potential alternative to multiresolution curves.

1. Introduction

One of the major topics in computer aided geometric design (CAGD) is to design a smooth curve that interpolates or approximates a given sequence of points. The sequence may be open, i.e., it forms a polyline, or it may be closed cyclically, forming a polygon. This topic has been researched very widely—see for example Farin et al. (2002).

Recent years have seen a growing interest in combining the creation of smooth curves together with the preservation of certain geometric properties, such as area preservation. Sapiro and Tannenbaum (1995) and Olver et al. (1997) study the problem of preserving volumes, areas and lengths under smoothing operations. They show that the geometric heat flow equation can be used to obtain smoothing without shrinkage. Elber (2001) combines multiresolution control and linear constraint satisfaction within the framework of multiresolution curve editing. He shows that preservation of area can be represented as a linear constraint, so this property can also be preserved. Hahmann et al. (2005), by using a wavelet decomposition, also achieve area preservation, together with some other properties, such as level-of-detail (LOD) display and progressive transmission. More recently, Sauvage et al. (2008) extended these two works to 3D B-spline surfaces with volume preservation by expressing the volume in trilinear form, enabling efficient LOD editing.

A fundamental technique used for smoothing curves is that of *corner cutting*; see de Boor (1987). In this paper, we present a new approach, called "corner cutting and augmentation" (CCA), to the smoothing and area preservation problems. Starting with a low resolution polygon or polyline, CCA progressively refines it, while maintaining a constant area. CCA uses

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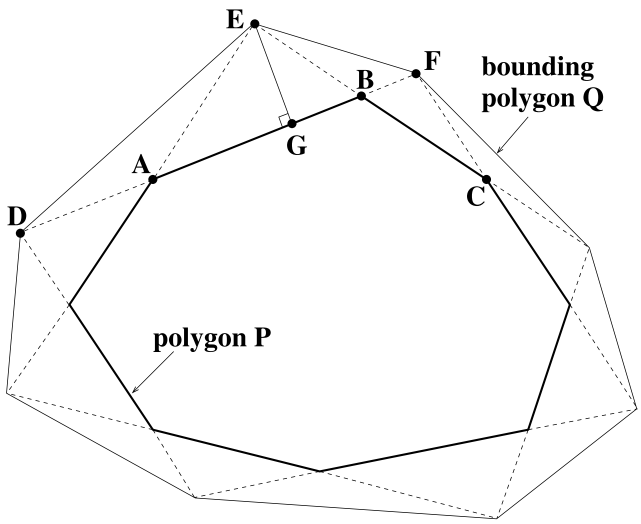

Fig. 1. Construction of the bounding hull: $Q = \mathbb{B}(P)$.

corner cutting together with an augmentation step in which the cut areas are added back to the polygon by constructing certain augmenting structures on the edges. Area is preserved in a local sense, meaning that whenever a corner is cut, the cut area is added to the other side of the line in immediate proximity to the cut corner. This approach can be applied to simple and to self-intersecting polygons and polylines.

Related issues in CAGD are those of multiresolution and subdivision. Elber (2001), Hahmann et al. (2005) and Sauvage et al. (2008) preserve area in the context of multiresolution, meaning that they maintain the same area when going from high resolution to low resolution, whereas CCA goes in the opposite direction. Subdivision schemes have a long history in CAGD; see for example the extensive review of Dyn and Levin (2002) or the more recent review of Dyn (2006). Our approach is constructive and bears some similarity to other constructions, e.g., Sabin and Dodgson (2005).

CCA has the following properties and potential applications:

- It is very simple and allows local control.
- Certain simple implementation restrictions ensure the existence and G^1-continuity of the limit curve, and the preservation of convexity.
- CCA can be useful for CAGD applications which require area preservation, e.g., the design of cross-sections of complicated tubes with a specified area.
- It can also be useful in drawing plots of functions, since it will maintain the area under a curve (the integral).
- CCA can be used in artistic design, where area preservation is a positive aesthetic feature. It can also be useful for online games by enabling a minimal representation of smooth curves bounding a given area.
- For some applications, CCA can be a simple alternative to multiresolution curves for purposes of LOD rendering and transmission; only the initial polygon needs to be transmitted, and it can be smoothed to any level by the receiving agent.

As an alternative to CCA, one could smooth the polygon by some standard method, and then scale up the result to preserve the area. However, this approach does not provide local control and it does not enable the user to smooth selected segments of a polygon while maintaining a constant area; for example, it would not preserve the integral of a line plot. Some preliminary practical results with CCA were presented in Gordon et al. (2007).

The rest of the paper is organized as follows. Section 2 develops the theoretical foundations of CCA. Section 3 presents CCA1, which implements CCA with triangular augmentation. Section 4 presents certain design issues, and Section 5 lists some extensions. Section 6 summarizes and presents some topics for further research.

2. Convex polygons and their bounding hulls

In this section, we describe how, given a convex polygon P, we can construct a certain bounding convex polygon Q, called the bounding hull of P. Certain properties of the bounding hull will be used to prove the existence of the limit curves of CCA. We denote the boundary of a set A in the plane by $\partial(A)$, and its interior by $\text{int}(A) = A - \partial(A)$.

Definition 1. Let P be a convex polygon with all internal angles $> \pi/2$. The *bounding hull*, $Q = \mathbb{B}(P)$, is defined as follows: consider an edge AB of P, as in Fig. 1, and let E be the intersection of the extensions of the two edges neighboring AB. Q is formed by connecting all such vertices in the same order as their corresponding edges in P.

$\mathbb{B}(\cdot)$ will also be used to represent the relation between vertices and edges of P and their corresponding edges and vertices in $\mathbb{B}(P)$, e.g., in Fig. 1, $E = \mathbb{B}(AB)$ and $EF = \mathbb{B}(B)$. Note that in Definition 1, E and P are on opposite sides of the line defined by AB because the internal angles at A and B are $> \pi/2$.

Lemma 1. *Let P be a convex polygon with all angles $> \pi/2$ and $Q = \mathbb{B}(P)$. Then Q is convex and $P \subset \mathrm{int}(Q)$.*

Proof. To show that Q is convex, we will show that all its internal angles are $< \pi$. Consider a vertex angle of Q, such as $\angle DEF$ in Fig. 1. Let G be the projection of E onto the line formed by A and B. Since the internal angles at A and B are $> \pi/2$, the base angles of $\triangle EAB$, $\angle EAB$ and $\angle EBA$, are both $< \pi/2$. Therefore, G is internal to AB and also to DF. In $\triangle EDG$, the angle at G is $\pi/2$, and so $\angle DEG < \pi/2$. Similarly, $\angle FEG < \pi/2$, so $\angle DEF < \pi$.

