Lesson 10:

Relations

- Definitions
- Properties
- Combining Relations
- Relation Closure

R	a	b	c
0	1	1	0
1	1	0	0
2	0	0	0
3	0	1	0

Chapter 7

Relations

Relations are structures that express a connection between 2 or more discrete objects.

- Family Name - Phone number
- Employee - Salary
- Number - Prime factors
- Programs - input parameters

Binary Relations - relate pairs of discrete objects.

Relations

Functions relate between pairs of objects:

$$f : A \rightarrow B$$

Every $a \in A$ is assigned an element $b \in B$.

Cartesian Product relates between pairs of objects:

$$A \times B$$

Every $a \in A$ is related to every $b \in B$.

Binary Relations - relate pairs of elements in A and in B without any constraints.

Relations

<u>Definition</u>:

An *ordered n-tuple* (n-יה סדורה) $(a_1,a_2,\ldots,a_n)$ is an ordered collection in which a_1 is first, a_2 is second, ... and a_n as it's n-th element.

2 n-tuples are equal iff they have equal corresponding elements.

$$(a_1,a_2,\ldots,a_n) = (b_1,b_2,\ldots,b_n) \quad \text{iff} \quad a_i=b_i \ \text{ for } i=1,\ldots,n$$

An *ordered pair* (זוג סדור) is an n-tuple with n = 2.

Relations

<u>Definition</u>:

A *Binary Relation* (יחס (רלציה) בינארית) from set A to set B is a subset of A X B.

$R \subseteq A \times B$ is a relation fro A to B then :

$(a,b) \in R$ a R b   "a is related to b by R"

"a נמצא ביחס R ל- b"

$(a,b) \notin R$ a $\not R$ b   "a is not related to b by R"

Relations

<u>Example:</u>

$A=\{0, 1, 2\}$ $B=\{a, b\}$

$R=\{(0,a), (0, b), (1, a), (2, b)\}$

R is a relation from A to B because $R \subseteq A \times B$.

$0 \, R \, a$	$(1, a) \in R$
$2 \not{R} \, a$	$(1, b) \notin R$

Relations

Example:

A = Cities in the world

B = Countries in the World

R is the relation of all (a,b) in which a $\in$ A is the capital of b $\in$ B.

$$\left.\begin{array}{l}\{\text{Jerusalem, Israel}\} \\ \{\text{Paris, France}\}\end{array}\right\} \in R$$

$$\left.\begin{array}{l}\{\text{Jerusalem, USA}\} \\ \{\text{TLV, Japan}\}\end{array}\right\} \notin R$$

Relations

Example:

R is a relation on the set of integers.

$R \subseteq \mathcal{Z} \times \mathcal{Z}$

$R = \{ (a,b) \mid a \leq b\}$

$(2,100) \in R$

$(-33, 10{,}002) \in R$

$(20,10) \notin R$

$(0.1 , 2) \notin R$

Representing Relations

$R \subseteq A \text{ X } B.$

I. List

A={0, 1, 2, 3}   B={a, b, c}

R={(0,a), (0, b), (1, a), (3, b)}

II. Array of size 2 x 1

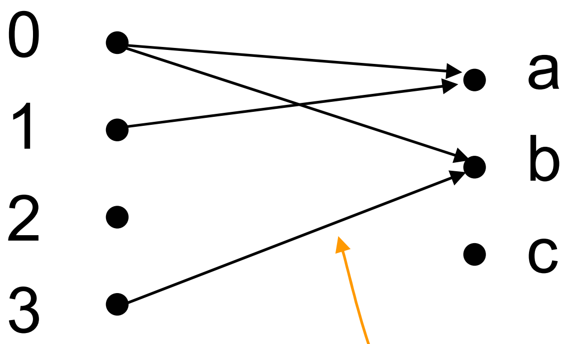
A={0, 1, 2, 3}   B={a, b, c}

$$R = \begin{pmatrix} 0 & 0 & 1 & 3 \\ a & b & a & b \end{pmatrix}$$

Representing Relations

III. Graph

$A=\{0, 1, 2, 3\}$ $B=\{a, b, c\}$ $R \subseteq A \times B$.



