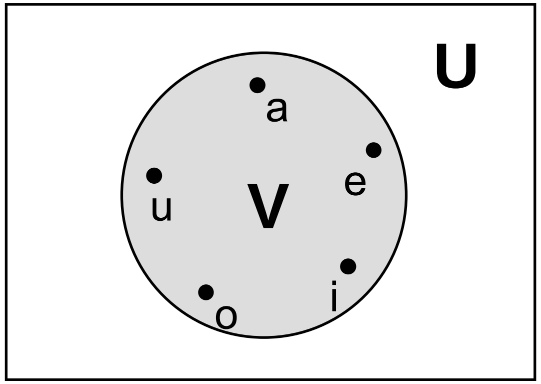
Lesson 1:

Set Theory

- What is a set?
- Elements of a set.
- Representing a set.
- Universal and Empty Set.
- Subsets.
- Set equality.
- Cardinality of sets
- Cartesian Product



Chapter 1.6

Why Set Theory?

In this course we study discrete objects :

Relations = ordered pairs of objects.

Combinations = collection of unordered elements.

Graphs = collection of vertices and edges.

The most basic discrete structure is the SET קבוצה.

Definition: a *set* is a group of objects.

Naive Set Theory (Georg Cantor 1845-1918)

Definition: the objects in a set are called *elements* or *members*. (אלמנטים, איברים)
A set *contains* its elements.

A set is denoted by the symbols '{ }'.

Examples:

{1, 3, 5, 7, 9} - Odd positive integers smaller than 10.

{a, 2, ∆, ♥, משה} - Unrelated objects:

Describing Sets

- list of the set's elements:

  $$S = \{1,2,3,4,5\}$$

- Use "…" $\quad S = \{1,\ldots,5\}$
- Specified by a predicate (property)
  - $S = \{x|p(x)\}$ S contain all the elements from U that satisfied the predicate P(x).
  - $S = \{n|1 \leq n \leq 5\}$

Examples:

$\{x \mid x \text{ is odd positive integer less than } 10\}$

$\{x \mid x < 100\}$

$\{x \mid x \text{ are English vowels}\}$

$\{x \mid x \text{ is prime and } x>1000\}$

Well Known Sets

$\mathcal{N} = \{0, 1, 2, 3, \ldots\}$	- Natural numbers
$\mathcal{Z} = \{\ldots, -2, -2, 0, 1, 2, \ldots\}$	- Integers
$\mathcal{Z}^+ = \{1, 2, 3, 4, \ldots\}$	- Positive integers
$\mathcal{R} = \{x \mid x \text{ is real}\}$	- Real numbers
$\mathcal{C} = \{x \mid x \text{ is complex}\}$	- Complex Numbers

Sets are typically denoted by Capital letters.

Special Sets

U - *The Universal Set* - contains all the elements in the world of discussion.

$\varnothing$ **= { }** - *The Empty Set* or *Null Set*

Examples:

$\{x \mid x > 10 \text{ and } x < 5\}$

Definition:

Two sets are *equal* iff (if and only if) they contain the same elements.

$\{1, 3, 5\} = \{5, 3, 1\} = \{1, 3, 3, 3, 5, 5, 3\}$

$\{0,1,2,3,4,5,6\} = \{0, \ldots, 6\} = \{x \mid x<7 \text{ and } x \text{ is natural } \}$

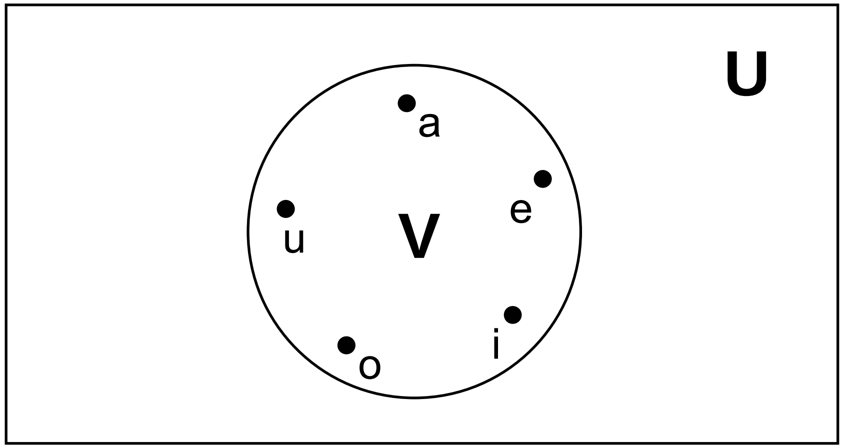
Notation:

$a \in A$ - a is an element of set A

$a \notin A$ - a is not an element of A.

Venn Diagrams

John Venn (1834-1923) - visualization of set operations.

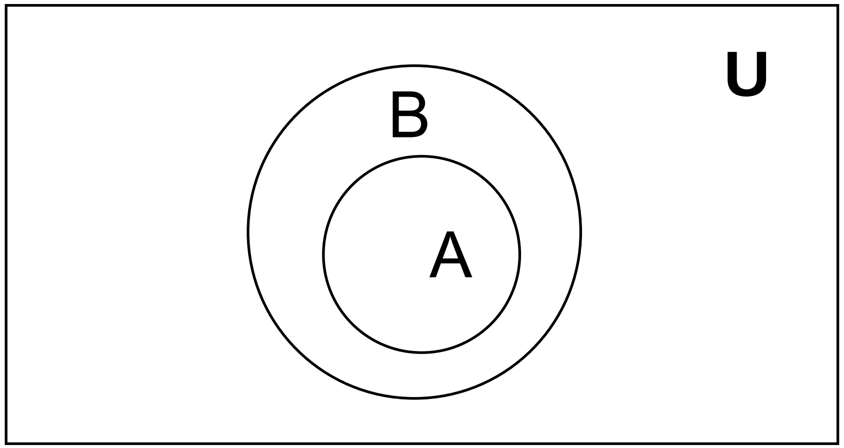


The set of English Vowels

Definition:

Set A is a *subset* of set B if every element of A is an element of B.

$$A \subseteq B$$



$A \subseteq B$

Definition:

Set A is a *proper subset* of set B if $A \subseteq B$ but $A \neq B$.

$$A \subset B$$

Theorem:

$A \subseteq B$ and $A \supseteq B$ iff $A = B$

For any set S:

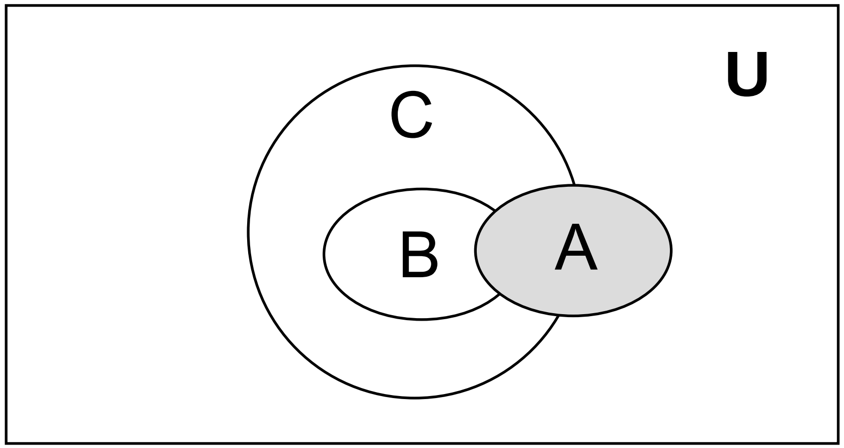
$\emptyset \subseteq S$ - all elements in $\emptyset$ are in S.

$S \subseteq S$ - all elements in S are in S.

For any set A, B, C:

if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

if $A \not\subseteq C$ and $B \subseteq C$ then $A \not\subseteq B$.