$P \subset \mathrm{int}(Q)$ because all of P's vertices are internal to Q. □

Let d denote the Euclidean distance. If x is a point and Y is a set of points, we denote $d(x, Y) = \inf\{d(x, y) \mid y \in Y\}$. The *Hausdorff distance* between two compact sets, X and Y, is defined as:

$$h(X, Y) = \max\Big\{\sup_{x \in X} d(x, Y),\ \sup_{y \in Y} d(y, X)\Big\}.$$

It is well known that $h(X, Y) = 0 \Leftrightarrow X = Y$; see, e.g., Henrikson (1999).

Lemma 2. *Let P, Q be as above, then $h(P, Q)$ is always obtained as the distance between a vertex of Q and its corresponding edge in P, i.e.,*

$$h(P, Q) = \max\big\{d(q, e) \mid q \text{ is a vertex of } Q,\ e \text{ is an edge of } P,\ \text{and } q = \mathbb{B}(e)\big\}.$$

Proof. By definition, $h(P, Q) = \max\{\sup_{p \in P} d(p, Q), \sup_{q \in Q} d(q, P)\}$. We will first show that the second element is larger than the first. P is convex and every $q \in Q$ is external to P, so the projection function $\pi : Q \to P$ is well-defined and surjective ($\pi(q)$ is the unique point in P that is closest to q). For all $q \in Q$, we have $d(\pi(q), Q) = \inf_{q' \in Q} d(\pi(q), q') \leqslant d(\pi(q), q)$. Since π is onto P, we get $\sup_{p \in P} d(p, Q) = \sup_{q \in Q} d(\pi(q), Q) \leqslant \sup_{q \in Q} d(\pi(q), q)$. Now, we have $\sup_{q \in Q} d(q, P) = \sup_{q \in Q} \inf_{p \in P} d(q, p) = \sup_{q \in Q} d(q, \pi(q))$. (The last equality follows from the definition of π.) Therefore, $\sup_{q \in Q} d(q, P) \geqslant \sup_{p \in P} d(p, Q)$, so $h(P, Q) = \sup_{q \in Q} d(q, P)$.

Consider an edge of Q, such as EF, and a point $q \in EF$. Since P is convex and the lines defined by AE and CF are supporting lines of P, $\pi(q) \in AB \cup BC$. From the proof of Lemma 1, $\pi(E)$ and $\pi(F)$ are internal to AB and AC, respectively. We can see from Fig. 1 that as q moves from E to F, $d(q, \pi(q))$ first decreases and then increases. Therefore, for $q \in EF$, the maximal distance between q and $\pi(q)$ is obtained at E or F, and the result follows. □

Lemma 3. *Let A be a bounded convex set in the plane with a non-empty interior, and $B = \partial A$ its boundary. Then B is a continuous curve.*

This result is well known, and some textbooks even define a "convex curve" in two dimensions as the boundary of a convex set with a non-empty interior. For the sake of completeness, a proof is presented in Appendix A.

Theorem 1. *Let $P_0, P_1, \ldots$ be a sequence of convex polygons such that all angles of P_n are $> \pi/2$ and all angles of P_n go to π as $n \to \infty$. Assume also that for some $A > 0$, the area of $P_n > A$. Let $Q_n = \mathbb{B}(P_n)$, and assume that $Q_{n+1} \subset \mathrm{int}(Q_n)$. Then the sequences $\{P_n\}$ and $\{Q_n\}$ both converge to the same continuous and convex curve P^*.*

Proof. Since the angles of P_n go to π, so do angles such as $\angle AEB$ in Fig. 1. Therefore, the angles of Q_n also go to π as $n \to \infty$. For every n, let d_n be the maximal distance between a vertex of Q_n and its corresponding edge in P_n. From Lemma 2, we have $d_n = h(P_n, Q_n)$. Since the Q_ns are nested, all edges are of bounded length. Consider triangles such as $\triangle EAB$ in Fig. 1: their bases (AB) are of bounded length, and the angles at their opposite vertices (E) tend to π. Hence, the heights (taken from E) also go to 0. Therefore, $h(P_n, Q_n) = d_n \xrightarrow[n\to\infty]{} 0$.

Define $Q = \bigcap_{n=0}^{\infty} \mathrm{int}(Q_n)$ and $P^* = \partial(Q)$. Q is convex, bounded, and has a non-empty interior, so by Lemma 3, P^* is a continuous curve. We will show that $\{Q_n\}$ and $\{P_n\}$ converge towards P^*. Let R be a ray originating from some interior point $a \in Q$, and cutting all the Q_ns. Denote $p = R \cap P^*$ and $q_n = R \cap Q_n$. Claim: $q_n \xrightarrow[n\to\infty]{} p$. The Q_ns are nested, so $\{q_n\}$ is a bounded sequence on R with monotonically decreasing distances from a, so for some point $p' \in R$, $q_n \xrightarrow[n\to\infty]{} p'$. Clearly, p and p' must coincide (otherwise, the open segment $\mathrm{int}(pp')$ on R does not intersect the interior of any Q_n, and this is impossible). Therefore, $Q_n \xrightarrow[n\to\infty]{} P^*$. Since $h(P_n, Q_n) \xrightarrow[n\to\infty]{} 0$, we also have $P_n \xrightarrow[n\to\infty]{} P^*$. □

The bounding hull concept can be extended to 3D as follows: assume that P is a convex polyhedron whose faces are triangles, and that any internal angle β between two adjacent triangles sharing an edge satisfies $\pi/2 < \beta < \pi$. For every

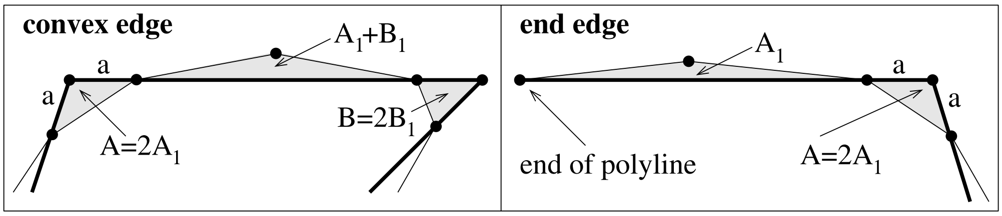

Fig. 2. CCA1 at a convex edge (left) and at an end edge (right).