$R=\{(0, a), (0, b), (1, a), (3, b)\}$

Representing Relations

III. Table / Binary Matrix

A={0, 1, 2, 3}   B={a, b, c}   R $\subseteq$ A X B.

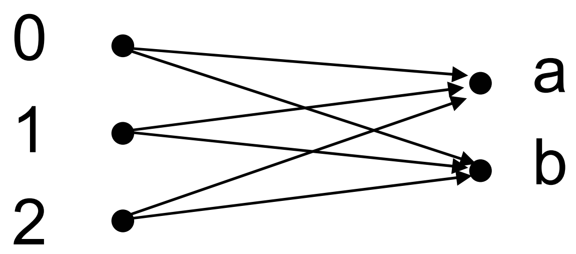
R	a	b	c
0	1	1	0
1	1	0	0
2	0	0	0
3	0	1	0

$$\longleftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

R={(0, a), (0, b), (1, a), (3, b)}

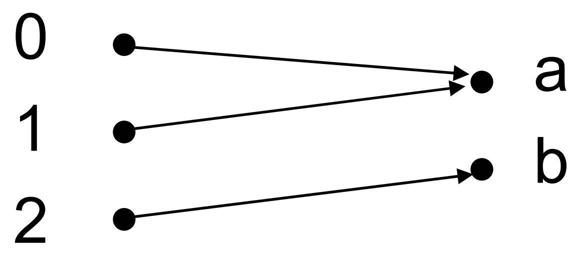
Relations

The Cartesian Product A X B is a relation:



AXB	a	b
0	1	1
1	1	1
2	1	1

Function f : A $\rightarrow$ B is a relation



f	a	b
0	1	0
1	1	0
2	0	1

Relations

Definition:

A relation **on** the set A is a relation from A to A .

Example:

$A = \{1, 2, 3, 4\}$

$R = \{ (a,b) \mid a \mid b \}$

$R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) \}$

$R \subseteq \mathcal{Z} \times \mathcal{Z}$

$R = \{ (a, b) \mid a < b \}$ infinite set

Relations

<u>Example:</u>

How many relations can be defined on a set A of size n ?

Answer:

A relation is a subset of A X A.

$| A \times A | = n^2$

Thus there are 2^{n^2} possible relations.

Relations

Definition:

The *Identity Relation* (יחס היחידה) over the set A,

$I_A \subseteq A \times A$ is defined as:

$$I_A = \{(a, a) \mid a \in A\}$$

Example:

$A=\{0, 1\}$

$I_A = \{(0, 0), (1, 1)\}$

Relations

Definition:

The *Inverse Relation* (יחס הופכי) of $R \subseteq A \times B$ is the relation $R^{-1} \subseteq B \times A$ such that:

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

Examples:

$A=\{0, 1, 2, 3\}$ $B=\{a, b, c\}$

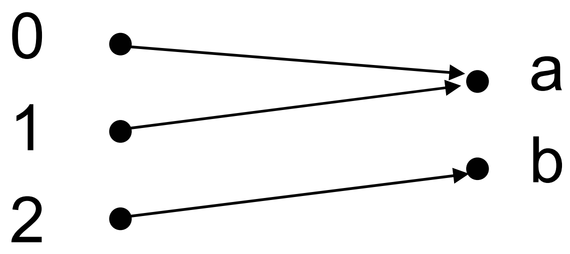
$R = \{(0, a), (0, b), (1, a), (3, b)\}$

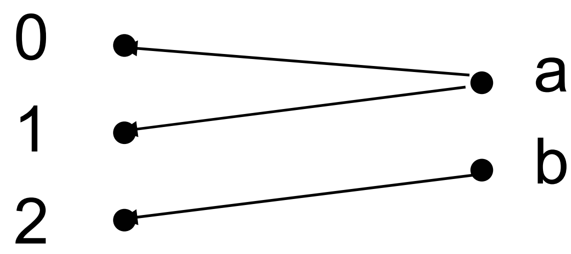
$R^{-1} = \{(a, 0), (b, 0), (a, 1), (b, 3)\}$

