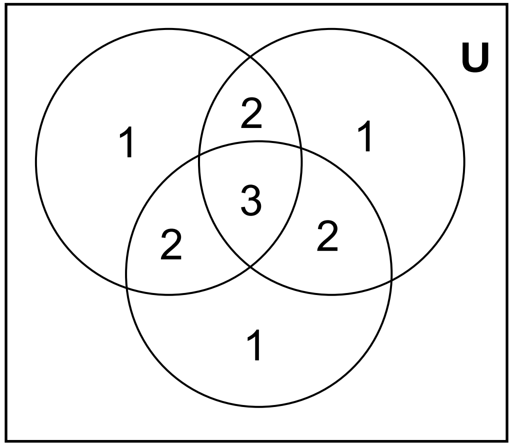
Lesson 8:

# Counting

- Sum & Product Rules
- Principle of Inclusion-Exclusion
- Pigeonhole Principle




Chapter 4.1-4.2

# Counting = Combinatorics

Dealing with discrete objects and processes, we need to:

- Count objects.
- Count # possibilities of ordering objects.
- Count # possibilities of choosing objects from a set.

Examples:

- How many internet addresses can be created under a given protocol.
- How many different network connections can be made in a computer cluster.
- How many operations are needed to complete an algorithm.

# The Sum Rule = חוק הסכום (חוק החיבור)

<u>Definition</u>: *The Sum Rule* חוק הסכום (חוק החיבור)

If a task can be applied in one of two ways and there are $n_1$ ways to perform the first way and $n_2$ to perform the second way then there is a total of $n_1 + n_2$ ways to perform the task.

# The Sum Rule = (חוק החיבור) חוק הסכום

<u>Example:</u>

Choose a representative for the CS dept. from year 2 or 3.

There are 107 students in year 2 and 91 students in year 3.

There are 107 possibilities of choosing a rep from year 2.

There are 91 possibilities of choosing a rep from year 3.

There is a total of 107+91 = 198 possibilities to choose a rep.

# The Sum Rule = חוק הסכום (חוק החיבור)

<u>Generalization:</u>

$T_1, T_2, \ldots, T_m$ are **independent** tasks that can be done in $n_1, n_2, \ldots, n_m$ ways respectfully.

then there are :

$n_1 + n_2 + \ldots + n_m$ ways to perform one of these task.

<u>Example:</u>

3rd year CS students are required to complete a project.
20 projects are offered by faculty
5 projects are offered by other departments.
10 projects are offered by industry.

Students have $20+5+10 = 35$ projects to choose from.

# The Sum Rule in Terms of Sets

If $A_1, A_2, \ldots, A_m$ are **disjoint** sets then the number of elements in the union of the sets equals the sum of the elements of the sets:

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$$

Number of ways to choose an element from $A_1$ or $A_2$ or ... $A_m$ (i.e. to choose from $A_1 \cup A_2 \cup \ldots \cup A_m$) :

$$\left.\begin{array}{l} |A_1| \text{ ways to choose from } A_1 \\ |A_2| \text{ ways to choose from } A_2 \\ \qquad\qquad \vdots \\ |A_m| \text{ ways to choose from } A_m \end{array}\right\} \quad |A_1| + |A_2| + \ldots + |A_m| \text{ ways to choose an element.}$$

# The Product Rule = חוק המכפלה

Definition: *The Product Rule* חוק המכפלה

If a task can be divided into 2 **independent** subtasks and there are $n_1$ ways to perform the first subtask and $n_2$ ways to perform the second subtask then there are $n_1 * n_2$ ways to perform the full task.

Intuition:

For every choice of subtask 1, one can choose ANY of the choices of subset 2 thus $n_1 * n_2$ .

## The Product Rule = חוק המכפלה

<u>Example:</u>

Choose a CS committed with 1 student from year 2 and 1 student from year 3.

There are 107 students in year 2.
There are 91 students in year 3.

There is a total of 107*91 = 9737 possibilities to choose a committee.

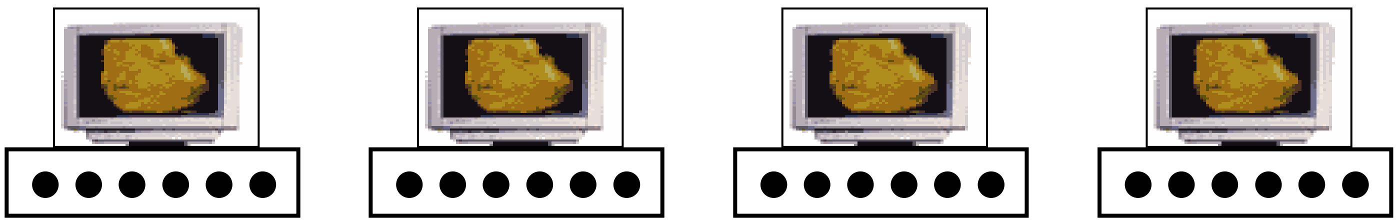
# The Product Rule = חוק המכפלה

Example:

There are 32 computers in the lab, each with 24 ports.
How many ports in the lab?

32 possibilities of choosing a computer
24 possibilities in each computer.

There is a total of 32*24 = 768 possibilities to choose port.



# The Product Rule = חוק המכפלה

<u>Generalization:</u>

If a task is performed by applying subtasks $T_1, T_2, \ldots, T_m$ in sequence and subtask $T_i$ can be performed in $n_i$ ways after $T_1, T_2, \ldots, T_{i-1}$ have been performed, then there are:

$n_1 * n_2 * \ldots * n_m$ ways to perform the full task.

<u>Example:</u>

How many bit strings are there of length 7?

$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{0/1}$

There are 2 possibilities for each bit (0/1).

Each of the 7 bits are chosen independently so there are $2*2*2*2*2*2*2 = 2^7 = 128$ bit strings.

# The Product Rule = חוק המכפלה

How many subsets does the set S have if $|S| = n$ ?

Answer:

There is a 1:1 and onto function between subsets of S and bit strings of length $|S|$.

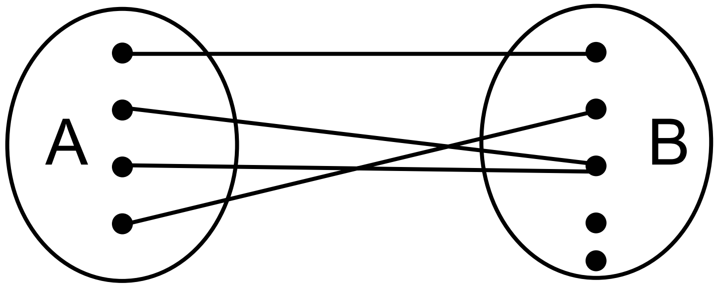
There are $2^{|S|}$ bit strings of length $|S|$ so there are $2^{|S|}$ subsets of S.

