Lesson 4:

Logics - II

- Propositional Equivalence
- Tautologies
- Predicates
- Quantifiers
- Nested Quantifiers
- Negation of Quantifiers

p	¬p
T	F
F	T

Chapter 1.2 - 1.4

Propositional Equivalences

Mathematical proofs and deduction is based on exchanging one proposition with another proposition of the same truth value (an equivalent proposition).

3 types of propositions.

Tautology

Definition:

A compound proposition whose truth value is always **T** regardless of the truth value of its components is called a *Tautology* (טאוטולוגיה).

Example:

$$p \vee \neg p$$

Contradiction

Definition:

A compound proposition whose truth value is always **F** regardless of the truth value of its components is called a *Contradiction* (סתירה).

Example:

$$p \wedge \neg p$$

Contingency

Definition:

A compound proposition that is not a Tautology nor a Contradiction is a *Contingency* (אפשרות).

Example:

$$p \rightarrow p$$

Tautology vs Contradiction vs Contingency

		Tautology	Contradiction	Contingency
p	q	$p \vee \neg p$	$p \wedge \neg p$	$p \rightarrow \neg p$
T	T	T	F	F
T	F	T	F	T
F	T	T	F	T
F	F	T	F	T

Logical Equivalences

<u>Definition</u>:

The propositions p and q are *Logically Equivalent* (שקולים לוגית) if $p \leftrightarrow q$ is a Tautology.

$$\boxed{p \Leftrightarrow q}$$

<u>Example</u>:

$$p \vee q \Leftrightarrow q \vee p$$

$$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$$

Logical Equivalences

• Identity:	$p \vee F \Leftrightarrow p$	$p \wedge T \Leftrightarrow p$
• Domination:	$p \vee T \Leftrightarrow T$	$p \wedge F \Leftrightarrow F$
• Idempotent:	$p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$
• Double Complement:	$\neg\neg p \Leftrightarrow p$	
• Commutative:	$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	
• Associative:	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	

Logical Equivalences Vs Set Identities

• Identity:	$A \cup \varnothing = A = A \cap U$
• Domination:	$A \cup U = U$, $A \cap \varnothing = \varnothing$
• Idempotent:	$A \cup A = A = A \cap A$
• Double Complement:	$\overline{(\overline{A})} = A$
• Commutative:	$A \cup B = B \cup A$, $A \cap B = B \cap A$
• Associative:	$A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

Logical Equivalences

- Distributive: $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$

  $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$
- De Morgan $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
- Tautology $p \vee \neg p \Leftrightarrow \mathbf{T}$
- Contradiction $p \wedge \neg p \Leftrightarrow \mathbf{F}$

Logical Equivalences

- Generalized Distributive:

$$p_1 \vee p_2 \vee \ldots \vee p_n$$

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n$$

- Generalized De Morgan

$$\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Leftrightarrow (\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n)$$

$$\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n)$$

- Transitivity $p \Leftrightarrow q$ and $q \Leftrightarrow r$ then $p \Leftrightarrow r$

More in book p. 24.

Proving Logical Equivalences

Method 1: Truth Table

<u>Example</u>: Prove de Morgan's Law $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Proving Logical Equivalences

<u>Another Example</u>: Prove $(p \to q) \Leftrightarrow \neg p \vee q$

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Proving Logical Equivalences

Method 2: Use logical equivalences

A proposition in a compound proposition can be replaced by one that is logically equivalent (maintaining the truth value of the compound proposition).

if $p \Leftrightarrow q$ then

$$p \rightarrow r \quad \Leftrightarrow \quad q \rightarrow r$$

Proving Logical Equivalences

Method 2: Use logical equivalences

Example: Prove $\neg(p \vee (\neg p \wedge q)) \Leftrightarrow \neg p \wedge \neg q$

$\neg(p \vee (\neg p \wedge q))$	$\Leftrightarrow \neg p \wedge \neg(\neg p \wedge q)$	De Morgan II
	$\Leftrightarrow \neg p \wedge (\neg\neg p \vee \neg q)$	De Morgan I
	$\Leftrightarrow \neg p \wedge (p \vee \neg q)$	Double Negation
	$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive
	$\Leftrightarrow \mathbf{F} \vee (\neg p \wedge \neg q)$	Contradiction
	$\Leftrightarrow (\neg p \wedge \neg q) \vee \mathbf{F}$	Commutative
	$\Leftrightarrow \neg p \wedge \neg q$	Identity Law for **F**

