Lesson 3:

Logic

- Mathematical Logic
- Propositions
- Logical Operators
- Logical Puzzles
- Boolean Search
- Binary Operations

p	$\neg p$
T	F
F	T

Chapter 1.1 - 1.2

Mathematical Logic

Mathematical Logic is a tool for:

- Working with statements.
- Constructing and understanding mathematical argument.
- Understanding mathematical reasoning.

Mathematical Logic includes*:*

- A language for expressing statements.
- A methodology for reasoning about the truth and falsity of statements
- The foundation for expressing formal proofs.

Prepositional Logic

The study of compound statements created from simpler statements.

<u>Applications in computer science:</u>

Design of electronic circuits.

Construction of computer programs.

Queries to databases & search engines.

Aristotle (384 b.c. - 322 b.c.) - prepositional logic

Propositions

<u>Definition</u>:
A *Proposition* (פסוק) is a sentence or statement that is either TRUE or FALSE (אמת או שקר) but not both.

TRUE	FALSE
T	F
1	0

A proposition has a *Truth Value* (ערך אמת) T or F.

Propositions

Examples:

Propositions:

"Tel-Aviv is west of Jerusalem."	TRUE
"Snakes can fly."	FALSE
1 + 1 = 2	TRUE
2 + 2 = 3	FALSE

NOT Propositions:

"What Time is it?"	question
"Close the door!"	command
x + 1 = 2	x unknown
x + y = z	x,y,z unknown

Compound Propositions

Denote propositions with small letters: p,q,r,s

e.g. p = “1 + 1 = 2”

Create new propositions from existing propositions:

Compound Proposition (פסוקים מורכבים)

George Boole (1815-1864) - Founder of Propositional Logic:

Boolean Algebra

Compound Propositions

<u>Definition</u>:
If p is a proposition then "it is not True that p" ("not p") is a proposition and is called the *Negation of p* (שלילת p). $\boxed{\neg p}$

"Today is Monday"

<u>Negation:</u>

$\neg$ "Today is Monday"

"It is not true that today is Monday"

"Today is not Monday"

Truth Tables

The truth value of $\neg p$ depends on the truth value of p.

Truth Table (טבלת אמת) shows the relationship between truth values of different propositions.

Truth Table for Negation

p	$\neg p$
T	F
F	T

Logical Operators - Conjunction

Definition:

If p,q are propositions then $\boxed{p \wedge q}$ ("p and q") ("q וגם p") is a proposition. Its truth value is T if p is T and q is T.

$p \wedge q$ is called the conjunction (קוניונקציה) of p and q.

Example:

"Today is Friday and it is raining".

Is true on rainy Fridays.
Is false on all other days of the week and on sunny Fridays.

Logical Operators - Conjunction

Truth Table for Conjunction (AND)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Operators - Disjunction

<u>Definition</u>:
If p,q are propositions then $\boxed{p \vee q}$ ("p or q") ("q או p") is a proposition. Its truth value is T if p is T or q is T or both are T.

$p \wedge q$ is called the disjunction (דיסיונקציה) of p and q.

Example:

"Today is Friday or it is raining".

Is true on all rainy days and is also true on sunny Fridays.
Is false only on sunny days that are not Friday.

Logical Operators - Disjunction

Truth Table for Disjunction (OR)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Operators - Exclusive Or

<u>Definition</u>:

If p,q are propositions then $\boxed{p \oplus q}$ ("p xor q") is a proposition. Its truth value is T if p is T or q is T but not both.

Example:

"Today is Friday xor it is raining".

Is true on sunny Fridays and true on all other rainy days.
Is false only on rainy Fridays and sunny other days.

Logical Operators - Exclusive Or

Truth Table for Exclusive Or (XOR)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical Operators - Implication

<u>Definition</u>:

If p,q are propositions then $\boxed{p \to q}$ ("p implies q") (p גורר q) is a proposition. Its truth value is F if p is T and q is F and is T otherwise.

p is called the antecedent/premise (הנחה) and
q is called the conclusion/consequence (מסקנה/תוצאה).

Consider $p \to q$ as a contract. Breaking of contract happens only when p is T but q is not T (is F).

Logical Operators - Implication

Example:

“If you spend more than $1000 you must fill out a form”.

Is **false** if you spent $1500 and did not fill out a form.

Notice:
If you spent less than $1000 then there is no breach of contract regardless of whether the form was filled or not.

Logical Operators - Implication

Truth Table for Implications

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Logical Operators - Implication

Additional Examples:

"“If it is hot today we will go to the beach”.

“If today is Friday then 2+3 = 5”. Is always TRUE.

“If today is Monday then 2+3 = 6”. Is TRUE on all days except Monday.

“If 2 + 3 = 6 then today is Sunday”. Is always TRUE.