How many non empty sets? $2^{|S|} - 1$

How many subsets with more than 1 element? $2^{|S|} - 1 - |S|$

# The Product Rule = חוק המכפלה

How many function f: $A \rightarrow B$ can be defined if $|A| = m$ and $|B| = n$ ?



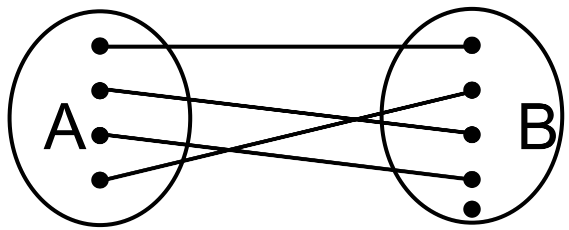

Answer:

For each of the m elements of the domain choose an element of the co-domain. Since there are no restrictions, there are n possibilities.

Thus there are $\underbrace{n*n*...*n}_{m} = n^m$

# The Product Rule = חוק המכפלה

How many **one-to-one** functions $f: A \rightarrow B$ can be defined if $|A| = m$ and $|B| = n$ ? ($m \leq n$)




Answer:

Suppose the domain elements are: $a_1, a_2, \ldots, a_m$.

$f(a_1)$ has n possibilities.

$f(a_2)$ has only n-1 possibilities left.

$f(a_3)$ has only n-2 possibilities left.

$f(a_m)$ has only n-m+1 possibilities left.

Thus there are $n*(n-1)*(n-2)*\ldots*(n-m+1)$

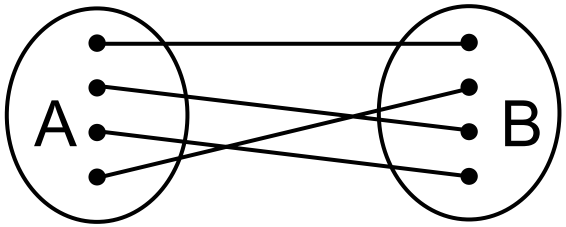
# The Product Rule = חוק המכפלה

How many **one-to-one** and **onto** functions f: $A \rightarrow B$ can be defined if $|A| = |B| = n$ ?

Answer:

In this case m = n so from previous result we have:

$$n*(n-1)*(n-2)*...*1 = n!$$



# The Product Rule in Terms of Sets

If $A_1, A_2, \ldots, A_m$ are finite sets then the number of elements in the Cartesian product of the sets equals the product of the number of elements in the sets:

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| * |A_2| * \ldots * |A_m|$$

Number of ways to choose one element from the Cartesian product equals the number of ways to chose one element from each set $A_i$ . There are $|A_i|$ possibilities for each i, thus a total of $\prod_{i=1}^{m} A_i$

# Sum & Product Rules in Programming

k := 0

for $i_1$ := 1 to $n_1$
  k:= k+1;
— $n_1$ ways for task 1

for $i_2$ := 1 to $n_2$
  k:= k+1;
— $n_2$ ways for task 2

•
•
•

for $i_m$ := 1 to $n_m$
  k:= k+1;
— $n_m$ ways for task m

---

k := 0

for $i_1$ := 1 to $n_1$ — for each of the $n_1$ choices for task 1..
  for i2 := 1 to $n_2$ — for each of the $n_2$ choices for task 2..
    for i2 := 1 to $n_2$ — for each of the $n_3$choices..
      •
      •
      •
        for $i_m$ := 1 to $n_m$
          k:= k+1;

endfor;

---

Each subtask (loop) is runs independently in sequence:
Total: $k = n_1+n_2+...+n_m$

Each subtask (loop) depends on previous. All loops at once.
Total: $k = n_1*n_2*...* n_m$

# Complex Counting Problems (Using both rules)

**Example**: Computer Passwords

Every user in a system has a password. The password must be 6-8 characters (letters or digits) and must contain at least one digit.
How many legal passwords are there?

Answer:

$p_n$ = number of legal passwords of length n.

$p = p_6 + p_7 + p_8$ Sum Rule

$p_6 =$ ?

# Complex Counting Problems (Using both rules)

To compute p6, count the number of strings of length 6 and subtract those with no digits.

Number of Characters = $\underbrace{26}_{\text{letters}} + \underbrace{10}_{\text{digits}} = 36$

Number of length 6 strings = $36^6$ Product Rule

Number of illegal length 6 strings (only letters) = $26^6$

$$p_6 = 36^6 - 26^6 \approx 1.8 \times 10^9$$

$$p_7 = 36^7 - 26^7 \approx 7 \times 10^{10}$$

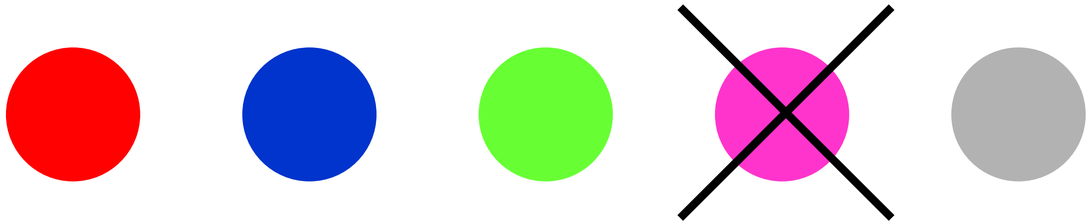
$$p_8 = 36^8 - 26^8 \approx 2.61 \times 10^{12}$$

$$p = p_6 + p_7 + p_8 \approx 2.68 \times 10^{12}$$

# Principle of Completion (Subtraction)
# (עקרון המשלים)

Count the number of elements that do NOT have a certain property.

Count the number of ways to perform a task but some of the ways are prohibited.



# Principle of Completion (Subtraction)
# (עקרון המשלים)

Definition:

*The Principle of Completion (Subtraction)*

(עקרון המשלים)

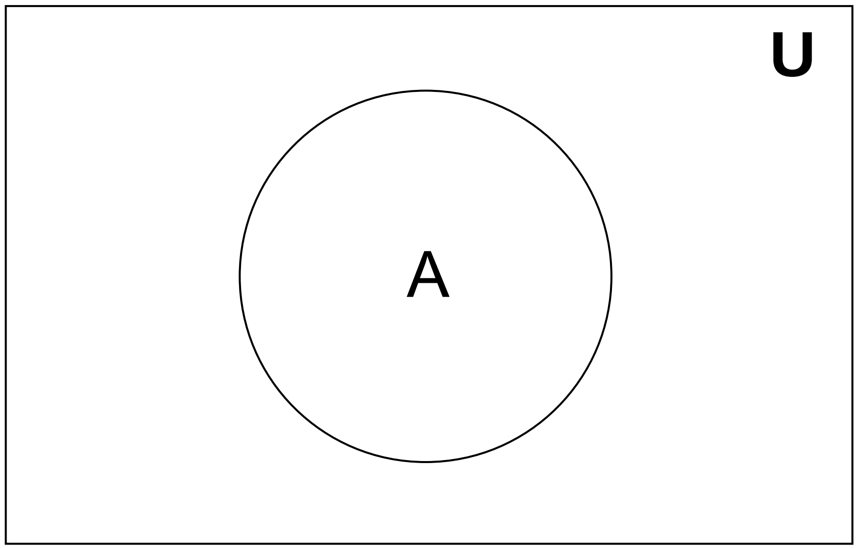
Suppose there are $N$ elements in the Universe of discussion, and $n_1$ of them have property P.