Proving Logical Equivalences

Example: Prove $(p \wedge q) \rightarrow p$ is a tautology

Method 1:

p	q	$p \wedge q$	$p \wedge q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Proving Logical Equivalences

Example: Prove $(p \wedge q) \rightarrow p$ is a tautology

Method 2:

$(p \wedge q) \rightarrow p$	Given
$\Leftrightarrow \neg (p \wedge q) \vee p$	Equivalence $(p \rightarrow q) \Leftrightarrow \neg p \vee q$
$\Leftrightarrow (\neg p \vee \neg q) \vee p$	De Morgan
$\Leftrightarrow (\neg q \vee \neg p) \vee p$	Commutative
$\Leftrightarrow \neg q \vee (\neg p \vee p)$	Associative
$\Leftrightarrow \neg q \vee \mathbf{T}$	Tautology of $\neg p \vee p$
$\Leftrightarrow \mathbf{T}$	Domination

Predicates & Quantifiers

$x + 1 = 3$ is not a proposition

Can be turned into a proposition via 2 methods:

1) Predicates - assign specific values to unknowns

2) Quantifiers - define a set of values for the unknowns.

Chapter 1.3

Predicates

$x + 1 = 3$

Assign a value to the unknowns.

<u>Example</u>:

Assign the value 2 to x: $x + 1 = 3$ is then True.

Assign the value 0 to x: $x + 1 = 3$ is then False.

<u>Definition</u>: x is called a *Variable* (משתנה).

<u>Definition</u>: $x+1=3$ is called a *Predicate* (פרדיקט) or *Propositional Function* (פונקציה פסוקית).
A statement that has no value because no assignment has been made.

Predicates

Example:

$P(x) =$ " $x > 3$" P(x) is a predicate

$P(x)$ - no truth value

$P(2) \Leftrightarrow$ **F**

$P(4) \Leftrightarrow$ **T**

Example:

$P(x) =$ "$x (x + 1) = 1$"

$P(x)$ - no truth value

$P(2) \Leftrightarrow$ **F**

Multi Varible Predicates

Example:

$P(x,y) =$ " $x + y = 5$" $P(x,y)$ is a predicate

$P(2,3) \Leftrightarrow$ **T**

$P(3,0) \Leftrightarrow$ **F**

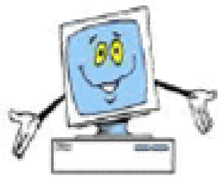
$P(4,4) \Leftrightarrow$ **F**

Examples:

$P(x,y,z) =$ "$x - y = z$"

$P(x_1,x_2,\ldots,x_n) =$ "$x_1+x_2+\ldots +x_n = 0$"

Predicates and the Computer



Example:

IF $x > 0$ THEN $x := x + 1$

$P(x) =$ "$x > 0$"   predicate

In a correct program an assignment has been made prior to this statement.

If $P(x) \Leftrightarrow$ **T** then command $x:=x+1$ is executed.

Quantifiers כמתים

$P(x,y) =$ " $x + y = y + x$" P(x,y) is a predicate

$P(2,3) \Leftrightarrow \mathbf{T}$ $P(100,0) \Leftrightarrow \mathbf{T}$ $P(-3,0.5) \Leftrightarrow \mathbf{T}$

$P(x) =$ " $x > 110 \wedge x < 130 \wedge$ x is prime"

$P(120) \Leftrightarrow \mathbf{F}$ $P(111) \Leftrightarrow \mathbf{F}$ $P(-125) \Leftrightarrow \mathbf{F}$

The Universal Quantifier

<u>Definition</u>: P(x) is a predicate then $\forall x P(x)$ ("for all x, P(x)" - "לכל x, p(x)") is a proposition whose value is **True** if $P(x) \Leftrightarrow$ **T** for all assignments x in the Universe of Discourse.

$\forall$ is called the *Universal Quantifier* (כמת ה-"לכל")

The Universal Quantifier

Examples:

$\forall x\ (x + 1 = 1 + x)$ Truth value is **T** for numbers

$\forall x\ (x * 0 = x)$ Truth value is **F** for Natural numbers

for Real Numbers ?

“All students in this class study Algebra.”

P(x) = “x studies Algebra”

$\forall x\ P(x)$ Universe x are students in this class.