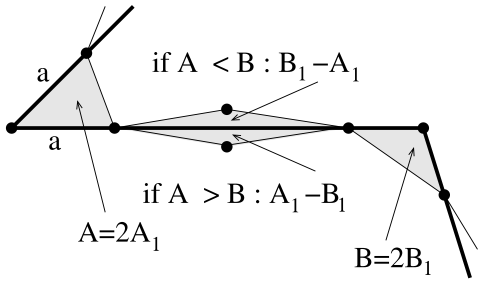

Fig. 3. CCA1 at an inflection edge.

triangle face T, consider the three planes formed by its adjacent triangles: they meet at a point A which is separated from P by the plane defined by T. We can define the bounding hull of P as the convex hull of all such points. We conjecture that Theorem 1 can be extended to 3D, and also to higher dimensions.

3. CCA1: triangular augmenting structures

Assume we are given a polygon or polyline with all angles $\neq \pi$. For every vertex (except the end vertices of a polyline), we mark two equidistant points on the incident edges, lying at a distance of $a = \alpha \times$ the shorter edge length, where $0 < \alpha < 0.5$. The corner at the vertex is then cut by joining these two points, and augmenting isosceles triangles are constructed on the incident edges. The cutting and augmenting operations are illustrated in Figs. 2 and 3, and the area of the augmenting triangle is determined as follows:

- Convex edge (the two cut areas are on the same side of the line). The area of the augmenting triangle is equal to half the sum of the two cut areas.
- End edge of a polyline. The augmenting triangle is equal to half the cut area.
- Inflection edge. If the two cut areas are equal, then no augmenting triangle is constructed; otherwise, it is constructed on one side so as to compensate for the net cut area.

This description raises the question of how α should be determined. This is taken up in Section 4. The following subsections show how certain limitations imply G^1-continuity and convexity preservation, and how non-convex polygons and polylines are handled.

3.1. Smoothness and preservation of convexity

In this section we will describe certain implementation restrictions which guarantee the existence of the limit curve defined by CCA1, its G^1-continuity, and also the preservation of convexity (when the original polygon is convex). We will also show that the limit curve does not contain any straight line segments. The treatment is geometric in nature, similarly to Paluszny et al. (1997).

We assume that the initial polygon, P_0, is convex with angles $< \pi$. Fig. 4 shows an edge AB of a convex polygon, with the cutting lines intersecting AB at points C and D, and meeting at E. Let F be the projection of E on AB, and assume that α_1, α_2 are two numbers satisfying $\alpha_1 \leqslant |AC|/|AB|, \alpha_2 \leqslant |DB|/|AB|$, respectively.

Definition 2. The CCA1 algorithm is required to satisfy the following conditions:

- The cut triangles and the augmenting triangles are isosceles.
- For some $\epsilon > 0$, each α_i satisfies the inequalities $\epsilon < \alpha_i < \frac{1}{3}$.
- The augmenting triangle, GCD in Fig. 4, satisfies the following:
  1. The area of GCD is equal to half the sum of the two cut corners.

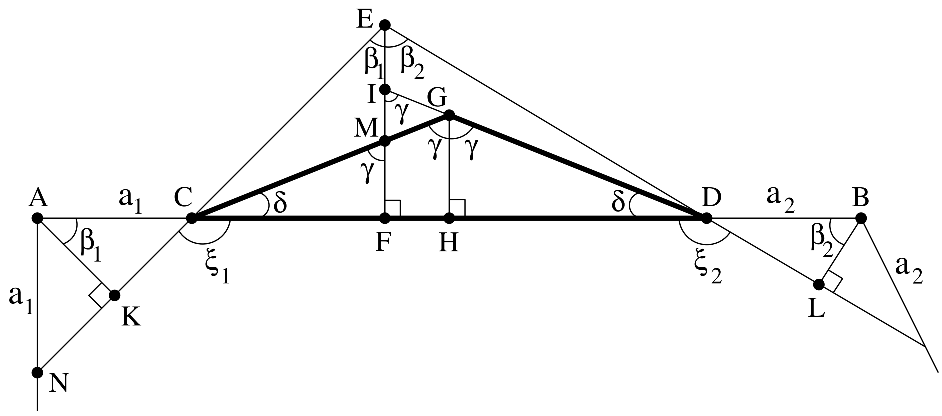

Fig. 4. Explanation of the requirements for CCA1 at a convex edge AB.

2. Let I be the intersection of EF with the extension of one of the sides of $\triangle GCD$ (if G lies on EF, we take $I = G$). Then for some constant $c > 1$, $|EF|/|IF| \geqslant c$.

Condition 3 implies that G is strictly interior to $\triangle ECD$. Note that the bound on α places each of the two αs in the Gregory–Qu convergence region (Gregory and Qu, 1996, Fig. 3; Paluszny et al., 1997, Fig. 3). Let β_1, β_2 be the half-angles at A and B, respectively. Then $\angle CEF = \beta_1$ and $\angle DEF = \beta_2$. We denote other relevant angles as shown in Fig. 4.

Let $P_1, P_2, \ldots$ be the sequence of polygons obtained by successive applications of CCA1. We denote $Q_n = \mathbb{B}(P_n)$. If the limit curve exists, we will denote it by P^*. Our main result for CCA1 is the following:

Theorem 2. *Under the above conditions, P^* exists, it is convex and G^1-continuous, and it does not contain any straight line segments.*

The proof will follow from a sequence of lemmas. The following lemma specifies some inequalities which imply the preservation of convexity, and are also necessary (but not sufficient) for the G^1-continuity of the limit curve.

Lemma 4. *For $i = 1, 2$, the following inequalities hold*:

$$\gamma > \beta_i, \tag{1}$$

$$\xi_i > 2\beta_i, \tag{2}$$

$$\pi/2 < \xi_i + \delta < \pi. \tag{3}$$

Proof. Inequality (1) follows immediately from the fact that G is strictly interior to $\triangle GCD$—see Fig. 4. We also have $\xi_i = \beta_i + \pi/2$ and $\beta_i < \pi/2$, from which inequality (2) follows. From $\xi_i = \beta_i + \pi/2$, we also get $\xi_i + \delta > \pi/2$. Since G is strictly interior to $\triangle ECD$, $\delta < \angle ECD$. Also, $\xi_1 + \angle ECD = \pi$, so $\xi_1 + \delta < \pi$. Similarly, $\xi_2 + \delta < \pi$. □

Inequality (1) means that the new angle formed at the vertex of an augmenting triangle is larger than the average of the neighboring angles of the previous iteration. Inequality (2) implies that $\xi_i + \delta > 2\beta_i$, i.e., a new angle which is not the vertex of an augmenting triangle is larger than the neighboring angle of the previous iteration. From (3) it follows that CCA1 preserves convexity. We can now state the following:

Lemma 5. *If P_0 is convex with initial angles $> \pi/2$, then all the angles tend upwards to π at an exponential rate as $n \to \infty$, and every P_n is convex.*

Proof. From Fig. 4, we can see that the two new half-angles which appear to the "right" of β_1 are $(\xi_1 + \delta)/2$ and γ. Now, $\xi_1 = \beta_1 + \pi/2$, so $\tan((\xi_1 + \gamma)/2) > \tan((\beta_1 + \pi/2)/2)$ (note that from Lemma 4, $(\xi_1 + \delta)/2 < \pi/2$). Using the trigonometric identity $\tan((\alpha + \pi/2)/2) = (1 + \sin\alpha)/\cos\alpha$, we get $\tan((\xi_1 + \delta)/2) > (1 + \sin\beta_1)/\cos\beta_1 \geqslant 2\tan\beta_1$. We also have $\tan\gamma = |CF|/|MF|$ and $\tan\beta_1 = |CF|/|EF|$. Therefore, $\tan\gamma/\tan\beta_1 = |EF|/|MF| \geqslant |EF|/|IF| \geqslant c > 1$, so $\tan\gamma \geqslant c\tan\beta_1$. Assume now that β_1 is the smallest half-angle of P_n, then it follows from the above that every half-angle ϵ of P_{n+1} satisfies $\tan\epsilon > \min(2, c)\tan\beta_1$, so all angles tend to π and P_n is convex.

To see why the rate of convergence is exponential, denote $\epsilon_0 = \beta_1$, and for $k \geqslant 1$, by ϵ_k the smallest half-angle of P_{n+k}. Denote also $d = \min(2, c) > 1$, and $b = 1/\tan\beta_1$. From the above, we get $\tan\epsilon_1 > d\tan\epsilon_0 = d/b$, and in general, $\tan\epsilon_n > d\tan\epsilon_{n-1}$. From this we get $\tan\epsilon_n > d^n/b$, and it follows that $\sin(\pi/2 - \epsilon_n) = \cos\epsilon_n < b\sin\epsilon_n/d^n < b/d^n$. Now, for any angle $\zeta < \pi/4$, we have $\zeta < 1.2 \times \sin\zeta$. After a finite number of iterations, we will have $\pi/2 - \epsilon_n < \pi/4$, so $\pi/2 - \epsilon_n < b/(1.2 \times d^n)$, i.e., ϵ_n converges to $\pi/2$ at an exponential rate. □

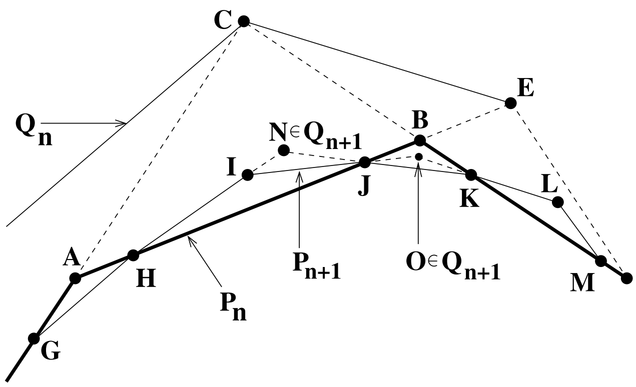

Fig. 5. Diagram showing that $Q_{n+1} \subset \text{int}(Q_n)$.

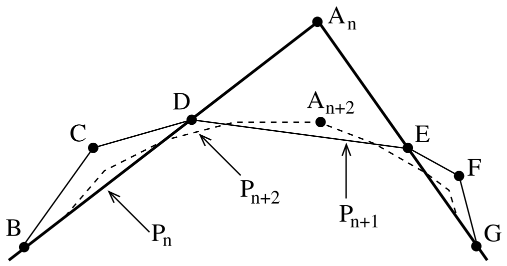

Fig. 6. A sequence of embedded cut corners.

Lemma 6. *All edge lengths go to zero at an exponential rate as the number of iterations goes to infinity.*

Proof. From the proof of Lemma 5 we know that eventually, all interior angles are $> \pi/2$. We assume that n is sufficiently large so that P_n satisfies this condition. Let ℓ be the length of the longest edge in P_n. We will show that in P_{n+1}, all edges are shorter than $\ell/\sqrt{2} \approx 0.707 \times \ell$, and the result follows from this.

Fig. 4 shows that there are two types of new edges: a corner-cutting edge such as CN, and the side of an augmenting triangle such as CG. Since every α used by CCA1 satisfies $\alpha < 1/3$, we have $|AC| < \ell/3$. Therefore, $|CN| = 2|CK| = 2|AC| \sin\beta_1 < 2|AC| < 2\ell/3 < \ell/\sqrt{2}$.

As to CG, we have $|CH| = |CD|/2 < \ell/2$, and $\gamma > \beta > \pi/4 \Rightarrow \sin\gamma > \sqrt{2}/2$. Therefore, $|CG| = |CH|/\sin\gamma < \ell/(2\sin\gamma) < \ell/\sqrt{2}$. □

Lemma 7. *For CCA1,* $Q_{n+1} \subset \text{int}(Q_n)$.

Proof. Fig. 5 shows parts of P_n, Q_n and P_{n+1}. The vertices of Q_{n+1} are of two types: N is formed by an augmenting edge HI and a cutting edge JK, and O is formed by two augmenting edges IJ and KL. We will show that both are interior to Q_n. Since $\angle GHI, \angle LKJ < \pi$, N is interior to $\triangle CAB$, so it is interior to Q_n. Also, $\angle IJK, \angle LKJ < \pi$, so the intersection of the extensions of IJ and LK meet inside $\triangle BJK$. Hence, O is interior to P_n, and therefore also to Q_n. □

Corollary 1. *The limit curve, P^*, exists, it is convex, and both $\{P_n\}$ and $\{Q_n\}$ converge to P^*.*

Proof. Immediate from Theorem 1. □

Lemma 8. *Every cut corner of P_n contains a point of P^* in its interior.*

Proof. The proof is explained with the aid of Fig. 6: let A_n be a vertex of P_n and DE the segment cutting the corner at A_n (in the construction of P_{n+1}). Consider the augmenting triangle constructed on DE: the vertex of that triangle, A_{n+2}, is a vertex of P_{n+2} and it is cut at the next stage. Note that the cut triangle at A_{n+2} is strictly contained in $\triangle A_n DE$. Thus, we get a sequence of strictly nested triangles whose edge lengths go to zero. Hence, the sequence $A_n, A_{n+2}, A_{n+4}, \ldots$ converges, and its limit is a point of P^*. □

Consider P_n and P^*: every cut corner of P_n contains a point of P^*. Also, every augmenting triangle constructed on an edge of P_n also contains a point of P^* (in the cut vertex of that triangle), so that point is external to P_n. Since both P_n and P^* are convex, P^* cuts every edge of P_n exactly twice. Those intersection points form a secant polygon of P^*, which we denote by S_n. We now have the following two results:

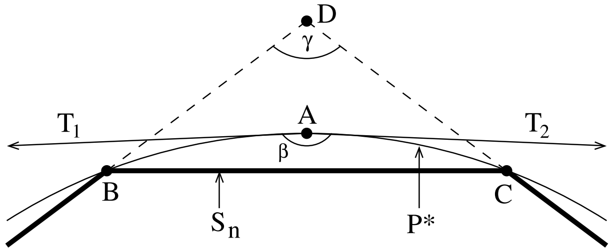

Fig. 7. Explanation of the single tangent property.