Then the number of elements that do **NOT** have property P equals $N - n_1$.

# Principle of Completion - in Terms of Sets

The number of elements **NOT** in the set A equals the number of elements in the Complement of A:

$$|\overline{A}| = |U| - |A|$$



# Principle of Completion (Subtraction)
# (עקרון המשלים)

<u>Example:</u>

How many positive integers smaller than 100 are **not** divisible by 3 ?

Answer:

There are 99 positive integers smaller than 100.

There are floor(100/3) positive integers divisible by 3.

Thus the number of positive integers smaller than 100 that are divisible by 3 is:

$$99 - \lfloor 100/3 \rfloor = 99 - 33 = 66$$

## Complex Counting Problems (Using three rules)

**Example**: IP Addresses

Every computer on the internet network is assigned an *Internet Address*. In Version 4 Internet Protocol (IPv4) an address is 32 bits and contains:

Network Number (Netid)

Host number (Hostid) = the id of the computer within the network.

There are 3 types of IP addresses dependent on network size.

# Complex Counting Problems (Using both rules)

Class A addresses - for very large networks (few bits for Netid (many bits for Hostid).

Class B addresses - for medium sized networks

Class C addresses - for small networks (many bits for Netid (few bits for Hostid).

| | | | | | |
|---|---|---|---|---|---|
| Class A | 0 | Netid 7-bit | | Hostid 24-bit | |
| Class B | 1 | 0 | Netid 14-bit | Hostid 16-bit | |
| Class C | 1 | 1 | 0 | Netid 21-bit | Hostid 8-bit |

Class A Netid can not be 1111111

Hostid (in all classes) can not be all 0 or all 1.

# Complex Counting Problems (Using both rules)

How many IP addresses are possible?

Answer:

p = number of legal IP addresses.

$p = p_A + p_B + p_C$ Sum Rule

$p_A = ?$

$2^7 - 1$ possible Netid (1111111 not allowed) Subtraction Rule

$2^{24} - 2$ possible Hostid (all 0 and all 1 not allowed)

$P_A = (2^7 - 1) * (2^{24}-2) \approx 2.1 \times 10^9$ Product Rule

$p_B = ?$ $p_C = ?$

$P_B = 2^{14} * (2^{16}-2) \approx 1.1 \times 10^9$

$P_C = 2^{21} * (2^8-2) \approx 5.3 \times 10^8$

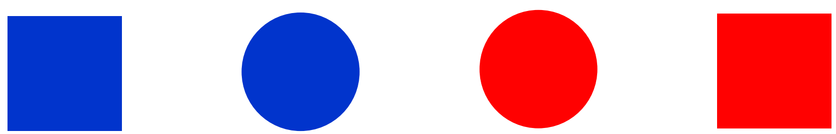
$p = p_A + p_B + p_C \approx 3.7 \times 10^9$ Sum Rule

# Principle of Inclusion - Exclusion
# (עקרון ההכלה וההדחה)

The Sum Rule assumes subtasks of a given task are disjoint.

If subtasks are not disjoint then counting possibilities of each subtask leads to over-counting.

Solution: Count possibilities of $1^{st}$ subtask, add the possibility of $2^{nd}$ subtask and **subtract** possibilities that are common to both.



# Principle of Inclusion - Exclusion
# (עקרון ההכלה וההדחה)

Definition: *The Principle of Inclusion-Exclusion*

עקרון ההכלה וההדחה

If a task can be applied in one of two ways and there are $n_1$ ways to perform the first way and $n_2$ to perform the second way and $n_{12}$ ways that are common then there is a total of $n_1 + n_2 - n_{12}$ ways to perform the task.

# Principle of Inclusion - Exclusion
# (עקרון ההכלה וההדחה)

Example:

Choose a representative for the CS dept. from year 2 or 3.

There are 107 students in year 2 and 91 students in year 3.

There are  8 students registered as both $2^{nd}$ and $3^{rd}$ year.

$$\# \text{ possibilities} = 107 + 91 - 8 = 190$$

# Principle of Inclusion - Exclusion
# (עקרון ההכלה וההדחה)

<u>Example:</u>

How many bit strings of length 8 start with 1 or end with 00 ?

Answer:

1) # of strings of length 8 that start with 1:

   $2^7 = 128$ Product Rule

2) # of strings of length 8 that end with 00:

   $2^6 = 64$ Product Rule

3) # of strings counted twice (start with 1 & end with 00):

   $2^5 = 32$ Product Rule

Total # of strings = 128 + 64 - 32 = 160

# Principle of Inclusion - Exclusion
# (עקרון ההכלה וההדחה)

<u>Example</u>: Geometry

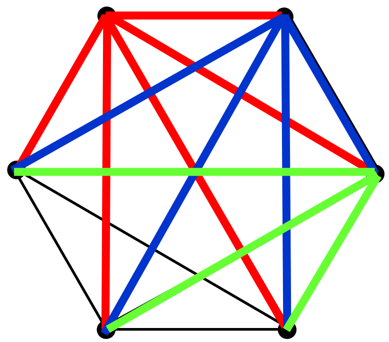
How many diagonals in an n-sided convex polygon?

Answer:

$1^{st}$ vertex: n-1 edges

$2^{nd}$ vertex: n-2 edges

$3^{rd}$ vertex: n-3 edges



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Total: (n-1)+(n-2)+(n-3)+ ... +1 $= \frac{n(n-1)}{2}$

Subtract edges $\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

# Principle of Inclusion - Exclusion
# (עקרון ההכלה וההדחה)

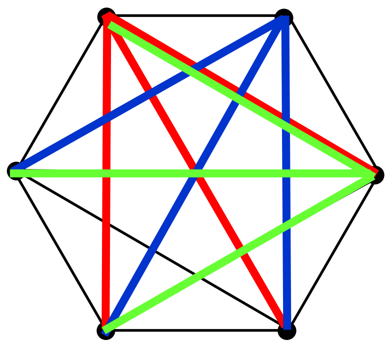
## Answer II:

Number of diagonals from each vertex: n-3

Number of vertices: n

Total = n*(n-3)

