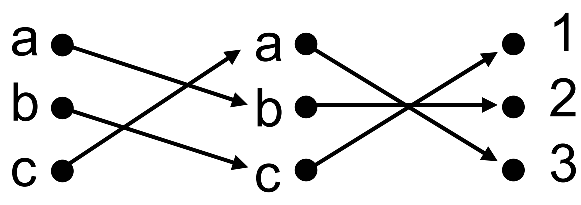
Lesson 6:

Functions

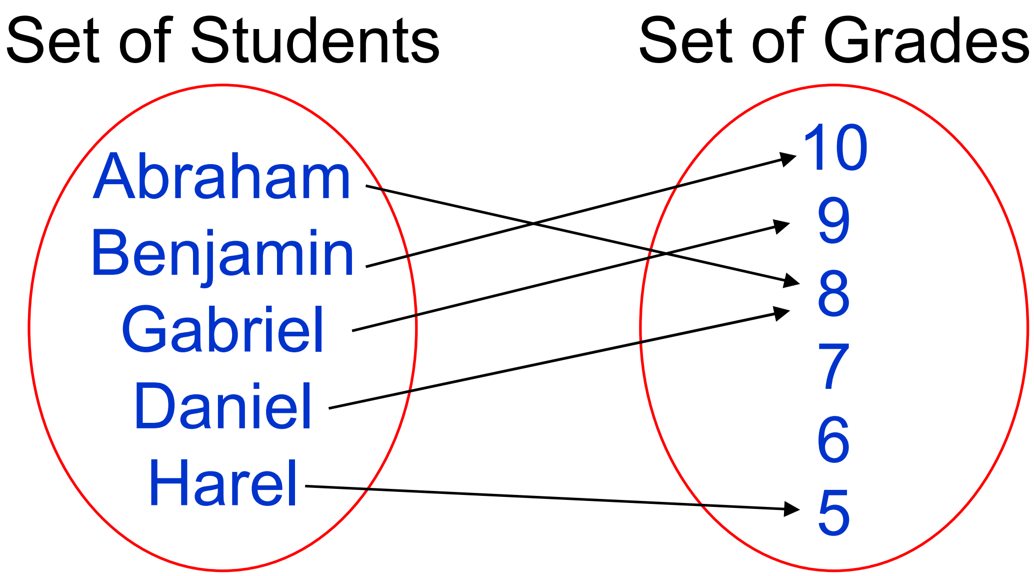
- Function Definition
- Function Types
  - one-to-one
  - onto
- Inverse Functions
- Composition of Functions



Chapter 1.8

Functions (פונקציות)

Functions are a tool to link between elements of one set and elements of another set.



Functions

<u>Functions are important in Mathematics and CS:</u>

- Defines Sequences and series.
- Used to compute time & memory complexity of algorithms.
- In programming - used to perform recurrent computations.
- Recursive Functions - throughout CS.

Functions

Definition:

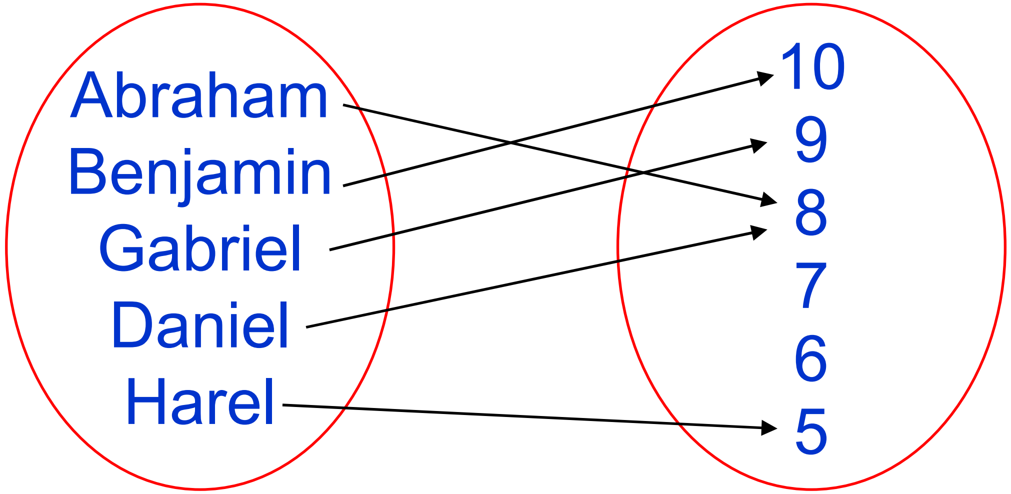
Let A and B be sets. A *function* (פונקציה) f from A to B assigns exactly one element of B to each element of A .