OR

S(x) = “x is in this class”

$\forall x\ (S(x) \rightarrow P(x))$ Universe x are students in general.

The Universal Quantifier

Examples:

$P(x)$ = "$x < 2$"
$\forall x\, P(x)$ x are real numbers.

$\forall x\, P(x)$ is False. Proof: $P(3) \Leftrightarrow$ **F**

$P(x)$ = "$x + 3 < 20$"
$\forall x\, P(x)$ x are natural numbers smaller than 5.

$\forall x\, P(x)$ is True.
Proof: $P(1) \Leftrightarrow$ **T,** $P(2) \Leftrightarrow$ **T,** $P(3) \Leftrightarrow$ **T,** $P(4) \Leftrightarrow$ **T**

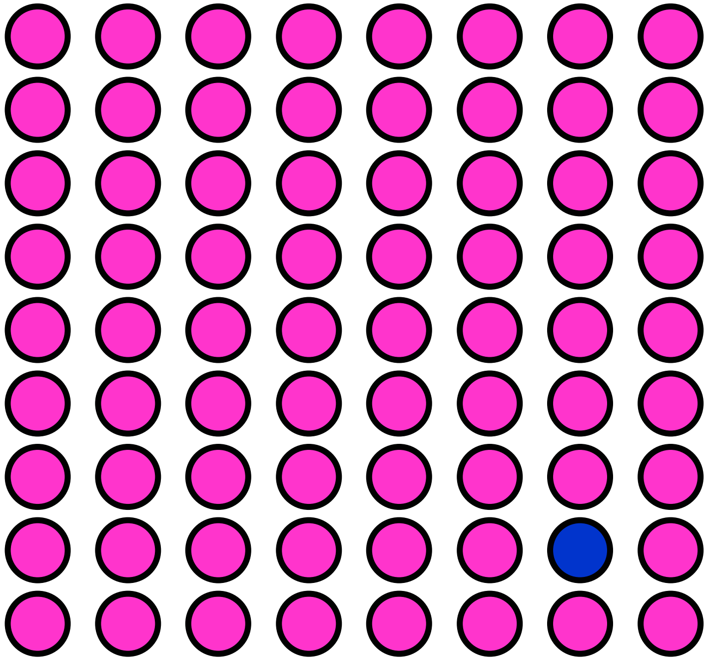
Note: If the domain of x is finite then

$$\forall x\, P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

The Universal Quantifier

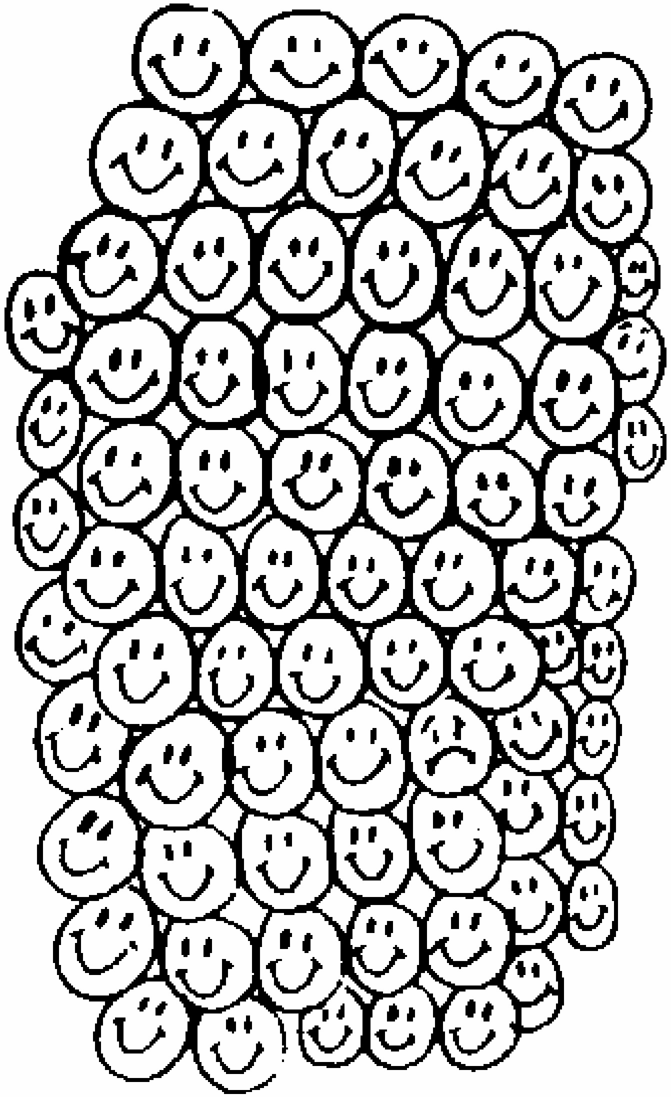
To prove $\forall x\, P(x) \Leftrightarrow$ **F** 1 example of x is enough.

