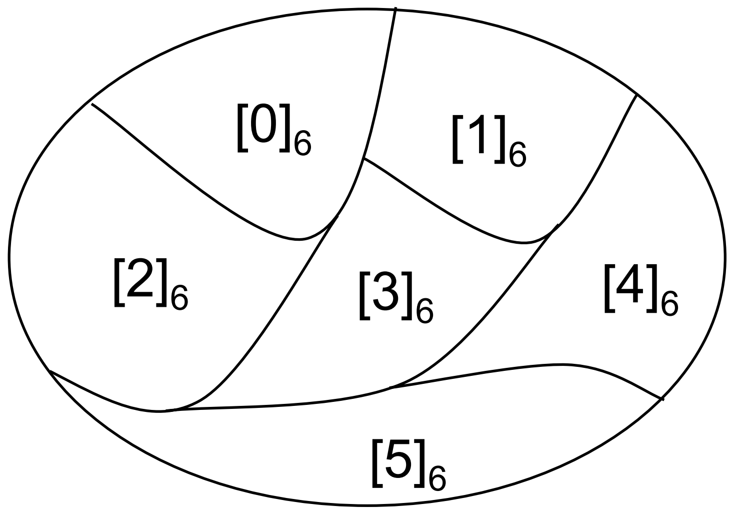
Lesson 11:

Relations II

- Equivalence Relations
- Equivalence Classes
- Partial Ordered Sets
- Linear Ordered Sets
- Well Ordered Sets
- Topological Sorting



Chapter 7.5-7.6

Equivalence Relations

Example:

DM students are divided into 3 classrooms for the final exam according to the first letter of their family name:

A - H - room 1

I - N - room 2

M - Z - room 3

R is a relation on the students. $(a,b) \in R$ iff a and b are in the same group of family names.

R is Reflexive, Symmetric and Transitive.

R divides the students into 3 groups.

Equivalence Relations

<u>Example:</u>

R is a relation on the set of Hebrew words.

a R b iff length of a equals length of b.

R is Reflexive, Symmetric and Transitive.

R divides the Hebrew words into groups.

Each group contains words of equal length.

אנציקלופדיה	משטרה	מכתב	אבא
קלסטרופוביה	שולחן	תפוח	אמא
	פסנתר	קורס	דשא

Equivalence Relations

Definition:

A relation R on a set A is called *Equivalence Relation* (יחס שקילות) if it is reflexive, symmetric and transitive.

Definition:

2 elements of A that are related by an equivalence relation are called *Equivalent* (שקולים).

- An element is equivalent to itself (reflexivity).
- If a and b are equivalent and b and c are equivalent then a and c are equivalent (transitivity).

Equivalence Relations

<u>Example:</u>

R is a relation on the real numbers.

aRb iff a-b is a whole number.

Is R an Equivalence relation?

Answer:

1) R is Reflexive.

Proof: For real a, a-a = 0 and 0 is whole.

So $\forall a\ (a,a) \in R$.

Equivalence Relations

Answer cont:

2) R is Symmetric.

Proof: if $(a,b)\in R$ then, $a-b = k$ s.t. k is whole. But then $b-a = -k$ and $-k$ is also whole. So $(b,a) \in R$.

3) R is Transitive.

Proof: if $(a,b)\in R$ and $(b,c)\in R$ then, $a-b = k$ and $b-c = l$ s.t. k and l are whole. But then $a-c = (a-b)+(b-c) = k+l$ is whole. So $(a,c) \in R$.

Equivalence Relations

Example:

R is a relation on $\mathcal{Z}$. $m \in \mathcal{Z}^+$, $m>1$.

$R_m = \{ (a,b) \mid \; m \mid (a-b) \}$

Is R an Equivalence relation?

Answer:

1) R_m is Reflexive.

Proof: $a-a = 0$ and $m|0$. So $(a,a) \in R_m$.

Equivalence Relations

Answer cont:

2) R is Symmetric.

Proof: if $(a,b) \in R_m$ then, $m|a-b$ i.e. $(a-b) = km$ s.t. $k \in \mathcal{Z}$.

But then $(b-a) = (-k)m$ and $-k$ is also whole. So $m|(b,a)$ and $(b,a) \in R_m$.

3) R is Transitive.