Relations

<u>Examples:</u>

$$R = \begin{pmatrix} 1 & 1 & 2 & 3 \\ a & c & c & d \end{pmatrix} \qquad R^{-1} = \begin{pmatrix} a & c & c & d \\ 1 & 1 & 2 & 3 \end{pmatrix}$$





$$\begin{array}{c|cc} & a & b \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{array} \qquad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline a & 1 & 1 & 0 \\ b & 0 & 0 & 1 \end{array}$$

Properties of Relations - Reflexivity

<u>Definition</u>:

A relation R on a set A is *reflexive* ((רפלקסיבי (חוזר)

iff $\forall a \in A, (a,a) \in R$.

A relation R is reflexive iff $I_A \subseteq R$.

<u>Examples:</u>

$R = \{(x, y) \mid x, y \in Z \text{ and } x \geq y\}$ reflexive

R is the relation over people s.t. reflexive
$(x, y) \in R$ if x and y have the same parents.

$R = \{(1, 1), (1, 2), (2, 1)\}$ not reflexive

Properties of Relations - Symmetry

Definition:

A relation R on a set A is *symmetric* (סימטרי)

if $\forall a,b \in A,\ (a,b) \in R \rightarrow (b,a) \in R$

Definition:

A relation R on a set A is *antisymmetric* (אנטי-סימטרי)

if $\forall a,b \in A,\ (a,b) \in R \wedge (b,a) \in R \rightarrow a = b$

A relation R is symmetric iff $R = R^{-1}$

A relation R is antisymmetric iff $R \cap R^{-1} \subseteq I_A$

Properties of Relations - Symmetry

Examples:

$R = \{(1, 1), (1, 2), (2, 1)\}$ symmetric

$R = \{(x, y) \mid x, y \in Z \text{ and } x*y = 6\}$ symmetric

$R = \{(1, 1), (1, 2), (2, 3), (4, 2)\}$ antisymmetric

R is the relation over students s.t.
$(x, y) \in R$ if x and y take the same course. symmetric

R is the relation over students s.t.
$(x, y) \in R$ if x's grade is higher than y. antisymmetric

Properties of Relations - Symmetry

The terms symmetric and antisymmetric are not opposites.

$R = \{(1, 2), (2, 1), (1, 3)\}$

NOT symmetric
NOT antisymmetric

$R = \{(1, 1), (2, 2), (3, 3)\}$

symmetric
& antisymmetric

Properties of Relations - Transitivity

Definition:

A relation R on a set A is *transitive* (טרנזיטיבי (חוזר))

iff $\forall a,b,c \in A,\ \ (a,b)\in R \wedge (b, c)\in R \rightarrow (a, c)\in R$.

Examples:

$R = \{(1, 1), (3, 1), (1, 2)\}$ — not transitive

R is the relation over students s.t.
$(x, y)\in R$ if x's grade is higher than y. — transitive

$R = I_A$ — transitive

Properties of Relations

$R \subseteq \mathcal{Z} \times \mathcal{Z}$	Reflexive	Symmetric	Anti-symetric	Transitive
$R_1 = \{(x, y) \mid x \leq y\}$	√	X	√	√
$R_2 = \{(x, y) \mid x > y\}$	X	X	√	√
$R_3 = \{(x, y) \mid x = y\}$	√	√	√	√
$R_4 = \{(x, y) \mid x = y \text{ or } x = -y\}$	√	√	X	√
$R_5 = \{(x, y) \mid x = y + 1\}$	X	X	√	X
$R_6 = \{(x, y) \mid x + y \leq 3\}$	X	√	X	X
$R_7 = \{(x, y) \mid x \mid y \}$	√	X	√	√
$R_8 = \{(x, y) \mid x+y=6 \wedge x>0 \}$	X	X	X	X