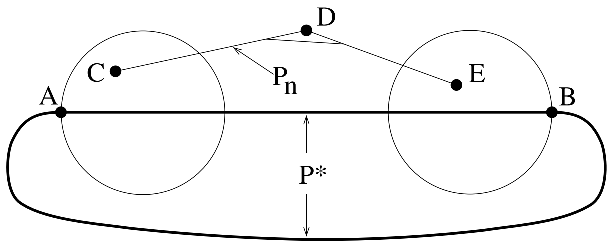

Fig. 8. Diagram showing that P^* does not contain a straight line segment.

Lemma 9. *For all n, every vertex of P_n is external to P^*.*

Proof. Immediate from the preceding discussion. □

Lemma 10. *P^* is G^1-continuous.*

Proof. Note that P^* cannot have a line segment that is colinear with an edge of P_n. Since the edges of P_n go to 0 and the angles of P_n go to π, it is easy to see that the same holds also for S_n. Therefore, the angles between two edges of S_n that are separated by one edge, also go to π. Suppose there are two different supporting rays, T_1 and T_2, emanating from some point A on P^*, with an angle β between them, as shown in Fig. 7. (We assume that the lines defined by T_1 and T_2 are tangents to P^*.) Assume that n is large enough so that the smallest angle between two edges of S_n separated by one edge is $> \beta$. We also assume that A does not coincide with a vertex of S_n (otherwise, we use S_{n+1}). Fig. 7 shows an edge BC of S_n and the part of P^* containing A inside the triangle DBC formed by BC and the extensions of its adjacent edges in S_n. By our choice of n, the angle at D, denoted by γ, is $> \beta$.

Consider T_1: it is a supporting ray of P^*, which is convex, so it cannot strictly separate between B and C because they are on P^*. Hence, T_1 cannot cut $\triangle DBC$ at an interior point of BC. Therefore, T_1 cuts $\triangle DBC$ at BD, and similarly, T_2 cuts $\triangle DBC$ at CD. Since A is interior to $\triangle DBC$, we have $\beta > \gamma$, and this is a contradiction. □

Lemma 11. *P^* does not contain any straight line segments.*

Proof. Assume otherwise, and consider two points A and B lying on such a segment. Consider two non-intersecting circles whose centers are on the segment AB and passing through A and B, as shown in Fig. 8. Since $\mathrm{int}(P^*) \neq \emptyset$, we can assume that these circles intersect P^* only at the points A and B, and at the diametrically opposite points. The edges of P_n tend to zero, so for n sufficiently large, each circle must contain a vertex of P_n.

Denote these vertices by C and E. From Lemma 9, C and E are external to P^*. The edges of P_n go to zero, so for n sufficiently large, there is at least one more vertex, D, between C and E. P_n is convex and AB is a straight line segment, so all edges of P_n between C and E are external to P^*, and the line formed by AB separates them from $\mathrm{int}(P^*)$. Therefore, the cut corner at D is also external to P^*, and this contradicts Lemma 8. □

Proof of Theorem 2. This follows from Corollary 1 and Lemmas 10 and 11. □

Note that in the proof of Theorem 2 we did not use the area condition at all. This condition is also not used in the next subsection. In other words, CCA1 is really quite a general smoothing operation, and area preservation is just one particular application. An additional property that can be proved is that the edges of Q_n go to zero as $n \to \infty$.

3.2. Polylines and non-convex polygons

We will call a polyline convex if, by connecting its two endpoints, we get a convex polygon. This terminology is similar to the one used in Paluszny et al. (1997). Assume for now that we are dealing with a convex polyline. We consider the

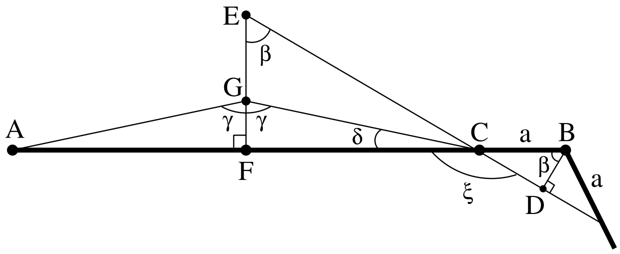

Fig. 9. The augmenting triangle, *GAC*, on an end edge.

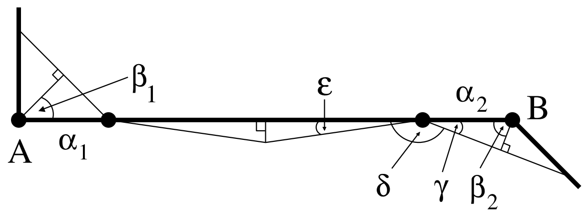

Fig. 10. Angles formed by CCA1 at an inflection edge.

requirements for an end edge of a polyline. Fig. 9 shows an end edge *AB* (*A* is the endpoint) with the cutting line intersecting it at *C*. In order to preserve convexity, the augmenting triangle, *GAC*, should lie on the opposite side of *AB* to the cut triangle. Similarly to the requirements of Definition 2, we stipulate that the augmenting triangle *GAC* should satisfy the following conditions:

1. The area of *GAC* is equal to half the sum of the cut corner.
2. $\triangle GAC$ should be isosceles.
3. For some constant $c > 1$, $|EF|/|GF| \geqslant c$.

Let β be the half-angle at B, and γ the half-angle at G, and the other angles as marked in Fig. 9. Recall that we assume throughout that $\alpha < 1/3$.