Proof: if $(a,b) \in R_m$ and $(b,c) \in R_m$ then, $m|(a-b)$ and $m|(b-c)$

i.e. $a-b = km$ and $b-c = lm$ s.t. k and l are whole.

But then $a-c = (a-b)+(b-c) = km+lm = (k+l)m$ and

$k+l$ is whole. So $m|(a-c)$ and $(a,c) \in R_m$.

Equivalence Relations

R_m divides $\mathcal{Z}^+$ into groups.

What are these groups?

$(a,b) \in R_m$ iff $(a-b)$ is a multiple of m

0, m, 2m, 3m, ...

1, 1+m, 1+2m, 1+3m, ...

$\vdots$

m-1, m-1+m, m-1+2m, m-1+3m, ...

$R_m = \{ (a,b) \mid a \equiv b \pmod{m} \}$

Equivalence Classes

<u>Definition</u>:

Let R be an equivalence relation on set A.

The set of all elements of A that are related R to $a \in A$ is called the *Equivalence Class* (מחלקת השקילות) of a.

The set is denoted $[a]_R$

If only the relation is obvious then denote by [a].

$$[a]_R = \{ s \mid (a, s) \in R \}$$

Equivalence Classes

<u>Example:</u>

$R = \{ (a,b) \mid a \equiv b \pmod 4 \}$

$[0]_R = \{ \ldots, -8, -4, 0, 4, 8, 12, \ldots \}$ integers with remainder 0

$[1]_R = \{ \ldots, -7, -3, 1, 5, 9, 13, \ldots \}$ integers with remainder 1

$R = \{ (a,b) \mid a \equiv b \pmod m \}$

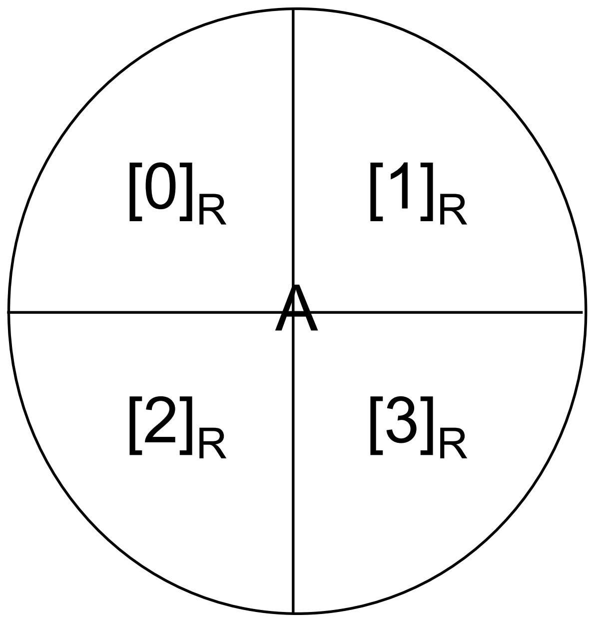
$[a]_m = \{ \ldots, a-2m, a-m, a, a+m, a+2m, 2+3m, \ldots \}$

Equivalence Classes

Example:

Equivalence R on set A divides A into disjoint equivalence classes.

$$R = \{ (a,b) \mid a \equiv b \pmod 4 \}$$



Equivalence Classes

Theorem: Let R be an equivalence relation on set A.

The following 3 statements are equivalent:

1) $(a,b) \in R$
2) $[a] = [b]$
3) $[a] \cap [b] \neq \emptyset$

Corollary: Every pair of equivalence classes are either identical or equivalent.

Equivalence Classes

Proof:

I. Prove 1) $\rightarrow$ 2)

Assume aRb and show equality of the sets [a]=[b].

A) Prove $[a] \subseteq [b]$.

Assume $c \in [a]$, thus

$(a,c) \in R$	(def. of equiv. class)
$(a, b) \in R$	(assumption 1))
$(b, a) \in R$	(R is symmetric)
$(b, c) \in R$	(R is transitive)
$c \in [a]$	(def. of equiv. class)

Thus $[a] \subseteq [b]$.

B) Prove $[b] \subseteq [a]$ similarly

Equivalence Classes

Proof cont.:

II. Prove 2) $\rightarrow$ 3)

Assume [a]=[b] and show $[a] \cap [b] \neq \emptyset$

Since [a]=[b], $[a] \cap [b] = [a] = [b]$

We must show that [a] is not empty.