Properties of Relations

Reflexive

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & \multicolumn{3}{c}{\left(\begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix}\right)} \\ b & & & \\ c & & & \end{array}$$

M is diagonal

Symmetric

$$\begin{array}{cc} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 1 & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \end{array}$$

$M = M^T$

AntiSymmetric

$$\begin{array}{cc} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 1 & 1 & \\ 0 & & 0 \\ & 1 & \end{pmatrix} \end{array}$$

M is anti-symmetric elements may or may not be on the diagonal

Properties of Relations

Question:

How many reflexive relations can be defined on a set A of n elements?

Answer:

There are 2^{n^2} relations that can be defined on A (the number of subsets of A X A).

Reflexive relations contain (a,a) for all $a \in A$.

The rest of the n^2-n = n(n-2) elements may or may not be in the relation thus the number of reflective relations on A is:

$$2^{n(n-1)}$$

Properties of Relations

Example: Reflexive relations

$n = 2 \qquad 2^{2*1/2} = 4$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$$

Properties of Relations

Question:

How many symmetric relations can be defined on a set A of n elements?

Answer:

Consider the matrix representation of the relation.

A symmetric matrix is defined by the upper triangle entries + the diagonal entries. (the symmetric entries are then automatically defined).

The number of entries in the upper triangle of the nxn Matrix is:

$$1+2+3+...+n=\frac{n(n+1)}{2}$$

Thus the number of symmetric relations is $2^{\frac{n(n+1)}{2}}$

Properties of Relations

Example: Symmetric relations

$n = 2 \qquad 2^{2*3/2} = 8$

1	1
1	1

0	1
1	1

1	1
1	0

0	1
1	0

1	0
0	1

0	0
0	1

1	0
0	0

0	0
0	0

Properties of Relations

Question:

How many anti-symmetric relations can be defined on a set A of n elements?

Answer:

Consider the matrix representation of the relation.

In an antisymmetric matrix the diagonal may or may not be 1: 2^n possibilities for the diagonal.

The upper triangle has $1+2+3+\ldots+(n-1)=\frac{n(n-1)}{2}$ entries.

Each entry 3 possibilities: Mij = 1 Mji = 0

Mij = 0 Mji = 1

Mij = 0 Mji = 0

Thus the number of anti-symmetric relations is $2^n 3^{\frac{n(n-1)}{2}}$

Properties of Relations

<u>Example:</u> Anti-symmetric relations

$n = 2$ $\quad\quad$ $2^2 * 3^{1*2/2} = 4 * 3 = 12$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$$

Combining Relations

Theorem: Let R_1 and R_2 be relations from A to B then the following relations are well defined:

$R1 \cup R2$, $R1 \cap R2$, $R1 \oplus R2$, $R1 - R2$, $R2 - R1$

Combining Relations

Using Matrix representation:

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$R_1 \cup R_2 \leftrightarrow M_1 \vee M_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$R_1 \cap R_2 \leftrightarrow M_1 \wedge M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1 \oplus R_2 \leftrightarrow M_1 \oplus M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Combining Relations

Using Matrix representation:

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$R_1 - R_2 \leftrightarrow M_1 - M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_2 - R_1 \leftrightarrow M_2 - M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R_1^{-1} \leftrightarrow 1 - M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Relation Composition

Definition:

Let R be a relation from A to B

and S be a relation from B to C

then the *Composition* (הרכבה) of R and S is the relation

$S \circ R$ of ordered pairs (a,c)

$$S \circ R = \{(a,c) \mid a \in A,\ c \in C \text{ and } \exists b\ (a,b) \in R \wedge (b,c) \in S\}$$