$f: A \rightarrow B$ ("f from A to B or "f maps A to B")

Notation: $f(a) = b$ if f assigns the element b to a.

Functions

Example: Set of Students Set of Grades



The function G assigns a grade to each student.

G: Students $\rightarrow$ Grades

G(Abraham) = 8
G(Harel) = 5

Functions

<u>Defining Functions:</u>

1) By defining every assignment specifically

G(Abraham) = 8, G(Benjamin) = 10, ..., G(Harel) = 5

2) By using a formula

$f(x) = x+1$

Functions

Example: $g: \{1, 2, 3, 4\} \rightarrow \{1, 3, 5, 7, 9\}$

$g(4) = 5$
$g(2) = 9$
$g(3) = 7$
$g(4) = 1$

Is g a function?

Example: $f: \mathcal{R} \rightarrow \mathcal{R}$

$f(x) = x^2$

Is f a function?

Example: $s: \mathcal{R} \rightarrow \mathcal{Q}$

$s(x) = \sqrt{x}$

Is s a function?

Functions

Definition:

Let f be a function from A to B : $f: A \rightarrow B$.

Then A is the *Domain* (תחום) of f and

B is the *CoDomain/Range* (טווח) of f.

If $f(a) = b$

then a is the *PreImage / Source* (מקור)

and b is the *Image* (תמונה)

Functions

If $S \subseteq A$ then the *Image* (תמונה) of S is the set of all elements in B assigned to elements of S:

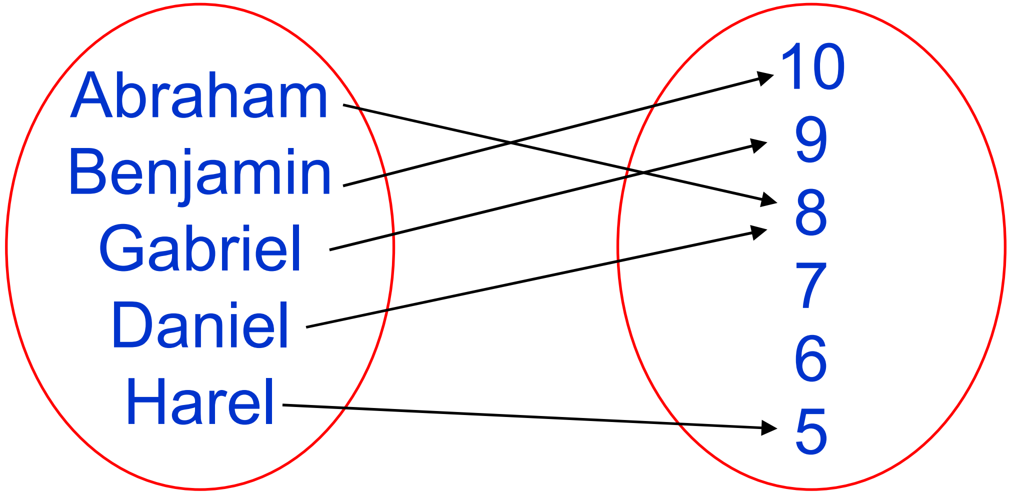
$$f(S) = \{b \mid b \in B \wedge \exists s\, (f(s) = b \wedge s \in S)\} = \{f(s) \mid s \in S\}$$

The set of all elements of B having a source in A is the *Image* of A.

note: $f(A) = image(A) \subseteq B$

Functions

Example: Set of Students Set of Grades



$$g : \text{Students} \to \text{Grades}$$

The Domain is the set of Students.

The Range is the set of Grades.

The Image of the Domain is the set $\{10,9,8,5\} \subseteq$ Grades

Functions

Example:

$f : \mathcal{Z} \rightarrow \mathcal{Z}$

$f(x) = x^2$

The Domain is $\mathcal{Z}$.

The Range is $\mathcal{Z}$.

The Image of the Domain $f(\mathcal{Z})$ is the set of all squares:

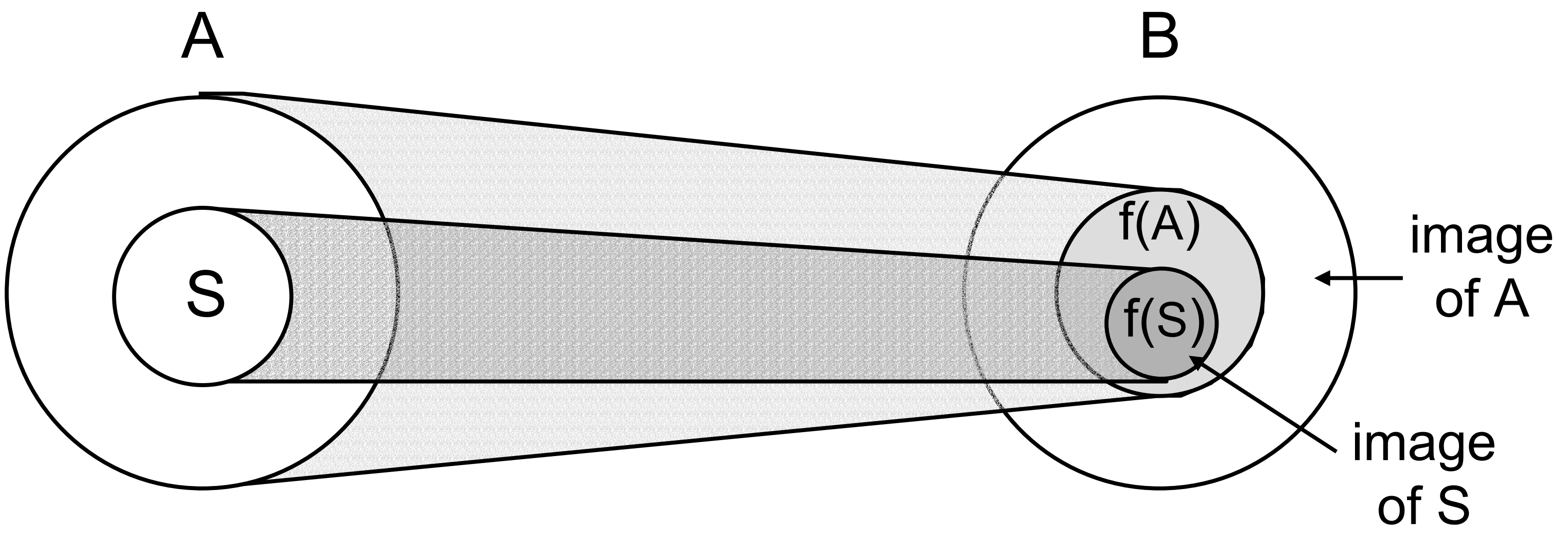
$\{0, 1, 4, 9, ...\} \subseteq \mathcal{Z}$

The Image of $\mathcal{Z}^+ \subseteq \mathcal{Z}$ is the set of all squares:

$\{0, 1, 4, 9, ...\} \subseteq \mathcal{Z}$

Functions

Diagrams



$$f : A \to B$$

$$S \subseteq A$$

Question: $S \subseteq A \overset{?}{\to} f(S) \subseteq f(A)$

$S \subset A \overset{?}{\to} f(S) \subset f(A)$

Functions and the Computer

In programming, the Domain and CoDomain are often specified.

Examples:

Java: int **floor**(float x)

Pascal: **function** floor(x : **real**) : **integer**

Functions

Definition:

Let f_1, f_2 be functions from A to $\mathcal{R}$ then f_1+f_2 and f_1f_2 are functions from A to R and defined by:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1f_2)(x) = f_1(x)\, f_2(x)$$

Examples:

$$f_1(x) = x^2 \qquad f_2(x) = x - x^2$$

$$(f_1 + f_2)(x) = x$$

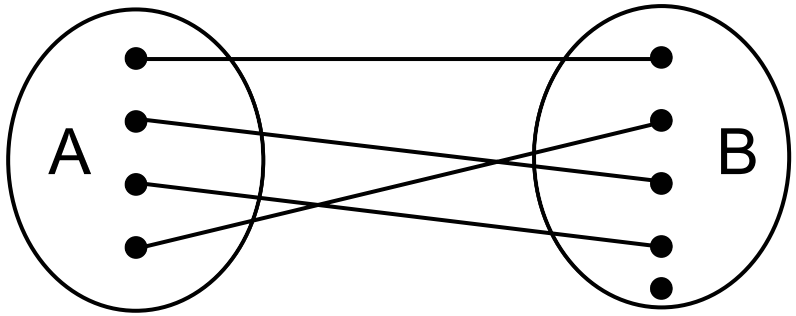
$$f_1f_2(x) = x^3 - x^4$$

One-to-One Functions

Definition:

A function f is called *one-to-one* (חד-חד ערכית) iff

$$f(x)=f(y) \rightarrow x=y$$



Question: $x \neq y \rightarrow f(x) \neq f(y)$?

$$\forall x \forall y \, (f(x)=f(y) \rightarrow x=y) \;\Leftrightarrow\; \forall x \forall y \, (x \neq y \rightarrow f(x) \neq f(y))$$

Universe $x,y \in$ Domain

One-to-One Functions

<u>Examples:</u>

$f : \{a, b, c\} \to \{1,2,3,4\}$

$f(a) = 1, f(b) = 3, f(c) = 4$ Is one-to-one ?

$f : \mathcal{Z} \to \mathcal{Z}$

$f(x) = x^2$ Is one-to-one ?