However $a \in [a]$, since R is reflexive and $(a,a) \in R$.

III. Prove 3) $\rightarrow$ 1)

Assume $[a] \cap [b] \neq \emptyset$ and show aRb.

There exists $c \in [a] \cap [b]$ not empty set.

$(a,c) \in R$, $(b,c) \in R$ def. of equiv. class.

$(c,b) \in R$ R is symmetric

$(a,b) \in R$ R is transitive

Equivalence Classes

Theorem: Every equivalence relation R on set A induces a *Partition* (חלוקה) of A.

<u>Definition</u>:

A *Partition* (חלוקה) of a set A is a collection of nonempty subsets A_i of A such that :

$$A_i \neq \varnothing$$

$$A_i \cap A_j = \varnothing$$

$$\bigcup_i A_i = A$$

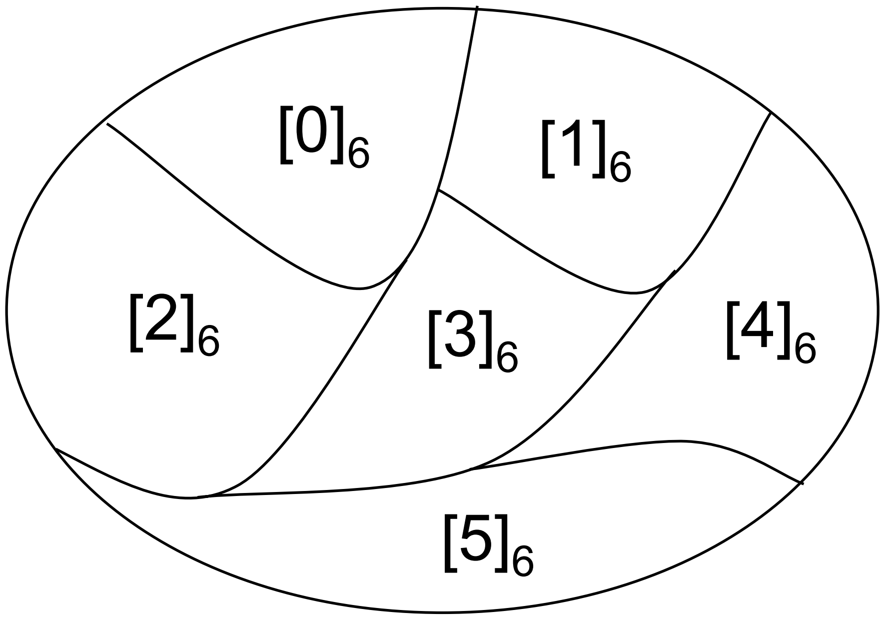
Proof : from previous theorem and corollary.

Equivalence Classes

Example:

$R_6 = \{ (a,b) \mid a \equiv b \pmod 6 \}$

Partition induced by R_6 :



Equivalence Classes

Theorem: Every partition of set A induces an equivalence relation R on A.

Proof : Assume $\{ A_i \}_{i = 1..n}$ is a partition of A.

Define a relation R on A: $(a,b) \in R$ if a and b are in the same subset A_i.

Prove that R is an equivalence relation.

1) R is reflexive:

$(a,a) \in R$ since $a \in A$

2) R is symmetric

if $(a,b) \in R$ then $(b,a) \in R$ by definition of R (same sets).

Equivalence Classes

Proof cont. :

2) R is symmetric

if $(a,b) \in R$ then $(b,a) \in R$ definition of R (same sets).

3) R is transitive:

assume $(a,b) \in R$ and $(b,c) \in R$ then

$a,b \in A_i$ $b,c \in A_j$ ($\{A_i\}$ is a cover of A)

$(b \in A_i) \wedge (b \in A_j) \rightarrow A_i = A_j$ (since A_i and A_j are either equal or disjoint)

Thus $a,b,c \in A_i$ and $(a,c) \in R$ (definition of R).

Partial Orderings

Relations can be used to order objects:

R = { (a,b) | if a precedes b alphabetically } — Alphabetical order

R = { (a,b) | if a is a task that must be completed before b} — Task scheduling

R = { (a,b) | if a < b} — Numerical order

Adding (a,a) $\in$ R for all a, R becomes:
Reflexive, Anti-Symmetric, and Transitive.