Lemma 12. *The sequence of angles formed at an end edge by successive iterations of CCA1 tends to π at an exponential rate as the number of iterations tends to infinity.*

Proof. From Fig. 9, the two new angles formed at an edge are 2γ and $\xi + \delta$. For 2γ, we have $\tan\gamma/\tan\beta = |EF|/|GF| \geqslant c > 1 \Rightarrow \tan\gamma \geqslant c\tan\beta$. For $\xi + \delta$, we have $\xi = \beta + \pi/2$, so $\xi/2 = (\beta + \pi/2)/2$. Similarly to the proof of Lemma 5, we get $\tan(\xi/2) > 2\tan\beta$. Therefore, $\tan((\xi + \delta)/2) > 2\tan\beta$, so the angles tend to π. The proof of the exponential rate of convergence is similar to that of Lemma 5. □

Lemma 13. *The end edges of a convex polyline go to zero at an exponential rate as $n \to \infty$.*

Proof. Referring to Fig. 9, we have $|AF| < |AB|/2$. By Lemma 12, the angles at an edge go to π, so we can assume that $\gamma > \pi/4$, so $\sin\gamma > 1/\sqrt{2}$. The new end edge is *AG*, and its length is $|AG| = |AF|/\sin\gamma$. Therefore, $|AG| < |AF|\sqrt{2} < |AB|/\sqrt{2}$, and the result follows. □

We turn now to non-convex polygons and polylines. Here, we also require that augmenting triangles should be isosceles. It can be seen that if there are several consecutive inflection edges, then after just one iteration, the inflection edges will become isolated.

Some care is required with inflection edges. Consider Fig. 10, which shows an inflection edge, *AB*, the two cut corners, and the augmenting triangle. In order for all the angles to increase in size, we should have $\delta > 2\beta_2$. It is simple to construct an example in which half the cut area at *A* is much larger than half the cut area at *B*, and $2\beta_2$ is very large. In such a case, we could get $\epsilon > \gamma$, and this is equivalent to $\delta < 2\beta_2$, i.e., a new angle could be smaller than its neighboring old angle. Such cases can be handled simply by requiring that for some fixed constant $C < 1$, a_1 should be decreased so that $\epsilon \leqslant C\gamma$.

As mentioned earlier, if the two cut areas are equal, we do not construct an augmenting triangle. However, if the inflection edge is larger than its two neighbors, then in the limit, such edges will not necessarily tend to zero. We can solve this problem as shown in Fig. 11: we construct two identical isosceles triangles on opposite sides of the inflection edge *AB* so that two edges of the triangles are colinear and form a single new inflection edge $A'B'$. The heights of the triangles can be arbitrarily small to satisfy any design requirement. For example, $A'B'$ is at some angle to *AB*, and in the

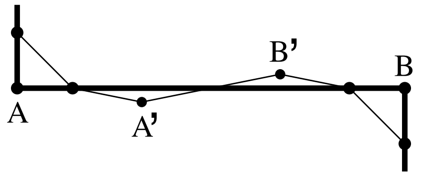

Fig. 11. Equal cut areas at a large inflection edge: two equal triangles are constructed so that inflection edges tend to zero with the number of iterations.

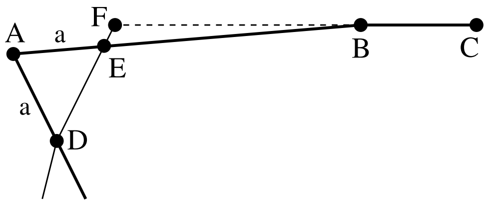

Fig. 12. An example showing that α cannot always depend only on the angle.

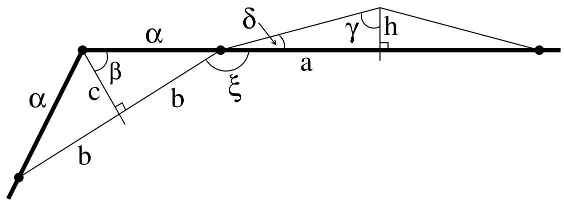

Fig. 13. A cut corner of a regular n-gon.

limit, the sum of such angles can be large. This sum can be bounded by choosing some angle β and requiring that at iteration n, the angle between AB and $A'B'$ is $\leqslant \beta/2^n$. This way, the limit angle will be bounded by β.

Also, similarly to the proof that end edges go to zero, so do inflection edges.

4. Angle-dependent choice of α

We use the term "α-profile" to denote any dependency of α on the angle of the cut corner (a constant α is also an α-profile). The choice of an α-profile is a design issue, to be determined by the user. The approach described below is motivated by striving to achieve, in some sense, curves that are close to arcs of a circle. This is done by considering regular polygons, because in the limit, regular polygons approach a circle. The α-profile that will be presented will satisfy the requirements of Definition 2. Even if the α-profile requires that $\alpha \to 0$ as the angle goes to zero, for any given polygon, the smallest angle will provide the ϵ value required in Definition 2.

In some extreme situations, it may not be possible to let α depend only on the angle, as shown in Fig. 12: the angle at B is very large and the extension of CB meets the cutting line through D and E at a point F so that $\triangle FEB$ is too small to contain the augmenting triangle. In such cases, one or both of the αs should be decreased until the augmenting triangle, as required by Definition 2, can be constructed.

A preliminary question is whether one iteration of CCA1 can change a regular polygon into another regular polygon. CCA1 can transform a square into a 12-sided regular polygon for a suitable choice of α, but it is easy to see that an equilateral triangle cannot produce a regular 9-gon.

Note first that every iteration of CCA1 triples the number of vertices. Consider a regular n-gon. Each internal angle is $\pi(n-2)/n$, and we denote by β one half of the internal angle, as shown in Fig. 13. From this, we have $\beta = \pi(n-2)/(2n)$. One iteration of CCA1 produces an m-gon with $m = 3n$, but it is not necessarily regular. Now, if the m-gon *were* regular, its internal half-angle would be $\gamma = \pi(3n-2)/(6n)$. From this, we derive the following relation:

$$\gamma = \frac{\beta + \pi}{3}. \tag{4}$$

Consider now the angle $\xi + \delta$ in Fig. 13: if the result of the iteration were a regular polygon, then we should also have $\xi + \delta = 2\gamma$. However, this condition is redundant since it follows immediately from Eq. (4). We assume that the edge length (of the original polygon) is 1. We choose α so that after one iteration, Eq. (4) holds. The condition for preserving the area is $ah = bc$. From Fig. 13, we have $b = \alpha \sin\beta$ and $c = \alpha\cos\beta$, so we get $ah = \alpha^2 \sin\beta\cos\beta$. We also have $h = a/\tan\gamma$, so $a^2/\tan\gamma = \alpha^2\sin\beta\cos\beta$. We now have $a = \frac{1}{2} - \alpha$, from which we get $(\frac{1}{2} - \alpha)^2/\tan\gamma = \alpha^2 \sin\beta\cos\beta$. From this we obtain a quadratic equation for α:

$$\bigl(4 - 2\tan\gamma\,\sin(2\beta)\bigr)\alpha^2 - 4\alpha + 1 = 0, \tag{5}$$

Table 1
Values used for creating the α-profile for CCA1.