$f : \mathcal{Z}^+ \to \mathcal{Z}$

$f(x) = x^2$ Is one-to-one ?

One-to-One Functions

$f : A \rightarrow B$

$|A| > |B|$ Can f be one-to-one ?

$|A| < |B|$ Must f be one-to-one ?

$|A| = |B|$ Can/Must f be one-to-one ?

<u>Example:</u> $f : \mathcal{R} \rightarrow \mathcal{R}$, $f(x) = x+1$

One-to-One Functions

Some families of functions are necessarily one-to-one.

<u>Definition</u>:

A function f : $\mathcal{R} \to \mathcal{R}$, is called *strictly increasing* (מונוטונית עולה ממש) iff x<y $\to$ f(x)<f(y)

<u>Definition</u>:

A function f : $\mathcal{R} \to \mathcal{R}$, is called *strictly decreasing* (מונוטונית יורדת ממש) iff x<y $\to$ f(x)>f(y)

One-to-One Functions

Theorem:

A strictly increasing or strictly decreasing function is necessarily one-to-one.

Proof: We prove for strictly increasing.

We prove $x \neq y \rightarrow f(x) \neq f(y)$.

If $x \neq y$ then

Case 1: $x < y$

$f(x) < f(y)$ (function is strictly increasing)

$f(x) \neq f(y)$

Case 2: $x > y$

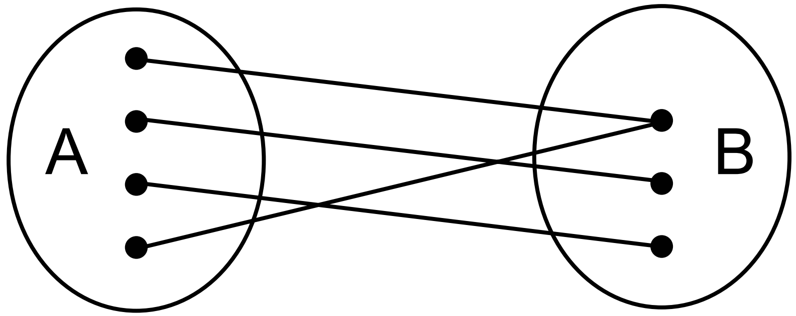
$f(x) < f(y)$ (function is strictly increasing)

$f(x) \neq f(y)$

Onto Functions

<u>Definition</u>:

A function $f : A \rightarrow B$ is called *onto* (על) iff for every $b \in B$ there is an element $a \in A$ such that $f(a)=b$.



$\forall y \exists x\, (f(x)=y)$

Universe $x \in$ Domain $y \in$ range

Onto Functions

Examples:

$f : \{a, b, c\} \rightarrow \{1,2\}$

$f(a) = 1,\ f(b) = 2,\ f(c) = 1$ Is onto ?

$f : \mathcal{Z} \rightarrow \mathcal{Z}$

$f(x) = x^2$ Is onto ?

$f : \mathcal{C} \rightarrow \mathcal{Z}$

$f(x) = x^2$ Is onto ?

$f : \mathcal{Z}^+ \rightarrow \mathcal{Z}^+$

$f(x) = x+1$ Is onto ?

Onto Functions

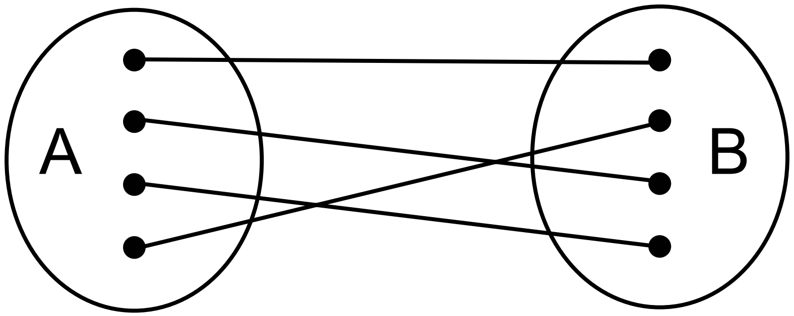
$$f : A \rightarrow B$$

$\|A\| < \|B\|$	Can f be onto ?
$\|A\| > \|B\|$	Must f be onto ?
$\|A\| = \|B\|$	Can/Must f be onto ?

Bijection Functions

Definition:

A function $f: A \rightarrow B$ is called a *bijection* or a one-to-one correspondence (פונקציה חד-חד ערכית ועל) if it is both one-to-one and onto.



Bijection Functions

Examples:

$f : \{a, b, c\} \rightarrow \{1,2,3\}$

$f(a) = 1, f(b) = 2, f(c) = 3$ Is a bijection ?