Relation Composition

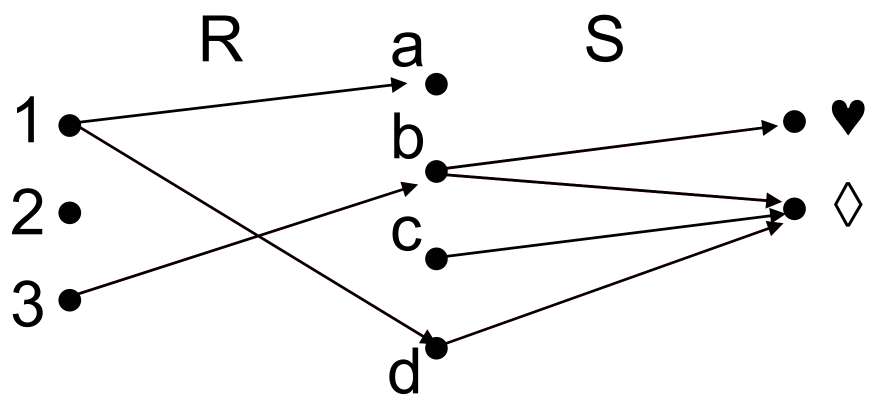
<u>Example:</u>

$A = \{1,2,3\}$ $B = \{a,b,c,d\}$ $C = \{♥,◊\}$

$R \subseteq A \times B$ $R = \{ (1,a), (1,d), (3,b) \}$

$S \subseteq B \times C$ $S = \{ (b,♥), (b,◊), (c,◊), (d,◊) \}$

$S \circ R \subseteq A \times C$ $S \circ R = \{(1,◊), (3,♥), (3,◊) \}$



Relation Composition

<u>Example:</u>

A = Students in DM   B = Grades (0..100) C = Pass/nopass

$R \subseteq A \times B$ R = { (a,b) | b is the final DM grade of a}

$S \subseteq B \times C$ S = { (b,c) | c=Pass for b<51 and
c=NoPass for $b \geq 51$}

$S \circ R \subseteq A \times C$ $S \circ R$ = {(a,c) | Student a passed the DM course }

Relation Composition

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{pmatrix} \end{array} \qquad M_S = \begin{array}{c|c} & \begin{array}{cc} \heartsuit & \Diamond \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} \quad & \quad \\ & \\ & \\ & \end{pmatrix} \end{array}$$

$$M_{S \circ R} = \begin{array}{c|c} & \begin{array}{cc} \heartsuit & \Diamond \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} \quad & \quad \\ & \\ & \end{pmatrix} \end{array}$$

$$M_{S \circ R} = M_R \odot M_S$$

Boolean Matrix Product

$$M_{S \circ R}(i,j) = \sum_k M_R(i,k)\, M_S(k,j)$$ Regular Matrix Product

$$M_{S \circ R}(i,j) = \bigvee_k [\, M_R(i,k) \wedge M_S(k,j)]$$ Boolean Matrix Product

Relation Composition

$$M_R = \begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{pmatrix} \end{array} \qquad M_S = \begin{array}{cc} & \begin{matrix} \heartsuit & \Diamond \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} \; & \; \\ \; & \; \\ \; & \; \\ \; & \; \end{pmatrix} \end{array}$$

$$M_{S \circ R} = \begin{array}{cc} & \begin{matrix} \heartsuit & \Diamond \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} \; & \; \\ \; & \; \\ \; & \; \end{pmatrix} \end{array}$$