Partial Orderings

<u>Definition</u>:

A relation R on a set A is called a *Partial Order* (יחס סדר חלקי) if it is reflexive, anti-symmetric and transitive.

<u>Definition</u>:

A set A together with a partial order R is called a Partially Ordered Set or Poset (קבוצה סדורה חלקית).

Denoted: (A,R)

Partial Orderings

<u>Example:</u>

The relation $\geq$ is a partial order on the integers.

Reflexive - since $a \geq a$

Antisymmetric - since $a \geq b$ and $b \geq a$ implies $a = b$

Transitive - if $a \geq b$ and $b \geq c$ then $a \geq c$.

$(\mathcal{Z}, \geq)$ - is a partially ordered set.

Partial Orderings

<u>Example:</u>

The relation | ("divides") is a partial order on $\mathcal{Z}^+$.

Reflexive - since a | a

Antisymmetric - since a | b and b | a implies a = b

Transitive - if a | b and b | c then a | c.

($\mathcal{Z}^+$, |) - is a partially ordered set.

Partial Orderings

<u>Example:</u>

The relation $\subseteq$ on the power set P(S) is a partial order .

Reflexive - since $T \subseteq T$ for any subset of S.

Antisymmetric - since $T \subseteq Q$ and $Q \subseteq T$ implies $T = Q$

Transitive - if $T \subseteq Q$ and $Q \subseteq U$ then $T \subseteq U$.

(P(S) , $\subseteq$) - is a partially ordered set.

Partial Orderings

Definition:

Two elements a,b of a partially ordered set (A, R) are called *Comparable* (ברי השוואה) if aRb or bRa. Otherwise they are called *Incomparable*.

Examples:

$(\mathcal{Z}^+, \mid)$	$2 \nmid 3$ and $3 \nmid 2$	2 and 3 are incomparable.
$(P(S), \subseteq)$	$\{1,2\} \not\subset \{2,3\}$ $\{2,3\} \not\subset \{3,2\}$	{1,2} and {2,3} are incomparable.

Linear Orderings

<u>Definition</u>:

A partially ordered set (A, R) in which every 2 elements are comparable is called a

Totally Ordered Set (קבוצה סדורה לגמרי)

or a

Linearly Ordered Set (קבוצה סדורה לינארית)

or a

Chain (שרשרת)

The relation R is called a

Total Order (יחס סדר מלא) or a Linear Order (יחס לינארי)

Linear Orderings

Example:

($\mathcal{Z}$, $\geq$) - is a linearly ordered set.

It is a partially ordered set (proven above) and all pairs of elements are comparable:

for all a,b $\in \mathcal{Z}$ a $\leq$ b or b $\leq$ a or both.

($\mathcal{Z}^+$, |) - is not a linearly ordered set (only partially)

because 2 and 3 are not comparable.

Minimal/Maximal Elements

<u>Definition</u>:

An element a of a partially ordered set (A, R) is called *Minimal* (מינימלי) if there is no element $b \in A$ s.t. $b \neq a$ and bRa.

<u>Definition</u>:

An element a of a partially ordered set (A, R) is called *Maximal* (מקסימלי) if there is no element $b \in A$ s.t. $b \neq a$ and aRb.

Minimal/Maximal Elements

Examples:

({2, 4, 5, 10, 12, 20, 25}, |)

Elements 2,5 are minimal - there is no a s.t. a|2 or a|5.

Elements 12, 20,25 are maximal - there is no a s.t.
12|a, 20|a or 25|a.

Minimal/Maximal Elements

Examples:

	Minimal	Maximal
$(P(B), \subseteq)$	$\varnothing$	B
$(P(B)-\varnothing, \subseteq)$	all elements of B	B
$(N-\{0\}, \mid)$	1	None
$(N-\{0,1\}, \mid)$	all primes	None
$(Z^-, \leq)$	None	-1
$(Z, \leq)$	None	None

Minimal/Maximal Elements

Theorem: Every Partially Ordered set (A,R) has at least one minimal (maximal) element .

Theorem: Every Linearly Ordered set (A,R) if it contains a minimal (maximal) element then it is unique.