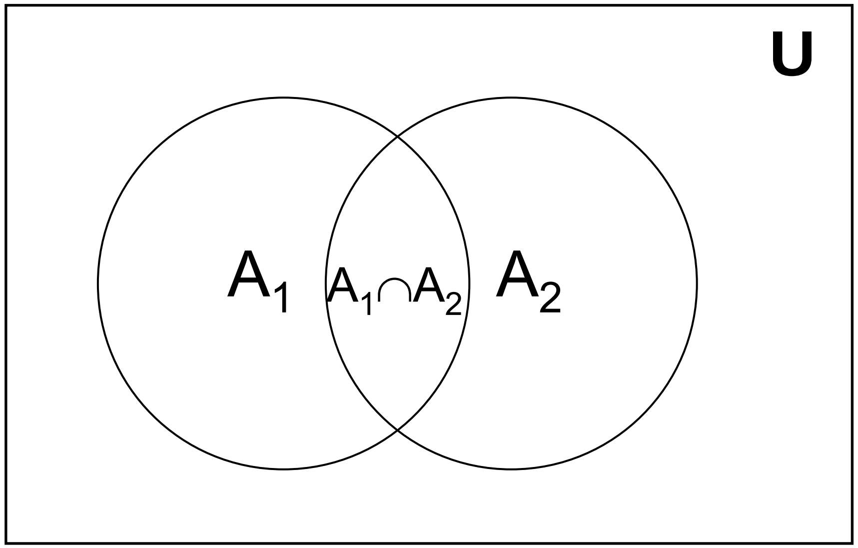
However diagonals were counted more than once (each diagonal is counted twice) so: $\frac{n(n-3)}{2}$



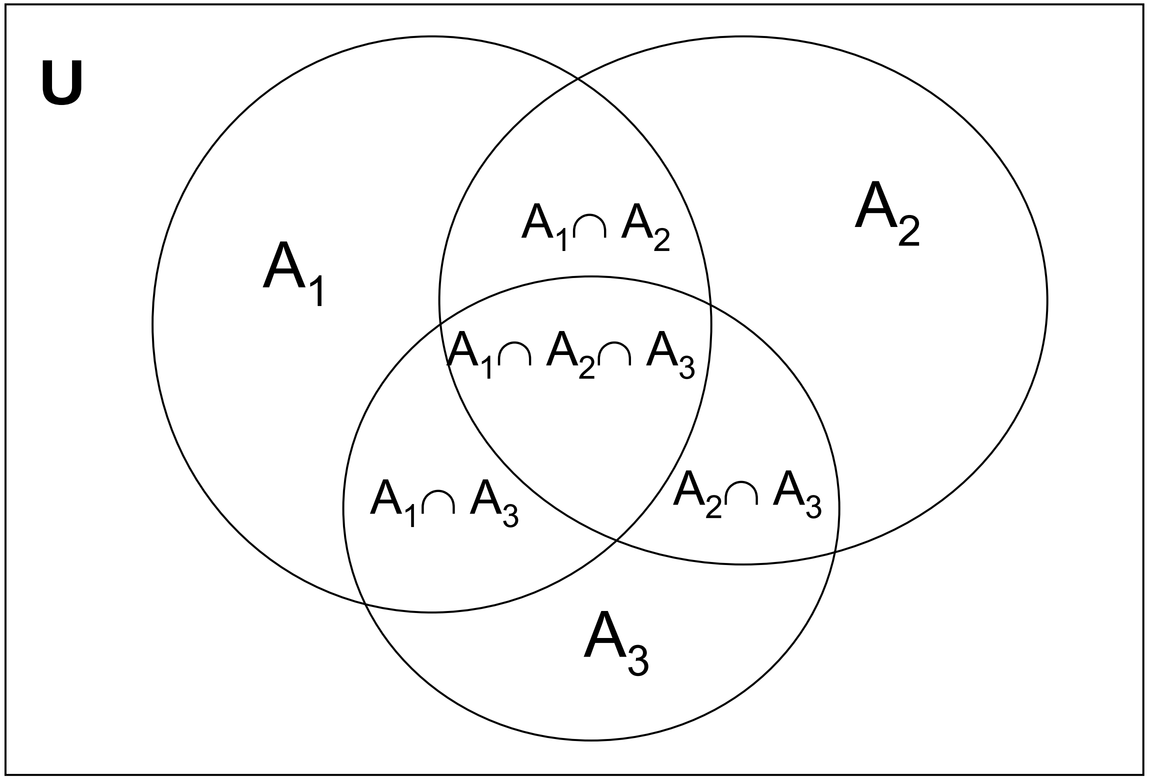
# Inclusion-Exclusion Principle in Terms of Sets

If $A_1$, $A_2$ are sets then the number of elements in the union of the sets equals the sum of the elements of the sets minus the elements in their intersection:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



# Generalizing the Inclusion-Exclusion Principle




$$|A_1 \cup A_2 \cup A_3| = ?$$

# Generalizing the Inclusion-Exclusion Principle

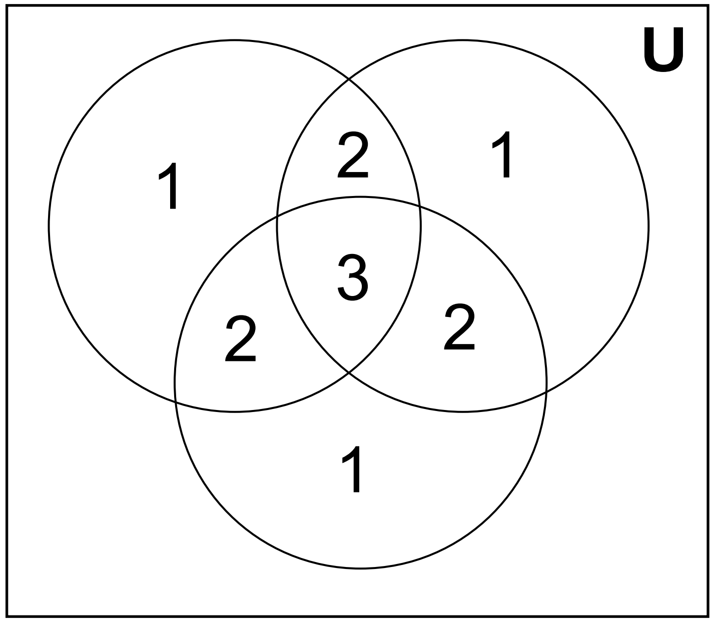
$|A_1| + |A_2| + |A_2|$

Counts:

Elements in 1 set : 1 time

Elements in 2 set : 2 times

Elements in 3 set : 3 times



# Generalizing the Inclusion-Exclusion Principle

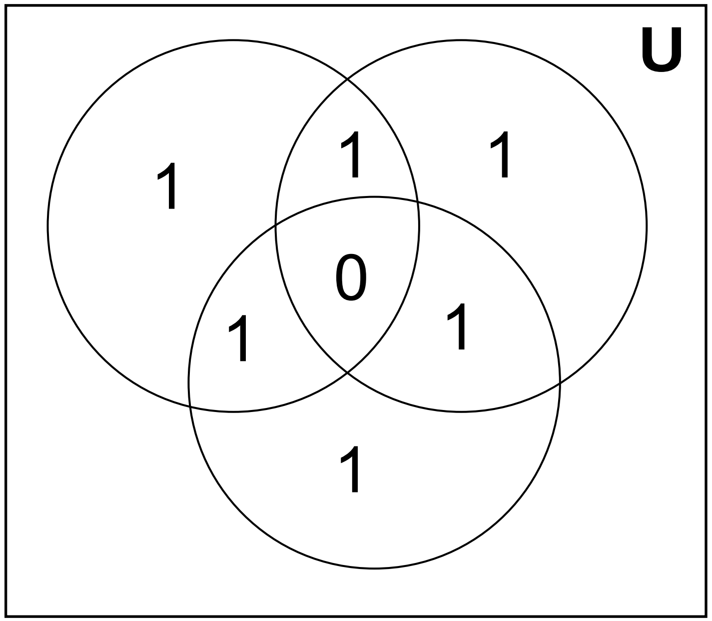
$|A_1| + |A_2| + |A_2| - |A_1 \cap A_2| - |A_1 \cap A_2| - |A_1 \cap A_2|$

Counts:

Elements in 1 set : 1 time

Elements in 2 set : 1 time

Elements in 3 set : 0 times



# Generalizing the Inclusion-Exclusion Principle

$$|A_1| + |A_2| + |A_2| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_3 \cap A_2|$$

$$+ |A_1 \cap A_2 \cap A_3|$$

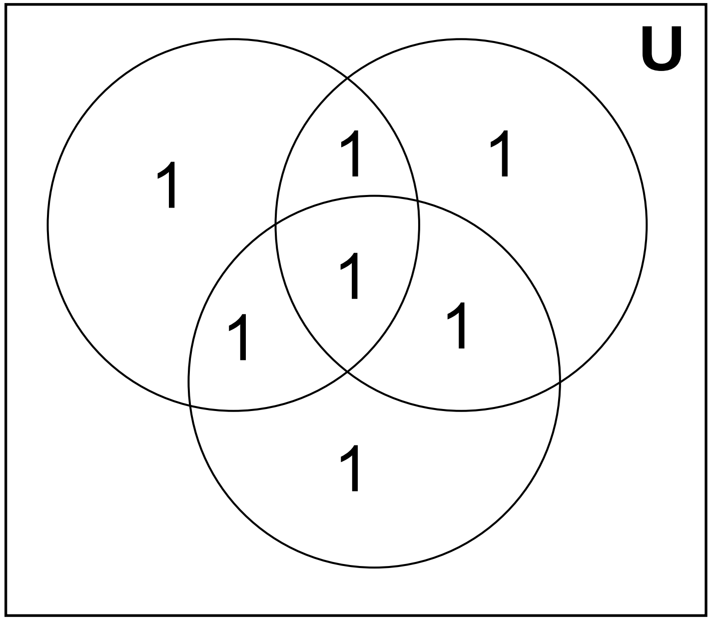
Counts:

Elements in 1 set : 1 time

Elements in 2 set : 1 time

Elements in 3 set : 1 time

$$= |A_1 \cup A_2 \cup A_3|$$



# Generalizing the Inclusion-Exclusion Principle

<u>Example</u>:

In the Faculty of Social Sciences the 156 students registered:

91 students in CS
82 students in EC
87 students in LAW

34 $CS \cap EC$
48 $EC \cap LAW$
27 $LAW \cap CS$

How many students registered in all three dept?

# Generalizing the Inclusion-Exclusion Principle

Answer:

$$|CS \cup EC \cup LAW| = |CS| + |EC| + |LAW| -$$

$$- |CS \cap EC| - |EC \cap LAW| - |LAW \cap CS|$$

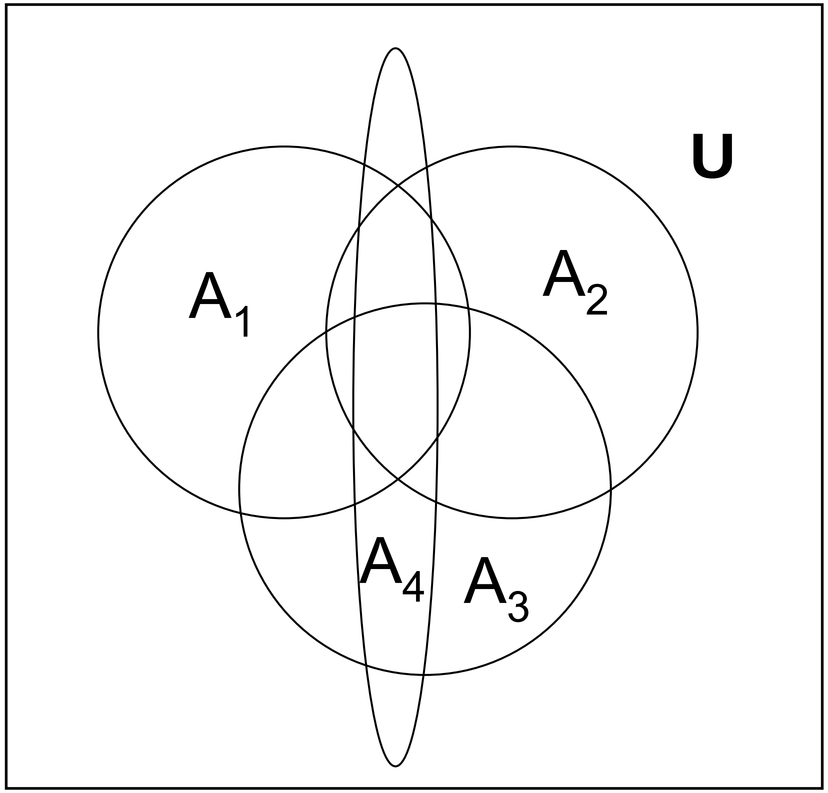
$$+ |CS \cap EC \cap LAW|$$

$$156 = 91 + 82 + 87 - 34 - 48 - 27 + |CS \cap EC \cap LAW|$$

$$|CS \cap EC \cap LAW| = 5$$

# Generalizing the Inclusion-Exclusion Principle

$|A_1 \cup A_2 \cup A_3 \cup A_4| = ?$



# Principle of Inclusion - Exclusion
# (עקרון ההכלה וההדחה)

**Theorem**: *The General Principle of Inclusion-Exclusion*

עקרון ההכלה וההדחה

Let $A_1, A_2, \ldots, A_n$ be finite sets then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots\ldots (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

# Principle of Inclusion - Exclusion
# &
# Principle of Completion

Count the number of elements in the set U of N elements that do not have any of the properties $P_1, ..., P_n$.