In Programming:

IF X THEN S (e.g. IF 1+1=2 THEN x:-x+1;)

Logical Operators - Implication

$p \rightarrow q$

"if p then q"	אם p אז q
"if p, q"	אם p , q
"p implies q"	p גורר q
"q follows from p"	q נובע מ-p
"q if p"	q אם p
"q when p"	q כאשר p
"q whenever p"	q כאשר p
"p only if q"	p רק אם q
"p is sufficient for q"	p מספיק עבור q
"q is necessary for p"	q הכרחי עבור p

Variations on $p \rightarrow q$

If p, q are propositions then:

$\boxed{\neg q \rightarrow \neg p}$ is the *Contraposition* (שלילת החיוב) of $p \rightarrow q$.

$\boxed{q \rightarrow p}$ is the *Converse* (ניגוד) of $p \rightarrow q$.

$\boxed{\neg p \rightarrow \neg q}$ is the *Inverse* (הופכי) of $p \rightarrow q$.

Contraposition of $p \rightarrow q$

$$\text{“If } \underbrace{\text{10 is divisible by 6}}_{p} \text{ then } \underbrace{\text{10 is divisible by 3.”}}_{q}$$

$$p \rightarrow q$$

$$\text{“If } \underbrace{\text{10 is } \textbf{not} \text{ divisible by 3}}_{\neg q} \text{ then } \underbrace{\text{10 is } \textbf{not} \text{ divisible by 6”.}}_{\neg p}$$

$$\neg q \rightarrow \neg p$$

Contraposition of $p \rightarrow q$

$p \rightarrow q$ is equivalent to the Contraposition of $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg q \rightarrow \neg p$
T	T	T
T	F	F
F	T	T
F	F	T

Converse of p→q

$$\underbrace{\text{“If David sits in the back}}_{\color{red}{p}}\ \text{then}\ \underbrace{\text{he cheats in the exam.”}}_{\color{red}{q}}$$

$$\color{red}{p \rightarrow q}$$

$$\underbrace{\text{“If David cheats in the exam}}_{\color{red}{q}}\ \text{then}\ \underbrace{\text{he is sitting in the back”.}}_{\color{red}{p}}$$

$$\color{red}{q \rightarrow p}$$

Converse of $p \rightarrow q$

$p \rightarrow q$ is not equivalent to the Converse of $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$q \rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

Inverse of $p \rightarrow q$

$$\underbrace{\text{“If it snows today}}_{p} \quad \text{then} \quad \underbrace{\text{I will stay home.”}}_{q}$$

$$p \rightarrow q$$

$$\underbrace{\text{“If it does not snow}}_{\neg p} \quad \text{then} \quad \underbrace{\text{I will not stay home”.}}_{\neg q}$$

$$\neg p \rightarrow \neg q$$

Inverse of $p \to q$

$p \to q$ is not equivalent to the Inverse of $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p \to \neg q$
T	T	T
T	F	T
F	T	F
F	F	T

Inverse of $p \to q$

Converse of $p \to q$ is equivalent to the Inverse of $p \to q$

p	q	$p \to q$	$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

Logical Operators - Biconditional

<u>Definition</u>:

If p,q are propositions then $\boxed{p \leftrightarrow q}$ ("p iff q") (q אם ורק אם p) is a proposition. Its truth value is T if p and q have the same truth value.

$p \leftrightarrow q$ is called the biconditional of p and q.

"p is necessary and sufficient for q"	p מספיק והכרחי עבור q
"if p, then q, and conversely"	אם p אז q, ולהיפך
"p iff q"	q אם"ם p

Logical Operators - Biconditional

Truth Table for biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Operators

Operator Name	Nickname	Parity	Symbol
Negation	NOT	Unary	$\neg$
Conjunction	AND	Binary	$\wedge$
Disjunction	OR	Binary	$\vee$
Exclusive-OR	XOR	Binary	$\oplus$
Implication	IMPLIES	Binary	$\rightarrow$
Biconditional	IFF	Binary	$\leftrightarrow$

Logical Operators Precedence

precedence

$(\;)$

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

$$\neg p \wedge q \; = (\neg p) \wedge q \; \neq \neg (p \wedge q)$$

$$p \vee q \leftrightarrow r \; = (p \vee q) \leftrightarrow r \neq p \vee (q \leftrightarrow r)$$

Logical Operators Vs Set Operations

Truth Table for
AND / OR

p	q	$\neg p$	$p \wedge q$	$p \vee q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

Membership Table for
Intersection/Union

A	B	$\overline{A}$	$A \cap B$	$A \cup B$
1	1	0	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	0

Translating Spoken Language to Logical Expressions (הצרנה לוגית)

- Removes ambiguity.
- Allows analysis, manipulation and reasoning.

Example:

$\overbrace{\text{You receive a course grade}}^{a}$ only if $\overbrace{\text{you passed the exam}}^{b}$ or $\underbrace{\text{you did not miss a class}}_{\neg c}$.

$$\boxed{a \rightarrow (b \vee \neg c)}$$

Consistency

Is the following set consistent (עקבי) ?