To prove $\forall x\, P(x) \Leftrightarrow$ **T** must test all possible x.



$P(x)$ = “Face x is smiling”.   x are faces on this page

$\forall x\ P(x) \Leftrightarrow$?



The Existential Quantifier

Definition: P(x) is a predicate then $\exists x P(x)$ ("there exists x, such that P(x)" - "קיים x כך ש- p(x)") is a proposition whose value is **True** if there exists at least 1 x for which $P(x) \Leftrightarrow$ **T**.

$\exists$ is called the *Existential Quantifier* ("כמת ה-"קיים)

The Existential Quantifier

<u>Examples</u>:

$\exists x\ (x > 3)$ Truth value is T for numbers

$\exists x\ (x + 1 = x)$ Truth value is F for Natural numbers

for Real Numbers ?

"There is a person in this room wearing glasses.

$P(x)$ = "x wears glasses"

$\exists x\ P(x)$ Universe x are people in this room.

OR

$S(x)$ = "x is a person in this room"

$\exists x\ (S(x) \rightarrow P(x))$ Universe x are people in general.

The Existential Quantifier

<u>Examples</u>:

$P(x)$ = "$x < 2$"
$\exists x\, P(x)$ x are real numbers.

$\exists x\, P(x)$ is True. Proof: $P(-1) \Leftrightarrow$ **T**

$P(x)$ = "$x^2 > 10$"
$\exists x\, P(x)$ x are natural numbers smaller than 4.

$\exists x\, P(x)$ is False.
Proof: $P(1) \Leftrightarrow$ **F** $P(2) \Leftrightarrow$ **F** $P(3) \Leftrightarrow$ **F**

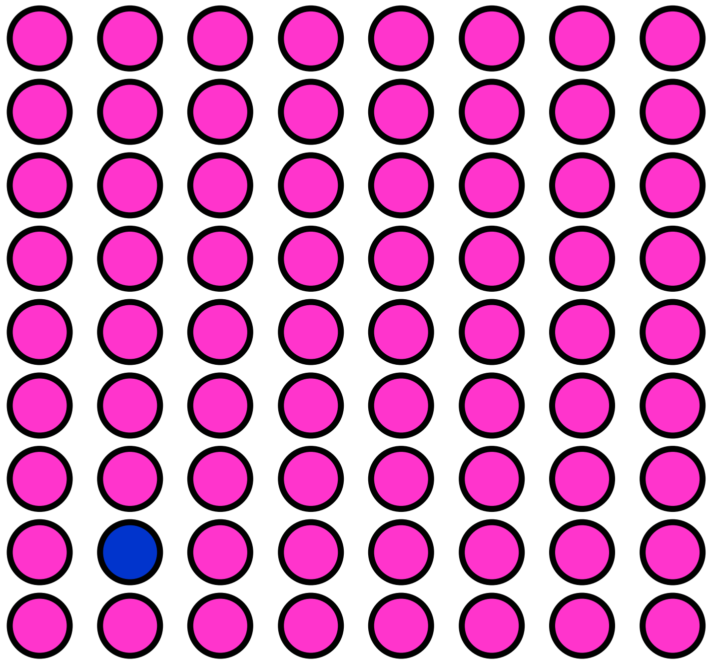
Note: If the domain of x is finite then

$$\exists x\, P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

The Existential Quantifier

To prove $\exists x\ P(x) \Leftrightarrow$ **T** 1 example of x is enough.

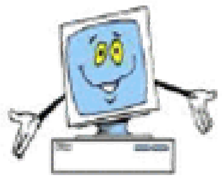
To prove $\exists x\ P(x) \Leftrightarrow$ **F** must test all possible x.



P(x) = “Face x on this Page”.   x are faces

$\exists x\, P(x) \Leftrightarrow$?