Let $A_i$ be the set of elements that have property $P_i$.

$A_1 \cup ... \cup A_n$ = the set of elements with 1 or more properties.

The # of elements without properties $P_1, ..., P_n$ :

$$|\overline{A_1 \cup ... \cup A_n}| = N - |A_1 \cup ... \cup A_n| = N - \Sigma|A_i| + \Sigma|A_i \cap A_j| + ...$$

$$.... \ (-1)^n|A_1 \cap A_2 \cap ... \cap A_n|$$

# Principle of Inclusion - Exclusion
# (עקרון ההכלה וההדחה)

<u>Example:</u>

How many positive integers smaller or equal to 1000 are **not** divisible by 7 or 11 ?

Answer:

$A_7$ = set of numbers divisible by 7

$A_{11}$ = set of numbers divisible by 11

$|A_7| = \text{floor}(1000/7) = 142$

$|A_{11}| = \text{floor}(1000/11) = 90$

$|A_7 \cap A_{11}| = \text{floor}(1000/(7*11) = 12$

$$|U - (A_7 \cup A_{11})| = 1000 - (|A_7| + |A_{11}| - |A_7 \cap A_{11}|)$$

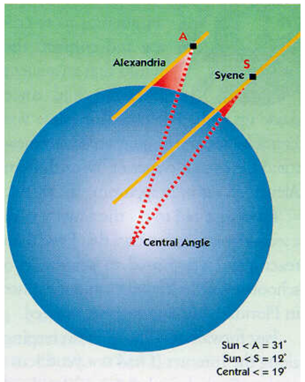
$$= 1000 - (142 + 90 - 12) = 780$$

# Principle of Inclusion - Exclusion
# &
# Principle of Completion

<u>Example</u>:

The Sieve of Eratosthenes   הנפה של ארטוסטנוס

Eratosthenes (276-194 B.C.E.)




# The Sieve of Eratosthenes

Find the number of primes smaller than 100.

Answer:

Composite integers have at least one prime divisor and it must be smaller than the square root of the integer.

Composite integers smaller than 100 must be divisible by 2,3,5 or 7.

The primes smaller than 100 are the integers that are **NOT** divisible by 2,3,5 or 7.

# The Sieve of Eratosthenes

## Answer cont.

$A_2, A_3, A_5, A_7 =$ the set of integers divisible by 2,3,5,7 respectively.

The # of primes smaller than 100 :

$$|\overline{A_2 \cup A_3 \cup A_5 \cup A_7}| + \underbrace{4}_{|\{2,3,5,7\}|}$$

According to the Principle of Inclusion - Exclusion & the Principle of Completion :

# The Sieve of Eratosthenes

Answer cont. # of primes =

$$= 100 - ( |A_2| + |A_3| + |A_5| + |A_7| )$$

$$+ (|A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7|)$$

$$- (|A_2 \cap A_3 \cap A_5| + |A_2 \cap A_3 \cap A_7| + |A_2 \cap A_5 \cap A_7| + |A_3 \cap A_5 \cap A_7|)$$

$$+ |A_2 \cap A_3 \cap A_5 \cap A_7| + 4$$

$$= 100 - \left( \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{7} \right\rfloor \right)$$

$$+ \left( \left\lfloor \frac{100}{2 \cdot 3} \right\rfloor + \left\lfloor \frac{100}{2 \cdot 5} \right\rfloor + \left\lfloor \frac{100}{2 \cdot 7} \right\rfloor + \left\lfloor \frac{100}{3 \cdot 5} \right\rfloor + \left\lfloor \frac{100}{3 \cdot 7} \right\rfloor + \left\lfloor \frac{100}{5 \cdot 7} \right\rfloor \right)$$

$$- \left( \left\lfloor \frac{100}{2 \cdot 3 \cdot 5} \right\rfloor + \left\lfloor \frac{100}{2 \cdot 3 \cdot 7} \right\rfloor + \left\lfloor \frac{100}{2 \cdot 5 \cdot 7} \right\rfloor + \left\lfloor \frac{100}{3 \cdot 5 \cdot 7} \right\rfloor \right)$$

$$+ \left\lfloor \frac{100}{2 \cdot 3 \cdot 5 \cdot 7} \right\rfloor + 4 = 100 - 117 + 45 - 6 + 4 = 26$$

# The Sieve of Eratosthenes

Find all the primes not exceeding a given number.

Algorithm:

- Erase all numbers divisible by 2 except 2.
- Look for next un-erased number x and erase all numbers divisible by x.
- Repeat

# The Sieve of Eratosthenes

Find all the primes not exceeding 100.

Erase all numbers divisible by 2 except 2.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | | 5 | | 7 | | 9 | |
| 11 | | 13 | | 15 | | 17 | | 19 | |
| 21 | | 23 | | 25 | | 27 | | 29 | |
| 31 | | 33 | | 35 | | 37 | | 39 | |
| 41 | | 43 | | 45 | | 47 | | 49 | |
| 51 | | 53 | | 55 | | 57 | | 59 | |
| 61 | | 63 | | 65 | | 67 | | 69 | |
| 71 | | 73 | | 75 | | 77 | | 79 | |
| 81 | | 83 | | 85 | | 87 | | 89 | |
| 91 | | 93 | | 95 | | 97 | | 99 | |

# The Sieve of Eratosthenes

Find all the primes not exceeding 100.

Erase all numbers divisible by 3 except 3.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | | 5 | | 7 | | | |
| 11 | | 13 | | | | 17 | | 19 | |
| | | 23 | | 25 | | | | 29 | |
| 31 | | | | 35 | | 37 | | | |
| 41 | | 43 | | | | 47 | | 49 | |
| | | 53 | | 55 | | | | 59 | |
| 61 | | | | 65 | | 67 | | | |
| 71 | | 73 | | | | 77 | | 79 | |
| | | 83 | | 85 | | | | 89 | |
| 91 | | | | 95 | | 97 | | | |

# The Sieve of Eratosthenes

Find all the primes not exceeding 100.