Interior angle	0	40	60	90	120	150	180
No. of edges	n/a	n/a	3	4	6	12	∞
α	0	0.211325	0.24443	0.211325	0.194774	0.185845	0.1830127

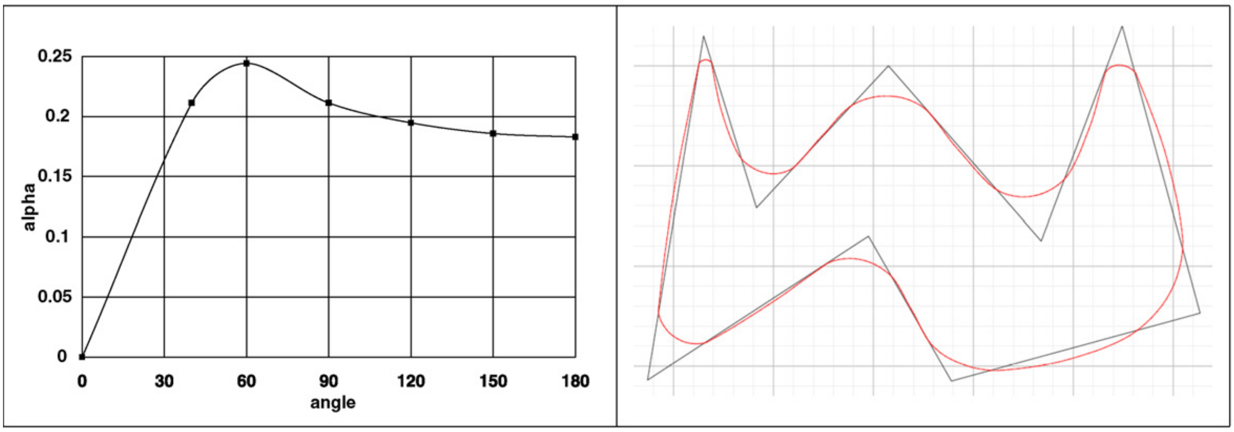

Fig. 14. A plot of the α-profile (left) and the result of CCA1 after three iterations (right).

where β and γ are related through Eq. (4). For every $\pi/6 \leqslant \beta < \pi$, the solution to Eq. (5) provides the α corresponding to the half-angle β.

When the original polygon is an equilateral triangle, then the solution to (5) with $\beta = 30°$ yields $\alpha = 0.23915$, and the resulting polygon is visibly not regular. For this particular case, we used a different approach. One iteration of CCA1 produces a 9-sided polygon, and we chose α so that the resulting 9-gon has an aspect ratio of 1. We forgo the technical details, but this approach yields the quadratic equation $(3\sqrt{3}-4)\alpha^2+(6-3\sqrt{3})\alpha+\sqrt{3}-2=0$, whose only positive solution is $\alpha \approx 0.24443$.

We wish to determine the limit of the solution to (5) as $\beta \to \pi/2$. Note first that from Eq. (4), we have $\beta = \pi/2 \Leftrightarrow \gamma = \pi/2$, so it makes no difference if $\beta \to \pi/2$ or $\gamma \to \pi/2$. Using elementary calculus techniques, we obtain the following equation when $\beta \to \pi/2$: $8\alpha^2 + 4\alpha - 1 = 0$. The only positive solution to this equation is $\alpha = (\sqrt{3}-1)/4 \approx 0.1830127$.

In creating our α-profile, we used the solutions of Eq. (5) for the internal angles of regular n-sided polygons, for $n = 4$, 6 and 12. For $n = 3$, we used the value 0.24443 obtained above. For π, we used the limit value 0.1830127. Also, for an angle of zero, we chose $\alpha = 0$; this is based on the reasoning that when a small α is used with a small angle, then the resulting curve matches the sharp corner more closely. We also added an additional control point at 40° in order to control the descent to 0 and ensure that the maximum is at 60°. Table 1 shows the angles that were used to determine certain values of α; other values of α were determined by cubic interpolation. Fig. 14 shows a plot of the α-profile and a sample result.

5. Additional properties and extensions of CCA

Local control. When the position of a vertex is changed, only its adjacent edges change. This affects the neighboring angles, and so the changes propagate to the augmenting structures on the four edges closest to the vertex, but no further. This provides the user with local control.

Anchor points. Given a polygon or polyline, one can determine one or more "anchor" points as vertices which remain fixed during the smoothing operations. This is done simply by considering each sequence of vertices between two anchor points as a polyline which is smoothed independently of other such sequences.

3D polylines. Consider the problem of smoothing a closed or open polyline in 3D. Let A, B, C, D be four consecutive vertices, with cut corners at B and C, as shown in Fig. 15. Let H_1 and H_2 be the two half planes determined by BC and containing AB and CD, respectively. Denote by B_1 and C_1 be the intersection of BC with the corner-cutting lines at B and C, respectively. If H_1 and H_2 are in one plane, then the problem is two-dimensional, and we deal with it in the usual way.

Otherwise, let P be the plane which bisects the angle formed by H_1 and H_2. The augmenting triangle is constructed on P with the base B_1C_1. This is done by considering the projections of the three edges and the cutting lines on P, and dealing with the 2D problem in the plane P. Note that if the problem is not two-dimensional, then the concept of an inflection edge is meaningless.

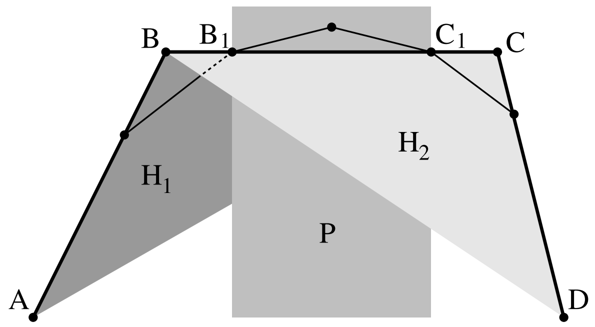

Fig. 15. CCA in 3D.

6. Conclusions

Corner cutting and augmentation (CCA) is a new subdivision method for smoothing polygons or polylines while maintaining the area. It is a local technique, meaning that area detracted from one side of the line is added to the other side in immediate proximity, and it allows local control. It is very simple and potentially useful for CAGD and artistic design. For some applications, it could replace the need for multiresolution curves for transmission purposes or LOD rendering, since the basic polygon can be smoothed as needed.