$i_A : A \rightarrow A$

$i_A(a) = a$ (identity זהות) Is bijection ?

$f : \mathcal{Z} \rightarrow \mathcal{Z}$

$f(x) = x+1$ Is bijection ?

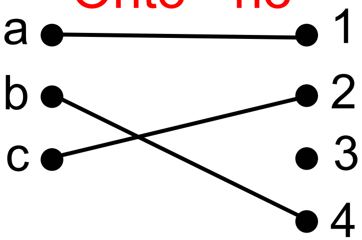
Bijection Functions

Theorem: Let $f : A \rightarrow A$. If A is a finite set then:
f is one-to-one **iff** f is onto.

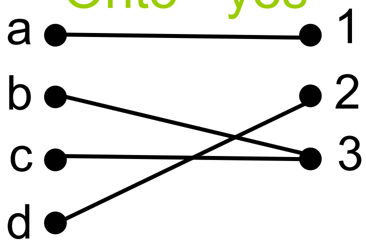
Is the theorem true for $f : A \rightarrow B$, where A,B are finite?

Function Types - Summary

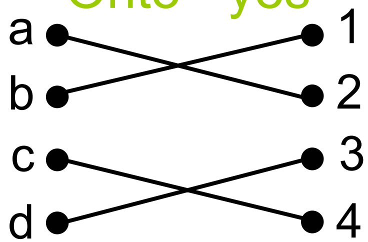
1-to1 - yes
Onto - no



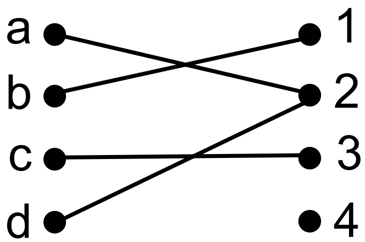
1-to1 - no
Onto - yes



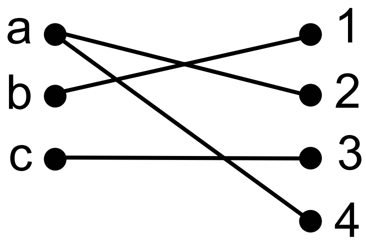
1-to1 - yes
Onto - yes



1-to1 - no
Onto - no



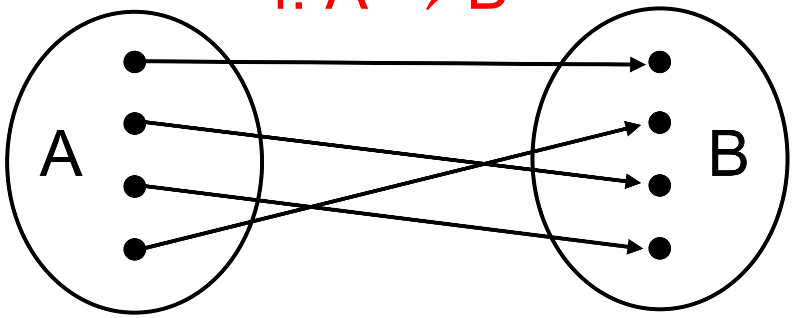
not a function



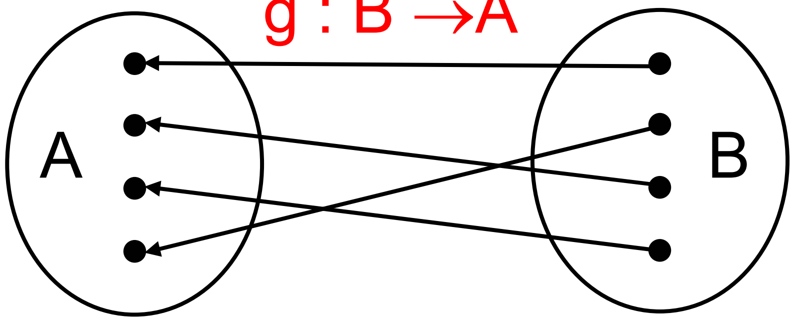
Inverse Functions

$f: A \rightarrow B$ then can we define $g = f^{-1} : B \rightarrow A$?

$f: A \rightarrow B$



$g : B \rightarrow A$



Inverse Functions

<u>Definition</u>:

Let $f : A \rightarrow B$ be a bijection (one-to-one and onto) then the *inverse function* (פונקציה הפוכה) of f exists and is defined as $f^{-1} : B \rightarrow A$ such that:

$$f^{-1}(b) = a \quad \leftrightarrow \quad f(a) = b$$

Function f is said to be invertible (הפיכה) if f^{-1} exists.

If f is not onto - there exists $b \in B$ with two sources.
If f is not 1-to-1 - there exists $b \in B$ without a source.