Quantifiers and the Computer



Write a program that outputs the truth value of a quantified proposition.

$\exists x\ P(x)$

% Loop over all values of x.
For every x evaluate P(x).
    If P(x) $\Leftrightarrow$ **T**
        return ($\exists x\ P(x) \Leftrightarrow$ **T**);
endfor
return ($\exists x\ P(x) \Leftrightarrow$ **F**);

$\forall x\ P(x)$

% Loop over all values of x.
For every x evaluate P(x).
    If P(x) $\Leftrightarrow$ **F**
        return ($\forall x\ P(x) \Leftrightarrow$ **F**);
endfor
return ($\forall x\ P(x) \Leftrightarrow$ **T**);

Negating Quantifiers

"Every student in this room studies Discrete Math"

$P(x)$ = "x studies DM"   Universe x is a student in this room.

$\forall x P(x)$ = "Every student in this room studies Discrete Math"

Negate the proposition:

"NOT every student in this room studies Discrete Math"

$$\neg \forall x P(x)$$

"There is a student in this room that does NOT study DM"

$$\exists x \neg P(x)$$

$$\boxed{\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)}$$

Negating Quantifiers

"There is a student in this room studying French"

$P(x)$ = "x studies French"   x is a student in this room.

$\exists x P(x)$ = "There is a student in this room studying French"

Negate the proposition:

"There is NOT a student in this room studying French"

$$\neg \exists x P(x)$$

"All students in this room do NOT study French"

$$\forall x \neg P(x)$$

$$\boxed{\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)}$$

Negating Quantifiers

Examples:

$$p = \forall x\,(x^2 > x)$$

$$\neg p = \neg\forall x\,(x^2 > x)$$
$$= \exists x\,\neg(x^2 > x)$$
$$= \exists x\,(x^2 \le x)$$

$$q = \exists x\,(x = x^2)$$

$$\neg q = \neg\,\exists x\,(x = x^2)$$
$$= \forall x\,\neg(x = x^2)$$
$$= \forall x\,(x \ne x^2)$$

Negating Quantifiers

$\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$ $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$

Can be deduced from De Morgan's Laws:

$$\begin{aligned}\neg\forall x P(x) &= \neg(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)) \\ &= (\neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n)) \\ &= \exists x \neg P(x)\end{aligned}$$

$$\begin{aligned}\neg\exists x P(x) &= \neg(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)) \\ &= (\neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n)) \\ &= \forall x \neg P(x)\end{aligned}$$

Translating Spoken Language to Logical Expressions (הצרנה לוגית)

"Every CS student studies Discrete Math"

$\Updownarrow$

"For every CS student, that student studies Discrete Math"

$\Updownarrow$

"For every CS student x, x studies Discrete Math"

C(x) = "x studies Discrete Math

$\forall x\, C(x)$ The universe: x is a CS student

OR

S(x) = "x is a CS student

$\forall x\, (S(x) \rightarrow C(x))$ The universe: x is a person

Translating Spoken Language to Logical Expressions (הצרנה לוגית)

"Every even number is divisible by 2.

$\Updownarrow$

"For every even number n, n is divisible by 2."

$\Updownarrow$

C(n) = "n is even"
D(n) = "n is divisible by 2"

$\forall n\ (C(n) \rightarrow D(n))$ n is a whole number

What about: "**Only** even numbers are divisible by 2" ?

Translating Spoken Language to Logical Expressions (הצרנה לוגית)

"There are some even numbers that are divisible by 4."

$\Updownarrow$

"There exists an even number n, s.t. n is divisible by 4."

$\Updownarrow$

C(n) = "n is even"
D(n) = "n is divisible by 4"

$\exists n\ (C(n) \wedge D(n))$ n is a whole number

Translating Spoken Language to Logical Expressions (הצרנה לוגית)

Charles L. Dodgeson (Lewis Carroll) (1832-1898)

"Begin at the beginning and go on till you come to the end; then stop."

Lewis Carroll, *Alice in Wonderland*

Logical Expressions - Lewis Carroll

premise
- "All lions are fierce"
- "Some lions do not drink coffee"

conclusion
- "Some fierce creatures do not drink coffee"

$P(x)$ = "x is a lion"

$Q(x)$ = "x is fierce"

$R(x)$ = "x drinks coffee"

$$\forall x\ (P(x) \rightarrow Q(x))$$

$$\exists x\ (P(x) \wedge \neg R(x))$$

$$\exists x\ (Q(x) \wedge \neg R(x))$$

Binding Variables

Definition:

When a quantifier is used with a variable we say the variable is *bound* (משתנה קשור). A variable that is not bound by a quantifier is *free* (משתנה חופשי).