$$M_{S \circ R} = M_R \odot M_S$$

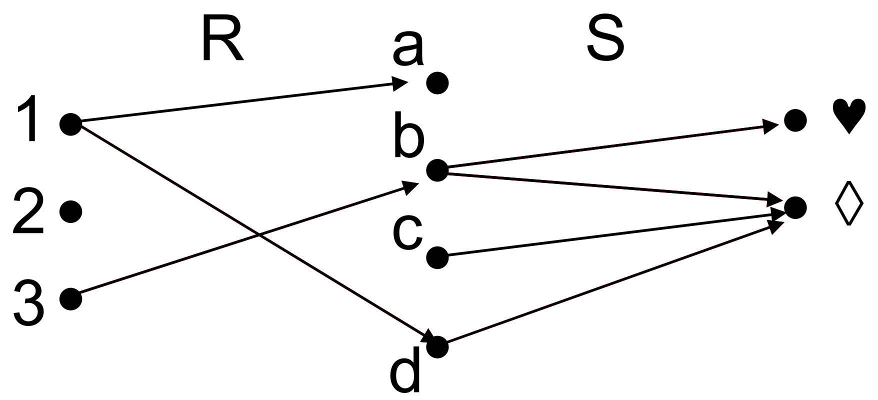
Boolean Matrix Product

$$M_{S \circ R}(a,c) = \sum_b M_R(a,b) * M_S(b,c)$$ Regular Matrix Product

$$M_{S \circ R}(a,c) = \bigvee_b [\, M_R(a,b) \wedge M_S(b,c)]$$ Boolean Matrix Product

Relation Composition

Example:



$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cc} \heartsuit & \lozenge \end{array} \\ \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$M_{S \circ R} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Relation Composition

<u>Definition</u>:

Let R be a relation on set A.

The *Power n* (n - חזקת) of R ($n \in \mathcal{Z}^{+}$) is defined recursively:

$$R^1 = R$$

$$R^n = R^{n-1} \circ R$$

$R^1 = R$, $R^2 = R \circ R$, $R^3 = (R \circ R) \circ R$ .....

Relation Composition

Example:

$R = \{ (1,1), (2,1), (3,2), (4,3) \}$

$R^2 = \{ (1,1), (2,1), (3,1), (4,2) \}$

$R^3 = \{ (1,1), (2,1), (3,1), (4,1) \}$

$R^4 = \{ (1,1), (2,1), (3,1), (4,1) \}$

$R^5 = R^6 = R^7 = ...$ for all $n \geq 3$

Relation Composition

<u>Example:</u>

R = { (1,1), (2,1), (3,2), (4,3) }

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_{R^2} = M_R \circ M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M_{R^3} = M_{R^2} \circ M_R = M_R \circ M_{R^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Relation Composition

Example:

$$M_{R^4} = M_{R^3} \circ M_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{R^5} = M_{R^6} = M_{R^7} = \ldots.$$

Combining Relations

Theorem: Relation R on set A is Transitive IFF $R^n \subseteq R$.

Proof:

1) prove $R^n \subseteq R \rightarrow R$ transitive :

$R^n \subseteq R$ so $R^2 \subseteq R$.

If $(a,b) \in R$ and $(b,c) \in R$ then by definition of composition $(a,c) \in R^2$ however $R^2 \subseteq R$ thus $(a,c) \in R$.

Transitivity is proven.

Combining Relations

Proof cont.:

2) prove R transitive $\rightarrow R^n \subseteq R$: Proof by induction.

$R^1 = R \subseteq R$.

Assume $R^n \subseteq R$ prove $R^{n+1} \subseteq R$

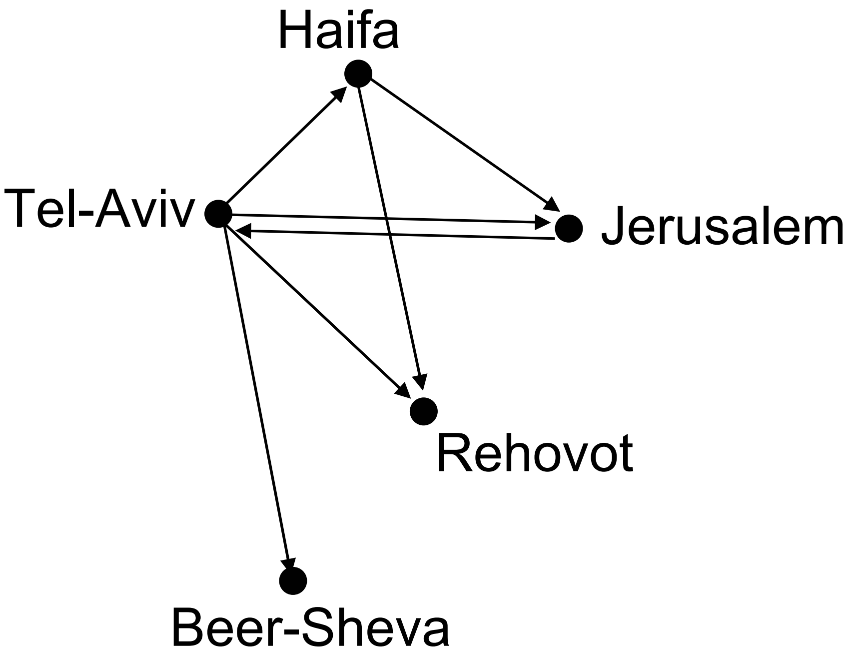
Let $(a,b) \in R^{n+1}$ and show $(a,b) \in R$.