Inverse Functions

Examples:

$f : \{a, b, c\} \to \{1,2,3\}$

$f(a) = 1,\ f(b) = 3,\ f(c) = 2$ invertable?

$f^{-1}(1) = a,\ f^{-1}(3) = b,\ f^{-1}(2) = c$

$f : \mathcal{Z} \to \mathcal{Z}$

$f(x) = x+1$ invertable?

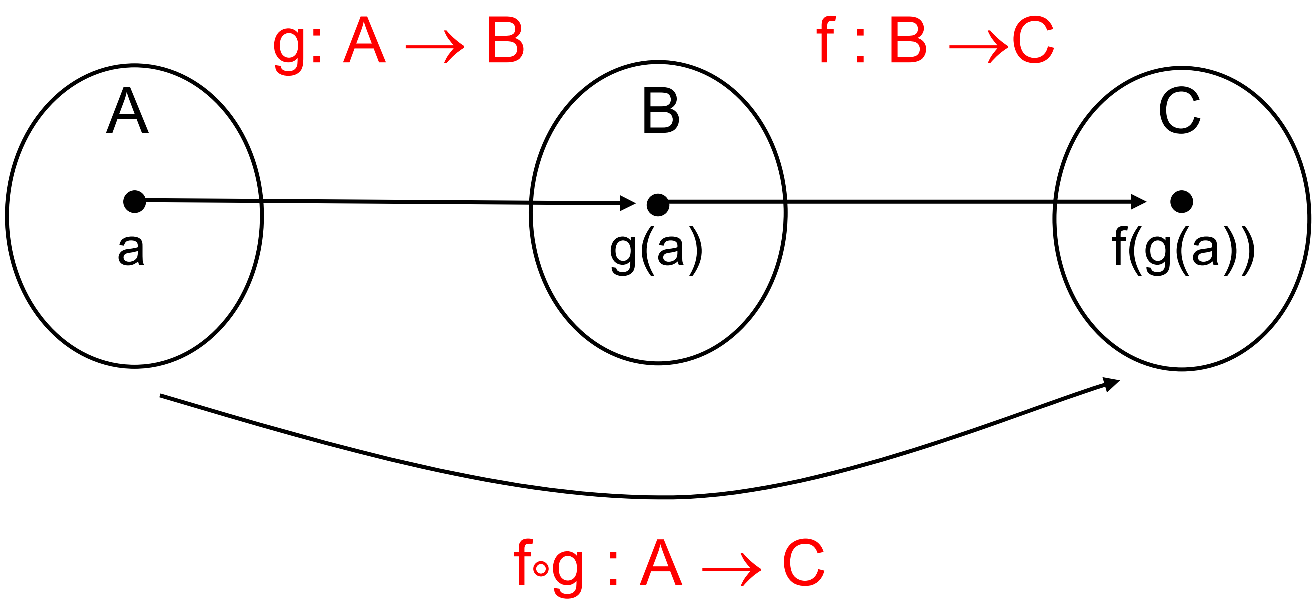
$f^{-1}(y) = y-1$

$f : \mathcal{Z} \to \mathcal{Z}$

$f(x) = x^2$ invertable?

Function Composition (הרכבה)

$g : A \to B$ and $f : B \to C$ then can we define $f \circ g : A \to C$?



Function Composition

Definition:

Let $g : A \to B$ and $f : B \to C$.

The *composition* (הרכבה) of the functions f and g

is denoted $f \circ g$ and is defined as:

$$f \circ g(a) = f(g(a))$$

Note: The image of domain A under g must be a subset of the domain B of f:

$$g(A) \subseteq B$$

Function Composition

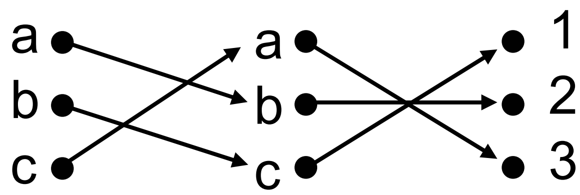
<u>Examples:</u>

$g : \{a, b, c\} \rightarrow \{a, b, c\}$

$f\ : \{a, b, c\} \rightarrow \{1, 2, 3\}$

$g(a) = b,\ g(b) = c,\ g(c) = a$

$f(a) = 3,\ f(b) = 2,\ f(c) = 1$



$f \circ g : \{a, b, c\} \rightarrow \{1, 2, 3\}$

$f \circ g(a) = f(g(a)) = f(b) = 2$

$f \circ g(b) = f(g(b)) = f(c) = 1$

$f \circ g(c) = f(g(c)) = f(a) = 3$

Question: if $g \circ f$ defined?