Erase all numbers divisible by 5 except 5.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | | 5 | | 7 | | | |
| 11 | | 13 | | | | 17 | | 19 | |
| | | 23 | | | | | | 29 | |
| 31 | | | | | | 37 | | | |
| 41 | | 43 | | | | 47 | | 49 | |
| | | 53 | | | | | | 59 | |
| 61 | | | | | | 67 | | | |
| 71 | | 73 | | | | 77 | | 79 | |
| | | 83 | | | | | | 89 | |
| 91 | | | | | | 97 | | | |

# The Sieve of Eratosthenes

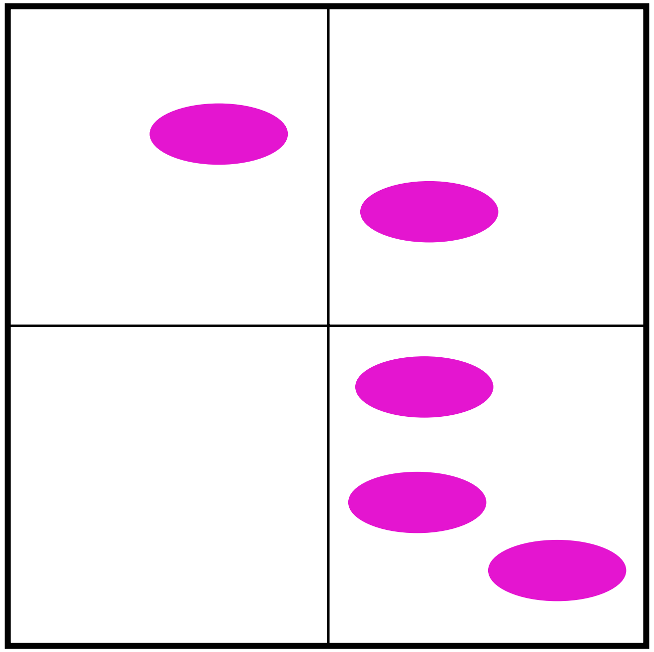
Find all the primes not exceeding 100.

Erase all numbers divisible by 7 except 7.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | | 5 | | 7 | | | |
| 11 | | 13 | | | | 17 | | 19 | |
| | | 23 | | | | | | 29 | |
| 31 | | | | | | 37 | | | |
| 41 | | 43 | | | | 47 | | | |
| | | 53 | | | | | | 59 | |
| 61 | | | | | | 67 | | | |
| 71 | | 73 | | | | | | 79 | |
| | | 83 | | | | | | 89 | |
| | | | | | | 97 | | | |

# The Pigeonhole Principle   עקרון שובך היונים

If 5 pigeons fly into a set of 4 pigeonholes then at least one hole will contain at least 2 pigeons.



# The Pigeonhole Principle עקרון שובך היונים

**Theorem**: *The Pigeonhole Principle* עקרון שובך היונים

If n+1 or more objects are placed into n boxes, then there is at least one box containing two or more of the objects.

**Proof:**

Suppose that none of the boxes has two or more objects then each box has at most 1 objects, and the total number of objects in boxes is at most n - contradiction!

# The Pigeonhole Principle   עקרון שובך היונים

Also called

The Dirichlet Drawer Principle   עקרון המגירה של דיריכלה

G. Lejeune Dirichlet  (1805-1859)

# The Pigeonhole Principle עקרון שובך היונים

Examples:

Among 367 people there must be at least 2 with the same birthday (because there are only 366 possibilities).

In any set of 23 Hebrew words, at least 2 words will start with the same letter (because there are only 22 letters).

How many students must be in a class to ensure that there are at least 2 with the same grade (1..100)?
Answer: 100

# The Pigeonhole Principle עקרון שובך היונים

**Theorem**:

A computer network has 6 computers. Each is connected to 0 or more of the other computers. Show that at least 2 have the same number of connections.

**Proof**:

A computer can be connected to 0..5 other computers.
However both 0 and 5 can not happen in the network.
So there are 5 possible number of connections (1..5/0).
There are 6 computers so by the Pigeonhole Principle there are at least 2 computers with the same number.

# The Generalized Pigeonhole Principle

**Theorem**: *The Generalized Pigeonhole Principle*

הכללת עקרון שובך היונים

If n objects are placed into k boxes, then there is at least one box containing at least

$\left\lceil \frac{n}{k} \right\rceil$ objects.

**Proof:**

Suppose that none of the boxes has $\left\lceil \frac{n}{k} \right\rceil$ or more objects then each box has at most $\left\lceil \frac{n}{k} \right\rceil - 1$ objects. Then, the total number of objects in boxes is at most:

$$k\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) < k\left(\left(\frac{n}{k} + 1\right) - 1\right) = n$$

# The Generalized Pigeonhole Principle

Examples:

Among 100 people there must be at least $\left\lceil \frac{100}{12} \right\rceil = 9$ born in the same month.

Suppose that in the DM class there are 25 students from years 1, 2 and 3. Then at least $\left\lceil \frac{25}{3} \right\rceil = 9$ are in same year.

## The Generalized Pigeonhole Principle

Examples:

In the DM class there are 25 students from years 1, 2 and 3.

Show that (at least) one of the following must be true:

At least 3 are in year 1

At least 19 are in year 2

At least 5 are in year 3

**Proof:**

If not then there are at most 2 in year 1, 18 in year 2 and 4 in year 3 giving a total of 2+18+4 = 24 contradiction to there being 25!

# The Generalized Pigeonhole Principle

What is the minimum number of objects n so that at least r of these objects must be in one of k boxes?

From the Pigeonhole Principle, n must satisfy: $\left\lceil \frac{n}{k} \right\rceil \geq r$

or $n/k > r-1$ or $n > k(r-1)$

Thus the smallest n must satisfy: $n = k(r-1) + 1$

# The Generalized Pigeonhole Principle

Examples:

How many cards must be selected from a deck of 52 to guarantee that at least 3 cards are from the same suit?

There are 4 suits. Assume 1 pile for each suit. Then from the Pigeonhole Principle, to ensure 3 cards in a pile the number of cards n must satisfy: $\left\lceil \frac{n}{4} \right\rceil \geq 3$

The smallest n is then: $4*2 +1 = 9$

How many cards must be selected to guarantee 3 hearts ?

# The Generalized Pigeonhole Principle

Examples:

How many area codes must be used so 25 billion people will have different phone numbers. A phone number has 10 digits and does not start with 1.

Answer:

The number of possible phone numbers is: $8*10^9$

From the Pigeonhole Principle at least $\left\lceil \frac{25*10^9}{8*10^9} \right\rceil = 4$

will have the same number thus

4 area codes are needed.

# Elegant use of The Pigeonhole Principle

**Theorem**:

For every set of n+1 positive integers not exceeding 2n there must be 2 integers such that one divides the other.

**Proof**:

Assume $a_1,\ldots, a_{n+1}$ integers. Write them as $a_j = 2^{kj} * q_j$ where $q_j$ is odd and $k_j$ is non-Negative.

Since $1 \le a_j \le 2n$ we have $1 \le q_j \le 2n$ and there are only n such odd integers.

From the Pigeonhole Principle at least 2 $q_j$'s are the same.