$R^{n+1} = R^n \circ R$ so there exists $x \in A$ such that

$(a,x) \in R$ and $(x,b) \in R^n$, however by the induction assumption $R^n \subseteq R$ and so $(x,b) \in R$.

R is transitive so $(a,x) \in R \wedge (x,b) \in R \rightarrow (a,b) \in R$.

Closure of Relations



Communication Network

R is a relation: (a, b) $\in$ R if a is directionally connected to b.

(Tel-Aviv, Jerusalem) $\in$ R

(Haifa, Tel-Aviv) $\notin$ R

Closure of Relations

<u>Definition</u>:

Let R be a relation on set A.

The *Closure* (סגור) of R in terms of property P is the relation S with property P s.t. $R \subseteq S$ and for any other relation $T \neq S$: $R \subseteq T \rightarrow S \subseteq T$.

S is the minimal relation with property P that contains R.

Reflexive Closure

<u>Example:</u>

R is a relation over A = {1,2,3}

R = { (1,1), (1,2), (2,1), (3,1) }

The Reflexive closure of R is obtained by adding:

(2,2) (3,3)

Note: any reflexive relation S over A must contain

(1,1),(2,2),(3,3).

Proof ?

Reflexive Closure

The reflexive closure of relation R on A is:

$R \cup \{(a,a) \mid a \in A\}$

<u>Example:</u>

$R = \{ (a,b) \mid a < b\}$ $R \subseteq \mathcal{Z} \times \mathcal{Z}$

$R = \{ (a,b) \mid a \leq b\}$ Reflexive closure of R

Symmetric Closure

Example:

R is a relation over A = {1,2,3}

R = { (1,1), (1,2), (2,3), (3,1), (3,2) }

The Symmetric closure of R is obtained by adding:

(2,1) (1,3)

Note: any symmetric relation S over A that includes R must contain (2,1),(1,3).

Proof ?

Symmetric Closure

The symmetric closure of relation R on A is:

$R \cup R^{-1} = R \cup \{(b,a) \mid (a,b) \in R\}$

<u>Example:</u>

$R = \{ (a,b) \mid a < b\}$ $\quad$ $R \subseteq \mathcal{Z} \times \mathcal{Z}$

$R = R \cup \{ (a,b) \mid a > b\}$ $\quad$ Reflexive closure of R

$= \{ (a,b) \mid a \neq b\}$

Transitive Closure

<u>Example:</u>

R is a relation over $A = \{1,2,3,4,5\}$

$R = \{ (1,2), (2,3), (3,4), (4,5) \}$

Add all (a,c) s.t. (a,b) , $(b,c) \in R$

$R \cup \{ (1,3), (2,4), (3,5) \}$

Transitive?

$= \{ (1,2), (2,3), (3,4), (4,5)\ (1,3), (2,4), (3,5) \}$

Transitive Closure

<u>Example Cont:</u>

$R' = \{ (1,2), (2,3), (3,4), (4,5)\ (1,3), (2,4), (3,5) \}$

Add all (a,c) s.t. (a,b) , $(b,c) \in R'$

$R'' = R' \cup \{ (1,4), (2,5) \}$

Transitive?