Smallest/Greatest Elements

<u>Definition</u>:

An element a of a partially ordered set (A, R) is called *The Smallest Element* (קטן ביותר) if for every element

$b \in A \quad aRb.$

<u>Definition</u>:

An element a of a partially ordered set (A, R) is called *The Greatest Element* (גדול ביותר) if for every element

$b \in A \quad bRa.$

Well Ordered Sets

Definition:

A set (A, R) is called a

Well Ordered Set (קבוצה סדורה היטב)

If it is linearly ordered and every subset of A has a smallest element.

$(\mathcal{Z}^+, \geq)$ - is a well ordered set.

$(\mathcal{R}, \geq)$ - is not well ordered.

$(\mathcal{Z}, \geq)$ - is not well ordered.

Topological Sorting

<u>Example:</u>

In task scheduling, some tasks can start only after other tasks have finished, other tasks are independent.
A scheduling (order) must be determined for the tasks.

Define a relation R on the tasks s.t. aRb iff task a must finish before b can start.
The set (Tasks,R) is a partial order.
The scheduler must produce a linear order which is compatible with this partial order.

Topological Sorting

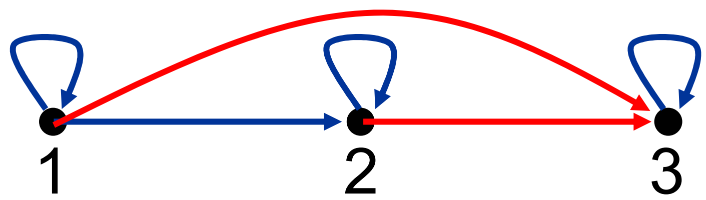
<u>Definition</u>:

A linear order R is *Compatible* (תואם) with a partial order S if $aSb \rightarrow aRb$.

<u>Example:</u>

S = { (1,1), (2,2), (3,3), (1,2) }

R = { (1,1), (2,2), (3,3), (1,2), (2,3), (1,3) }

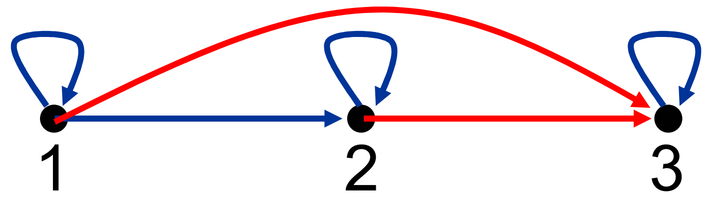


Topological Sorting

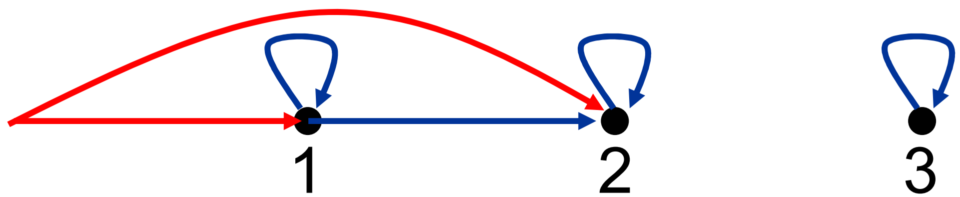
Compatible Linear Orders are not necessarily unique:

S = { (1,1), (2,2), (3,3), (1,2) }

R = { (1,1), (2,2), (3,3), (1,2), (2,3), (1,3) }



R_2 = { (1,1), (2,2), (3,3), (1,2), (3,1), (3,2) }



Topological Sorting

Definition:

Constructing a linear ordering from a partial ordering is called *Toplogical Sorting* (מיון טופולוגי).

Lemma: Every finite partial ordered set (A,R) has a minimal element (at least one).

Topological Sorting

Topological Sorting Algorithm:

Given a partially ordered set (A,S).