Assume $q_i = q_j = q$. We have $a_j = 2^{kj} * q$ $a_i = 2^{ki} * q$

Then: $\begin{cases} a_j \mid a_i & \text{if } k_j < k_i \\ a_i \mid a_j & \text{if } k_i < k_j \end{cases}$

# Elegant use of The Pigeonhole Principle

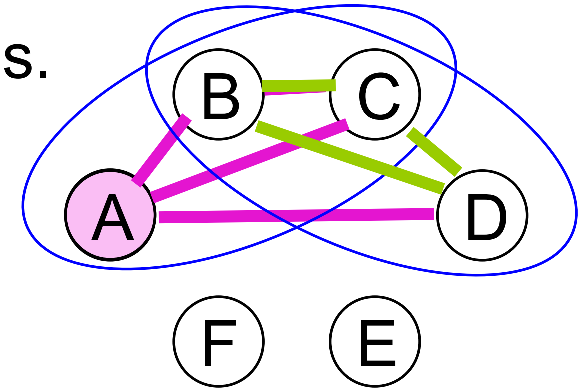
**Theorem**:

In a party of 6 each pair are either mutual friends or mutual enemies. Show that there must be 3 people that are mutual friends or mutual enemies (3-clique).

**Proof**:

Let A be one of the party. Amongst the 5 others there are $\lceil 5/2 \rceil = 3$ that are friends/enemies of A. Call them B,C,D.

If 1 pair of B,C,D are friends then, with A they are a 3-clique else they are a 3-clique of enemies.



# Elegant use of The Pigeonhole Principle

**Theorem:**

Every sequence of $n^2+1$ different numbers, must contain a subsequence of length n+1 which is strictly increasing or strictly decreasing.

Example: n = 2 Sequence of $2^2+1$ : 3 1 4 2 5

(3,2) (3,1) (2,2) (2,1) (1,1)

**Proof:**

Let $a_1, a_2, \dots, a_{n^2+1}$ be a sequence of $n^2+1$ numbers.

For each $a_k$ assign an ordered pair $(i_k, d_k)$

$i_k$ = the longest **increasing** sequencing starting at $a_k$

$d_k$ = the longest **decreasing** sequencing starting at $a_k$

# Elegant use of The Pigeonhole Principle

**Proof Cont**:

Assume there are no increasing and decreasing/sequences of length n+1. Thus $i_k, \leq n$, $d_k \leq n$.

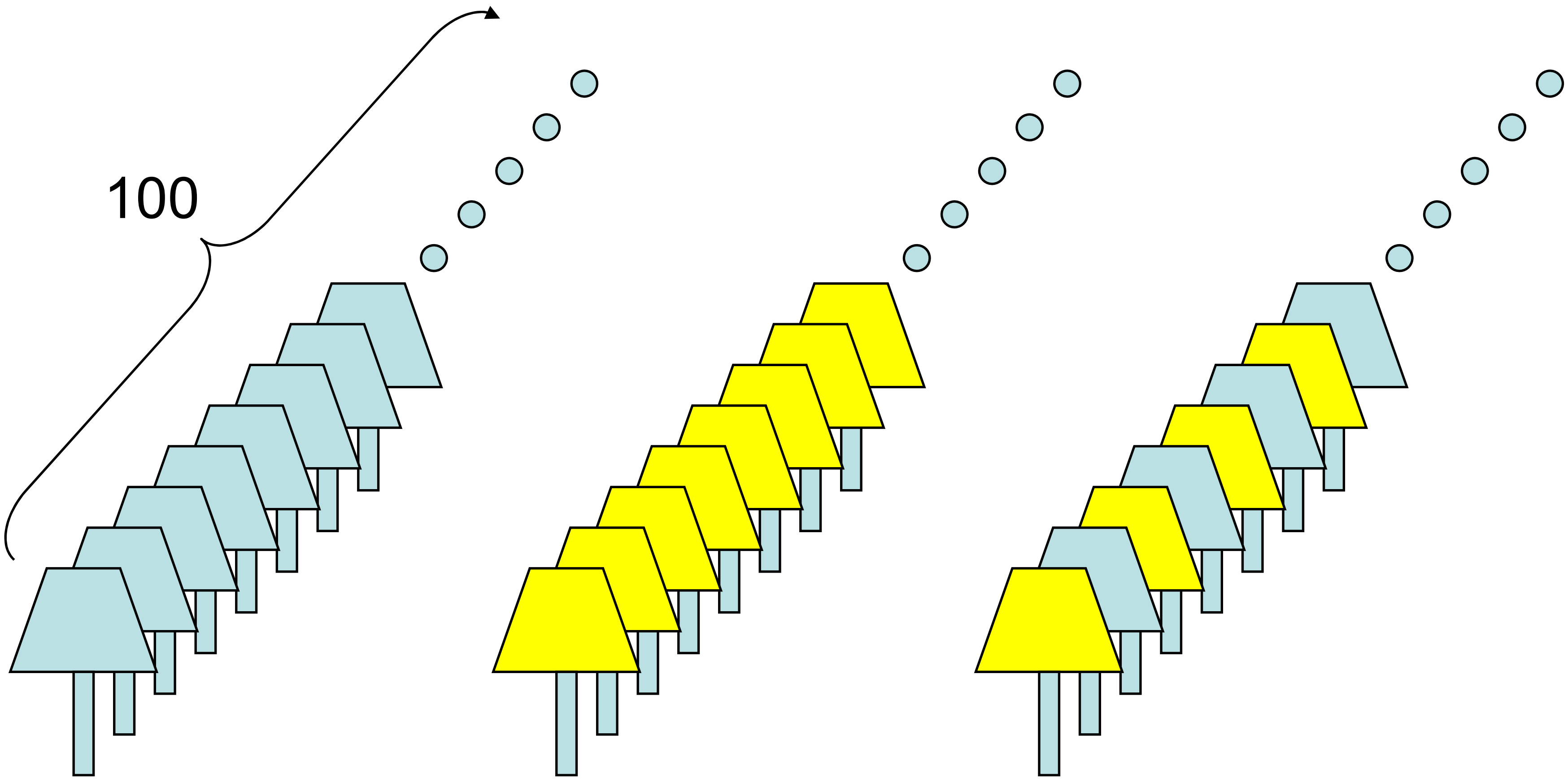
From the product rule there are $n^2$ possible pairs $(i_k, d_k)$.

But we have $n^2+1$ pairs, thus from the Pigeonhole Principle there must be 2 pairs that are the same: $(i_s, d_s) = (i_t, d_t) = (i, d)$ corresponding to $a_s$ and $a_t$ for $s < t$.

Case 1: if $a_s < a_t$ then we add $a_s$ to the beginning of the increasing sequence starting at $a_t$ and we have a sequence of length i+1 - contradiction!

Case 2: if $a_s > a_t$ then we add $a_s$ to the beginning of the decreasing sequence starting at $a_t$ and we have a sequence of length i+1 - contradiction!

# Counting Fun - Riddle




The $i^{th}$ dwarf changes state of all lamps in position k*i

How many lamps will be 'on' after the $100^{th}$ dwarf went by?