Definition:

The part of the proposition to which the quantifier is applied is called the *scope* (תחום הגדרה) of the quantifier.

∴ A variable is free if it is outside the scope of all quantifiers using the variable.

Binding Variables

<u>Examples:</u>

$\forall x P(x,y)$ x is bound. y is free.

$$\underbrace{\exists x\ P(x)}_{\text{scope of } \exists x} \wedge \underbrace{\forall y\ (Q(y) \rightarrow R(y))}_{\text{scope of } \forall y}$$

All variables are bound.

Variables of different scope can be the same letter:

$$\exists x\ P(x) \wedge \exists x\ (Q(x) \rightarrow R(x))$$

Nested Quantifiers

<u>Examples:</u>

$\forall x\ \forall y\ (x + y = y + x)$ The Commutative Law

$\forall x\ \exists y\ (x + y = 0)$ The Inverse Law

$\forall x\ \forall y\ \forall z\ (\ x+(y+z)=(x+y)+z\ )$ The Associative Law

Chapter 1.4

Translating Spoken Language (הצרנה לוגית)

$\forall x\, (C(x) \vee \exists y\, (C(y) \wedge F(x,y)))$

$C(x)$ = "x has a computer"

$F(x,y)$ = "x and y are friends"

Universe: x,y students

"Every student has a computer or has a friend with a computer."

Can $\exists y$ be moved to front?

Translating Spoken Language (הצרנה לוגית)

"Everyone has exactly one best friend"

B(x,y) = "x and y are best friends"   Universe: x,y people

$$\forall x\, \exists y\, (B(x,y) \wedge \forall z\, (z \neq y \rightarrow \neg B(x,z))$$

$$\overset{?}{\Leftrightarrow}$$

$$\forall x\, \exists y\, \forall z\, (B(x,y) \wedge (z \neq y \rightarrow \neg B(x,z))$$

$$\overset{?}{\Leftrightarrow}$$

$$\forall x\, \exists y\, (B(x,y) \wedge \neg \exists z(\, z \neq y \wedge B(x,z))$$

Can someone be best friend of self?

Order of Quantifiers

$$P(x,y) = \text{“}x + y = 0\text{”}$$

I) $\exists y \, \forall x \, P(x,y)$ There exists a y for which every x, p(x,y).

$\Leftrightarrow$ F

II) $\forall x \, \exists y \, P(x,y)$ For every x there exists a y for which p(x,y).

$\Leftrightarrow$ T

<u>Note:</u>

$$\exists y \, \forall x \, P(x,y) \rightarrow \forall x \, \exists y \, P(x,y)$$

$$\exists y \, \forall x \, P(x,y) \not\leftarrow \forall x \, \exists y \, P(x,y)$$

Negating Quantifiers

For every x there exists a y for which "x+y =0"

$P(x,y) = \text{"}x + y = 0\text{"}$

$\forall x\ \exists y\ P(x,y)$

It is **not true** that for every x there exists a y for which "x+y =0"

$\neg\forall x\ \exists y\ P(x,y) \Leftrightarrow$

$\exists x\ \neg\exists y\ P(x,y) \Leftrightarrow$

$\exists x\ \forall y\ \neg P(x,y) \Leftrightarrow$

There exists an x for which "$x+y \neq 0$" for all y.

Nested Quantifiers - Summary

Statement	Negation
$\forall x\ \forall y\ P(x,y) \Leftrightarrow$ $\forall y\ \forall x\ P(x,y)$	$\exists x\ \exists y\ \neg P(x,y) \Leftrightarrow$ $\exists y\ \exists x\ \neg P(x,y)$
$\forall x\ \exists y\ P(x,y)$	$\exists x\ \forall y\ \neg P(x,y)$
$\exists x\ \forall y\ P(x,y)$	$\forall x\ \exists y\ \neg P(x,y)$
$\exists x\ \exists y\ P(x,y) \Leftrightarrow$ $\exists y\ \exists x\ P(x,y)$	$\forall x\ \forall y\ \neg P(x,y) \Leftrightarrow$ $\forall y\ \forall x\ \neg P(x,y)$