Function Composition

Examples:

$g : \mathcal{Z} \rightarrow \mathcal{Z}$

$f : \mathcal{Z} \rightarrow \mathcal{Z}$

$g(x) = 3x + 2$

$f(x) = 2x + 3$

Are $f \circ g$ and $g \circ f$ defined?

$f \circ g(x) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$g \circ f(x) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

$\boxed{\text{Note : } f \circ g \neq g \circ f}$

Function Composition

$f \circ f^{-1} = ?$ $\qquad$ $f^{-1} \circ f = ?$

$f \circ f^{-1} = i_A, \quad f^{-1} \circ f = i_B$ $\qquad$ The identity function

f^{-1} exists iff f is one-to-one and onto. Thus f^{-1} is also one-to-one and onto and

$$f(b) = a \leftrightarrow f(a) = b$$

$$i_A = f^{-1} \circ f\ (a) = f^{-1}\ (f\ (a)) = f^{-1}(b) = a$$

$$i_B = f \circ f^{-1}\ (b) = f\ (f^{-1}(b)) = f(a) = b$$

Functions and the Computers

Functions can also be defined as a series of commands.

<u>Example:</u>

f is the function that maps natural numbers to the set of library books:

$f(x) =$ the book that results from the following steps:

* Enter the Library
* Go to the first set of shelves
* Count x books (left-to-right) on top shelf
* The x-th book is the output (image of x).

Functions and the Computers

In programming only a limited set of commands are allowed.

<u>Example:</u>

```
function f(x : integer) : integer
    if  x > 0   return(1)
    else        return(0)
    endif
end
```

Returns 1 if the number is positive, 0 otherwise.

Cardinality of Sets

Informally : The cardinality (עוצמה) of a set is the number of elements in the set.

For finite sets A and B : A and B have the same Cardinality IFF they have the same number of elements.

<u>Definition</u>:

The sets A and B have the same *cardinality* (עוצמה) IFF there exists a Bijection $f : A \rightarrow B$.
(i.e. **f** is 1:1 and onto).

Cardinality of Sets

Examples:

$|\{a, b, c, d\}| = |\{1, 2, 3, 4\}|$ same cardinality?

$f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4$

$|\mathcal{N}_{even}| = |\mathcal{N}_{odd}|$ same cardinality?

$f : \mathcal{N}_{even} \rightarrow \mathcal{N}_{odd}$

$f(n) = n+1$

Cardinality of Sets

Definition:

A set that is either finite or has the same Cardinality of the positive integers $\mathcal{Z}^+$ s called *Countable* (בת מניה).

Cardinality of an infinite countable set is denoted $\aleph_0$.

Intuition: a countable set can be 'counted' by following the the bijection from $\mathcal{Z}^+$. 1, 2, 3, ...

Intuition: a countable set can be ordered as a sequence .

Cardinality of Sets

<u>Examples</u>:

The set **S** of odd positive integers is countable.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 3 & 5 & 7 & 9 & 11 & \dots \end{array}$$

$$f : \mathcal{Z}^+ \to \mathcal{N}_{\text{odd}}$$

$$f(n) = 2n - 1$$

Cardinality of Sets

Examples:

The set $\mathcal{Z}$ Is it countable?

Yes: 0, 1, -1, 2, -2, 3, -3, 4, -4, ...

f : $\mathcal{Z}^+ \rightarrow \mathcal{Z}$

$f^{-1}(m) = 2|m| + (1-\text{sign}(m))/2$

The set $Q^+ = \{x/y \mid x,y \in \mathcal{Z}^+\}$ Is it countable?

Yes: $$\underbrace{\frac{1}{1}}_{x+y=2}, \underbrace{\frac{1}{2}, \frac{2}{1}}_{x+y=3}, \underbrace{\frac{1}{3}, \frac{2}{2}, \frac{3}{1}}_{x+y=4}, \ldots.$$

Cardinality of Sets

<u>Examples</u>:

The set $\mathcal{R}$ Is it countable?

NO!

Proof : Cantor's Diagonalization Argument (1879)

Assume by contradiction that $\mathcal{R}$ is countable.

Thus all subsets of $\mathcal{R}$, specifically [0..1] is countable and they can be linearly ordered:

$$
\begin{aligned}
r_1 &= 0\,.\,d_{11}\,d_{12}\,d_{13}\,d_{14}\,\ldots \\
r_2 &= 0\,.\,d_{21}\,d_{22}\,d_{23}\,d_{24}\,\ldots \\
r_3 &= 0\,.\,d_{31}\,d_{32}\,d_{33}\,d_{34}\,\ldots \\
r_4 &= 0\,.\,d_{41}\,d_{42}\,d_{43}\,d_{44}\,\ldots \\
&\vdots
\end{aligned}
$$