Set Cardinality

<u>Definition</u>:

If S contains n elements ($n \in N$) then S is a *finite set* and n is the *Cardinality* (עוצמה) of S.

$$n = | S |$$

Examples:

A = set of even positive integers smaller than 10

$|A| = 4$

B = set of the letters of the Hebrew Alphabet

$|B| = 22$

$|\varnothing| = 0$

Set Cardinality

Definition:
If S is not a finite set then it is an *infinite* set.

Examples:

$\mathcal{N}$ = the set of natural numbers

$\mathcal{R}$ = the set of real numbers

$\{ x \mid x \in \mathcal{Z} \text{ and } x > 58\}$

Q: does the cardinality of $\mathcal{N}$ differ from that of $\mathcal{R}$?

Sets as Elements

A set may contain sets as elements.

- The following set has 3 elements. Each element is a set of even numbers.

$$\{ \{2, 4\}, \{2, 6, 8\}, \{8, 2, 4, 10\} \}$$

- $\{ \varnothing \}$ is a set with 1 element which is the empty set.

  Note: $\{ \varnothing \} \neq \varnothing$

- $\{ a \} \neq \{\{ a \}\} \neq \{\{\{ a \}\}\} \neq \{\{\{\{\{\{ a \}\}\}\}\}\}$

- $\{ x \mid x \text{ is a subset of } \{a,b\} \} = \{\varnothing, \{a\}, \{b\}, \{a,b\} \}$

Russel's Paradox

Let S be the set whose elements are sets that do not contain themselves:

$$S = \{ x \mid x \notin x \}$$

Question: Does S contain itself? $S \in S$?

Answer:

case 1: suppose $S \in S$.

By definition S is a set of sets that do NOT contain themselves thus $S \notin S$ - contradiction!

case 2: suppose $S \notin S$.

By definition S is a set of sets that do NOT contain themselves thus $S \in S$ - contradiction!

Power Sets

<u>Definition</u>:
Given a set S, the *Power Set* (קבוצת החזקה) of S is the set of all subsets of S.
The power set of S is denoted *P(S).*

Examples:

$P(\{1, 2, 3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2,3\}, \{1,2,3\}\}$

$P(\varnothing) = \{\varnothing\}$

$P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$

Q: if $|S| = n$ then $|P(S)| = ?$

Cartesian Products

<u>Definition</u>:
The ordered *n-tuple* (n-יה סדורה) $(a_1,a_2,\ldots,a_n)$ is an ordered collection in which a_1 is first, a_2 second, ... and a_n as it's n-th element.

2 n-tuples are equal iff they have equal corresponding elements.

$$(a_1,a_2,\ldots,a_n) = (b_1,b_2,\ldots,b_n) \quad \text{iff} \quad a_i=b_i \ \text{ for } i=1,\ldots,n$$

Ordered pair:

$$(b , a) \neq (a , b)$$

Cartesian Products

<u>Definition</u>:
Let A and B be sets. The *Cartesian Product* (מכפלה קרטזית) of A and B is the set of all ordered pairs (a, b) where a is in A and b is in B.

$$A \times B = \{(a,b) \mid a \in A \, , \, b \in B \}$$

René Descartes 1596-1650

Examples:

$A = \{1, 2\}$, $B = \{a, b, c\}$

$A \times B = \{ (1,a),(1,b),(1,c),(2,a),(2,b),(2,c) \}$

Q: $A \times B = B \times A$? when is this true?

Cartesian Products

Generalization to n-tuples:

<u>Definition</u>:
The Cartesian Product of the sets $A_1,A_2,\ldots,A_n$ is the set of n-tuples $(a_1,a_2,\ldots,a_n)$ such that $a_i \in A_i$ for $i=1\ldots n$.

$A_1 x A_2 x \ldots x A_n = \{(a_1,a_2,\ldots,a_n) \mid a_i \in A_i \text{ , for } i=1\ldots n\}$

Examples:

$A = \{\Delta, \circ\}$, $B = \{1, 2\}$, $C = \{z\}$

$A \times B \times C = \{ (\Delta,1,z),(\Delta,2,z),(\circ,1,z),(\circ,2,z)\}$

Q: $|A \times B \times C| = ?$

Find the odd-man-out