- The mail message is either spam or is transmitted
- The mail message is not spam.
- If the message is spam then it is not transmitted.

p = “The mail message is spam”.

q = “The mail message is transmitted”.

Consistency

- The mail message is either spam or is transmitted

  $p \vee q$

- The mail message is not spam.

  $\neg p$

- If the message is spam then it is not transmitted.

  $p \rightarrow \neg q$

Answer:

if p is False then q must be True and False p implies all.

Consistency

Can also be verified using Truth Tables

p	q	$\neg q$	$\neg p$	$p \vee q$	$p \rightarrow \neg q$
T	T	F	F	T	T
T	F	T	F	T	F
F	T	F	T	T	T
F	F	T	T	F	T

Logical Puzzles

Raymond Smullyan (1919-)

An island has 2 kinds of inhabitants:
Knights - always tell the truth
Knaves - always lie.

Question:

You meet 2 people A and B. What are A and B if
A says: “B is a knight”
B says: “The two of us are of different types”

Logical Puzzles

Solution:

p = “A is a knight”
q = “B is a knight”

* Suppose p = True
  Then A is a knight and tells the truth: “B is a knight”
  so B is also a knight and q = True. Thus B tells the truth:
  “The two of us are of different types” $(\neg p \wedge q) \vee (p \wedge \neg q)$
  But this is False because A and B are of the same type.
  Contradiction! Thus p = False.
* Since p = False and A is a knave, A lies, thus:
  “B is a knight” is False and B is a knave:
  q = False.
  If B is a knave then B lies and $(\neg p \wedge q) \vee (p \wedge \neg q)$
  should be false and indeed A and B are of the same type.

Logical Puzzles

Additional Questions:

You meet 2 people A and B. What are A and B if

A says: “At least one of us is a knave”
B says: nothing

A says: “We are both knights”
B says: “A is a knave”

A says: “I am a knight”
B says: “I am a knight”

Logical Puzzles - II

5 friends have access to a chat room. It is known that:

(1) **K**eren or **H**ava (or both) are chatting.

(2) **R**oey or **V**ered are chatting - but not both.

(3) If **A**rik is chatting then so is **R**oey.

(4) **V**ered and **K**eren are either both chatting or both not.

(5) If **H**ava is chatting then **A**rik and **K**eren are chatting.

Question: Who is chatting in the chat room?

Logical Puzzles - II

Solution:

k = “Keren is in the chat room”.

h = “Hava is in the chat room”.

r = “Roey is in the chat room”.

v = “Vered is in the chat room”

a = “Arik is in the chat room”.

(1) $k \vee h$

(2) $r \oplus v$

(3) $a \rightarrow r$

(4) $v \leftrightarrow k$

(5) $h \rightarrow (a\ k)$

Logical Puzzles - II

Solution cont.:

From (2) we have that r and v are of different truth values:

(2)	$\begin{cases} r = \mathbf{T} \\ v = \mathbf{F} \end{cases}$	**or**	(2)	$\begin{cases} r = \mathbf{F} \\ v = \mathbf{T} \end{cases}$
(4)	$k = \mathbf{F}$		(4)	$k = \mathbf{T}$
(1)	$h = \mathbf{T}$	X	(3)	$a = \mathbf{F}$
(5)	$h = \mathbf{F}$		(5)	$h = \mathbf{F}$

Keren and Vered are in the chat room.

Logic and Computer Science:
Boolean Searches

Search in Databases, search engines, image retrieval using propositional logic (i.e. logical operators).

Boolean Search (חיפוש בוליאני).

Google, Yahoo, Lycos, AltaVista, Excite....

Logic and Computer Science:

Boolean Searches

Example: Search for actor Patrick Star.

Search: Patrick $\wedge$ Star

Returns: many sites with Patrick Stewart Star Trek

Search: Patrick $\wedge$ Star $\wedge$ $\neg$Stewart $\wedge$ $\neg$Trek

Returns: sites with Patrick Star

Note: As of 2004 :

Patrick Star of SpongeBob SquarePants movie

Demo: Image Retrieval

Logic and Computer Science:

Binary Operations

Computer information is represented as Bits = Binary digits.

A bit has 2 possible values: 0 and 1.

Numbers are represented as bit strings (e.g. 13 $\rightarrow$ 01101)

John Tukey (1915 - 2000)

Binary Operations

Logical operations on bits - Binary Operations.

$\boxed{0 \leftrightarrow \mathbf{F}}$ $\boxed{1 \leftrightarrow \mathbf{T}}$

x,y - boolean variable - whose value is 0 or 1.

		NOT	AND	OR	XOR
x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \oplus y$
T	T	F	T	T	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	F	F

Binary Operations

Binary operations on bit strings.

Bitwise NOT

Bitwise AND

Bitwise OR

Bitwise XOR

Example:

1 0 0 1	
0 1 0 1	
1 1 0 1	Bitwise AND
0 0 0 1	Bitwise OR
1 1 0 0	Bitwise XOR