1) Choose a minimal element a_1 in A

2) Choose a minimal element a_2 in A-$\{a_1\}$

3) Choose a minimal element a_2 in A-$\{a_1,a_2\}$

$\vdots$

Continue until: $\{a_1,a_2, \ldots , a_n\} = A$

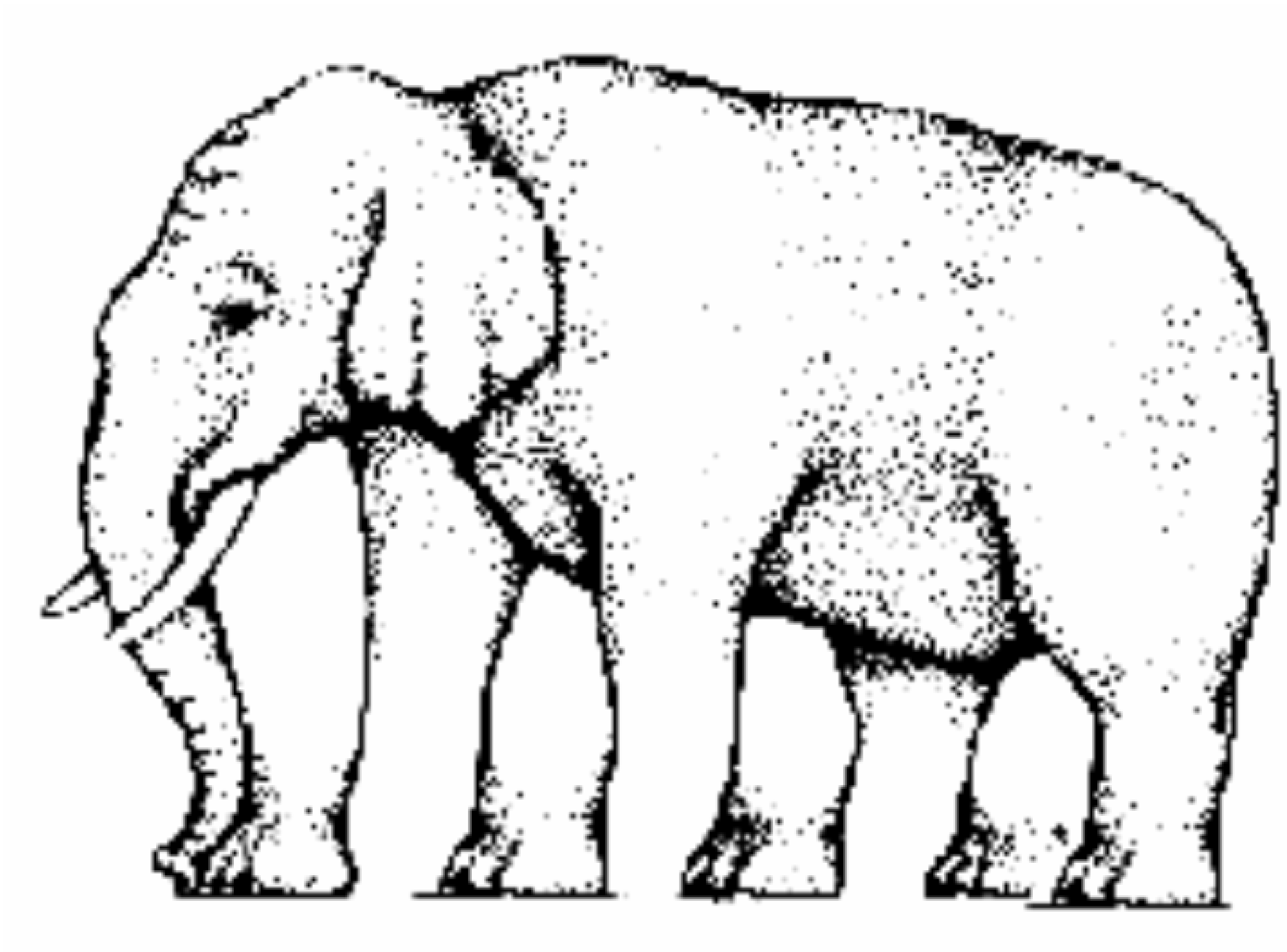
Topological Sorting

<u>Example:</u>

$$S = (\{1,2,4,5,12,20\}, \mid)$$

1) 1 is minimal in S — $a_1 = 1$
2) 2,5 are minimal in S-{1} — $a_2 = 5$
3) 2 is minimal in S-{1,5} — $a_3 = 2$
4) 4 is minimal in S-{1,5,2} — $a_4 = 4$
5) 12,20 are minimal in S-{1,2,4,5} — $a_5 = 20$
6) 12 is minimal in S-{1,2,4,5,20} — $a_6 = 12$

Topological Fun



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Topological Fun