$d_{ij} \in \{0,1,2,\ldots,9\}$

Cardinality of Sets

Proof cont. :

$$
\begin{aligned}
r_1 &= 0\,.\,d_{11}\,d_{12}\,d_{13}\,d_{14}\,\ldots \\
r_2 &= 0\,.\,d_{21}\,d_{22}\,d_{23}\,d_{24}\,\ldots \\
r_3 &= 0\,.\,d_{31}\,d_{32}\,d_{33}\,d_{34}\,\ldots \\
r_4 &= 0\,.\,d_{41}\,d_{42}\,d_{43}\,d_{44}\,\ldots \\
&\vdots
\end{aligned}
$$

Form a new real number : $r = 0\,.\,d_1\,d_2\,d_3\,d_4\,\ldots$

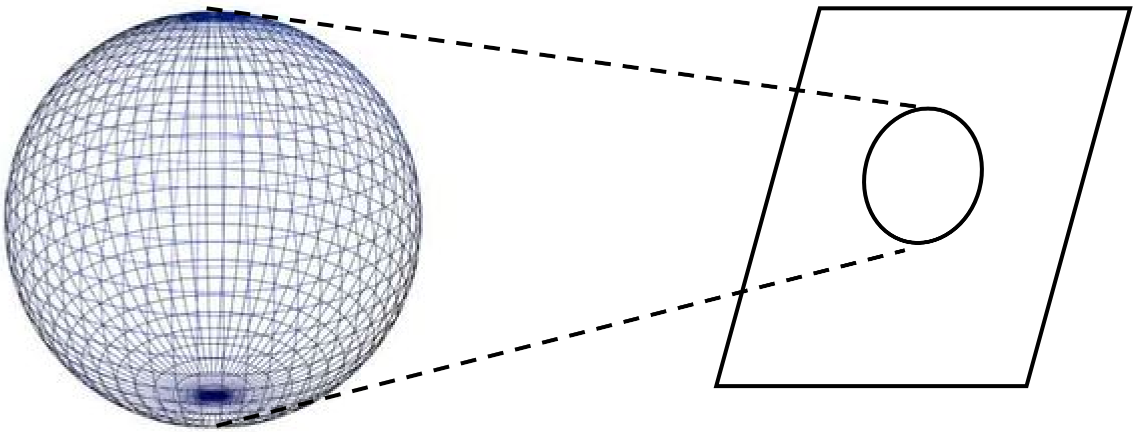
$$\text{where } d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

$r \neq r_i$ since they differ in the i-th decimal.

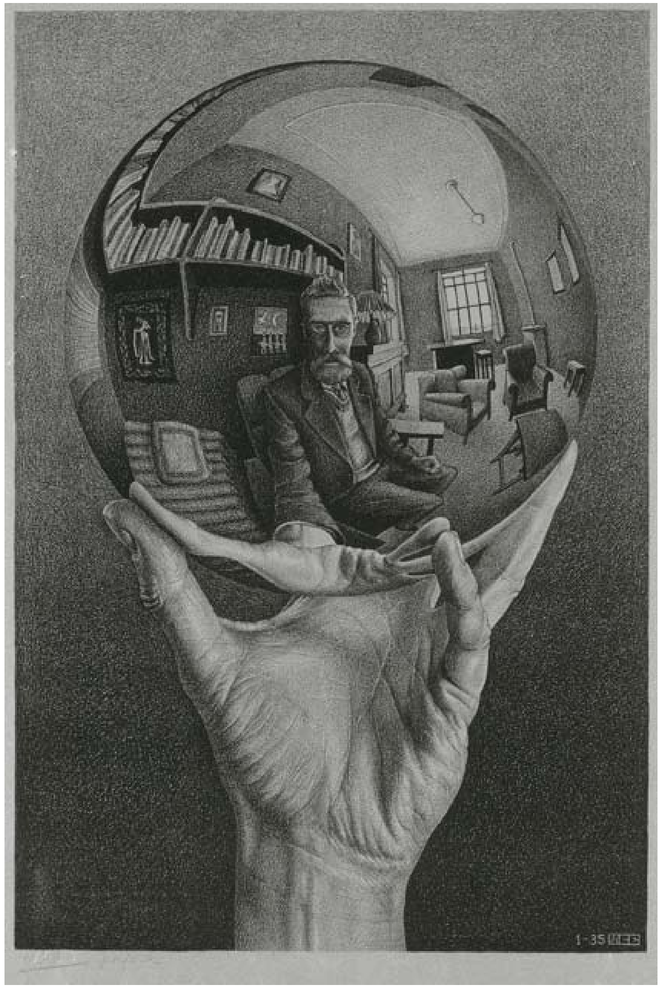
Thus there exists an $r \in [0..1]$ that is not listed

contradiction!

3D to 2D Spatial Functions



Anamorphosis



Sphere
MC Escher