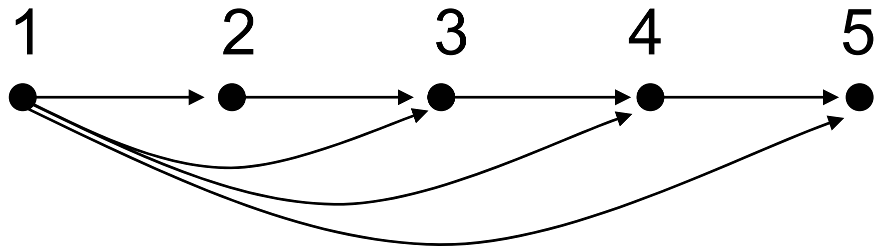
$R''' = R'' \cup \{ (1,5) \}$

Transitive!

Transitive Closure

Consider the graph representation of R.

R = { (1,2), (2,3), (3,4), (4,5) }



Add (1,3)

Add (1,4)

Add (1,5)

Add (1,x) for all x that is ‘reachable’ from 1.

i.e. there is a ‘path’ from 1 to ‘x’.

Transitive Closure

Definition:

Let R be a relation on set A.

A *Path* (מסלול) in R from $a \in A$ to $b \in A$ is a sequence of elements of A :

$x_0 = a, x_1, x_2, \ldots, x_n = b$

s.t. $(x_i, x_{i+1}) \in R$ for $i = 0 \ldots n-1$

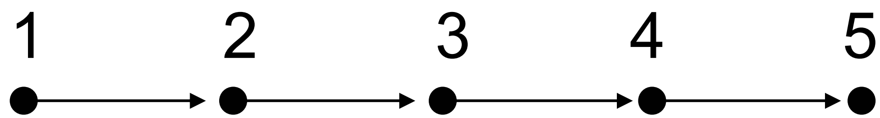
The length of the path is n.

Transitive Closure

Example:

R is a relation over A = {1,2,3,4,5}

R = { (1,2), (2,3), (3,4), (4,5) }



There is a path from 1 to 4: 1, 2, 3, 4

There is a path from 2 to 5: 2, 3, 4, 5

There is **no** path from 5 to 3.

Transitive Closure

Theorem: Let R be a relation on set A.

There is a path of length n ($n \in \mathcal{Z}^+$) from a to b IFF

$(a,b) \in R^n$.

Proof: By induction on n.

Transitive Closure

Transitive closure of R on set A is obtained by adding all (a,b) s.t. there is a path (of any length) from a to b.

Add all pairs (a,b) with a path of length ≤ 2 : R^2

Add all pairs (a,b) with a path of length ≤ 3 : R^3

Add all pairs (a,b) with a path of length ≤ 4 : R^4

$\vdots$

Transitive Closure

Theorem: Let R be a relation on set A.

The transitive closure of R is:

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Proof: Use previous theorem.

Transitive Closure

In practice there is no need to to unify ∞ sets $R^{n.}$

Theorem: Let R be a relation on set A, |A|= n.

If there is a path between a and b in R then

there is a path between a and b of length $\leq$ n.

If a $\neq$ b then there is a path of length n-1.

Transitive Closure

Proof:

Assume a,b $\in$ A and a path exists between a and b.
Assume, on the contrary, that the shortest path is of length m > n-1:

$$a = x_0, x_1, x_2, \ldots, x_m=b$$

there are m+1 > n elements of A in the path.
From the Pigeonhole principle 2 of these elements must be equal: $x_i = x_j$. Create a new path:

$$a = x_0, x_1, x_i, x_{j+1} \ldots, x_m=b$$

with at least 1 element less - contradiction that the shortest path is m.

Transitive Closure

In matrix representation: $R = \{ (1,3), (2,1), (3,2) \}$

$$M_R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$R^* = R^1 \cup R^2 \cup R^3$

$$M_{R^2} = M_R \odot M_R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{R^3} = M_{R^2} \odot M_R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{R^*} = M_R \vee M_{R^2} \vee M_{R^3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$