An implementation of CCA with triangular augmenting structures, called CCA1, is presented. Certain restrictions guarantee that the limit curve produced by CCA1 is G^1-continuous, preserves convexity, and contains no straight line segments. A design technique, called an α-profile, determines how closely the smooth result follows the original.

The CCA approach gives rise to some interesting and challenging topics for further research:

- Can the smooth curves obtained with CCA be characterized algebraically, e.g., as piecewise polynomial or rational splines?
- A related issue is the question of G^k-continuity of the limit curve for $k > 1$. We conjecture that CCA1 is G^2-continuous.
- The following inverse problem is also interesting: given a smooth curve, or a very tight polygonal approximation of such a curve, can one find a basis polygon of minimal size so that a few CCA iterations will produce a good approximation to the curve? A solution to this problem could provide an alternative to multiresolution curves.
- An open problem, raised by a reviewer, is the question of the convergence and smoothness of CCA applied to 3D polylines. Our bounding hull techniques rely on the planarity of the original polygon, so they do not apply directly to the 3D case.
- Extension to 3D polyhedra: as mentioned in Section 2, the bounding hull can be defined for convex polyhedra with triangular faces, but the general question of extending CCA to 3D remains open.

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Appendix A. Proof of Lemma 3

Let $a \in \mathrm{int}(A)$, and let R be a ray originating from a. Since A is bounded, R must intersect B at least once. We will show that this intersection is unique. Assume otherwise, and let p, q be two different points in $R \cap B$. Since $A \cup B$ is convex, the entire segment $pq \subset A \cup B$. Clearly, pq cannot intersect $\mathrm{int}(A)$, because R originates from an interior point. Hence, $pq \subset B$, and since $A \cup B$ is convex, the entire line defined by R is a line of support of $A \cup B$. This contradicts the fact that R contains an interior point (a). Therefore, B is a curve.

To show that B is a continuous curve, consider some fixed ray R_0 emanating from a. Denote by R_α the ray emanating from a and forming an angle α with R_0. For every α, denote $p(\alpha) = R_\alpha \cap B$. With this notation, B can be considered as a parametric curve: $B = \{p(\alpha) \mid 0 \leqslant \alpha < 2\pi\}$. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of angles such that $\alpha_n \xrightarrow[n\to\infty]{} \alpha_0$. We will prove that $p(\alpha_n) \xrightarrow[n\to\infty]{} p(\alpha_0)$.

B is bounded, so the set $\{p(\alpha_n)\}_{n=1}^{\infty}$ has at least one accumulation point p_1. Therefore, there exists a sequence of indices $n_1, n_2, \ldots$ such that $p(\alpha_{n_k}) \xrightarrow[k\to\infty]{} p_1$. We will show that $p_1 \in B$. Since $A \cup B$ is a closed set, $p_1 \in A \cup B = \mathrm{int}(A) \cup B$. If $p_1 \notin B$, then $p_1 \in \mathrm{int}(A)$, and so there is a disk D (of positive radius) around p_1 such that $D \subset \mathrm{int}(A)$. Hence, for some integer K, $p(\alpha_{n_k}) \in D$ for all $k > K$, and this contradicts the fact that $p(\alpha_n) \in B = \partial(A)$ for all n. Therefore, $p_1 \in B$. Furthermore, $p_1 \neq a$, because $a \in \mathrm{int}(A)$. Since $\alpha_n \xrightarrow[n\to\infty]{} \alpha_0$, the ray from a through p_1 forms an angle α_0 with R_0. Now, every ray from a

intersects B exactly once, so $p_1 = p(\alpha_0)$. This means that all the accumulation points of $\{p(\alpha_n)\}_{n=1}^{\infty}$ coincide with $p(\alpha_0)$, so $p(\alpha_n) \underset{n\to\infty}{\longrightarrow} p(\alpha_0)$. Therefore, B is a continuous curve.

References

de Boor, C., 1987. Cutting corners always works. Computer Aided Geometric Design 4, 125–131.

Dyn, N., 2006. Linear subdivision schemes for the refinement of geometric objects, in: Proc. International Congress of Mathematicians, Madrid, Spain, pp. 1201–1226.

Dyn, N., Levin, D., 2002. Subdivision schemes in geometric modeling. Acta Numerica 11, 73–144.

Elber, G., 2001. Multiresolution curve editing with linear constraints. Journal of Computing and Information Science in Engineering 1 (4), 347–355.

Farin, G., Hoschek, J., Kim, M.-S., 2002. Handbook of Computer Aided Geometric Design. Elsevier, Amsterdam.

Gordon, D., Sharon, O., Shavit, Z., 2007. Area preserving smoothing of polygons and polylines, in: Proc. Israel–Italy Bi-National Conf. on Shape Modeling & Reasoning for Industrial & Biomedical Applications. Lab. for CAD and Lifestyle Eng., Dept. of Mech. Eng., The Technion–Israel Inst. of Technology, Haifa, Israel, May 7–9, 2007, pp. 116–119.

Gregory, J.A., Qu, R., 1996. Nonuniform corner cutting. Computer Aided Geometric Design 13, 763–772.

Hahmann, S., Sauvage, B., Bonneau, G.-P., 2005. Area preserving deformation of multiresolution curves. Computer Aided Geometric Design 22, 349–367.

Henrikson, J., 1999. Completeness and total boundedness of the Hausdorff metric. The MIT Undergraduate Journal of Mathematics 1, 69–79. http://www-math.mit.edu/phase2/UJM/vol1/HAUSF.PDF.

Olver, P., Sapiro, G., Tannenbaum, A., 1997. Invariant geometric evolutions of surfaces and volumetric smoothing. SIAM J. on Applied Mathematics 57 (1), 176–194.

Paluszny, M., Prautzsch, H., Schäfer, M., 1997. A geometric look at corner cutting. Computer Aided Geometric Design 14, 421–447.

Sabin, M.A., Dodgson, N.A., 2005. A circle-preserving variant of the four-point subdivision scheme. In: Dæhlen, M., Mørken, K., Schumaker, L.L. (Eds.), Mathematical Methods for Curves and Surfaces. Tromsø, 2004. Nashboro Press, Brentwood, TN, USA, pp. 1–12.

Sapiro, G., Tannenbaum, A., 1995. Area and length preserving geometric invariant scale-spaces. IEEE Trans. on Pattern Analysis and Machine Intelligence 17 (1), 67–72.

Sauvage, B., Hahmann, S., Bonneau, G.-P., Elber, G., 2008. Detail preserving deformation of B-spline surfaces with volume constraint. Computer Aided Geometric Design 25, 678–696.