Lesson 9:

Combinatorics

- Permutations
- Combinations
- Binomial Coefficients
- Combinations with Repetitions

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

Chapter 4.3-4.5

Combinatorics

Choosing subsets from a set, ordering elements in a sequence, choosing combinations.

<u>Examples:</u>

- The dept has 10 computers we must choose 5 for the lab. How many possibilities are there?
- There are 5 computers in the lab, internet addresses must be assigned to each. How many possibilities are there?
- Choose 3 books from the shelf to donate.
- Choose 3 books to give as presents to kids A B and C.

Permutations

Definition:

A *Permutation* פרמוטציה of a set of distinct objects is an **ordered** arrangement of these objects.

An ordered arrangement of r elements of a set is called an *r-Permutation* (r - פרמוטציה).

The number of r-permutations of a set with n elements is denoted P(n,r)

Permutations

<u>Example:</u>

S = { 3, 1, 2}

3,1,2 = a permutation of S

3,2 = a 2-permutation of S

2,3 = a 2-permutation of S

different permutations

The 2-permutations of S :

1,2 1,3 2,1 2,3 3,1 3,2

P(3,2) = 6

Permutations

Theorem: The number of r-permutations of a set with n distinct elements ($0 \le r \le n$) equals:

$$P(n,r) = n*(n-1)*(n-2)* \ldots (n-r+1) = \frac{n!}{(n-r)!} \qquad \frac{n!}{(n-r)!}$$

Proof:

n possibilities to choose the 1st element.

n-1 possibilities to choose the 2nd element.

n-2 possibilities to choose the 3rd element.

⋮

n-r+1 possibilities to choose the r-th element.

apply the Product Rule

Permutations

Example:

Number of possibilities to give out 3 of the 10 books as prizes to 3 different people: $P(10,3) = \frac{10!}{(10-3)!} = 720$

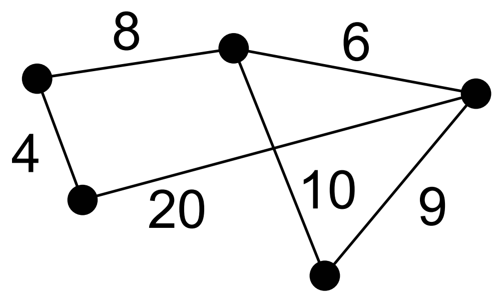
In the Olympics 100m run there are 8 competitors, 1 each will be given: gold medal, silver medal, bronze medal. How many combinations of winners are there?

$$P(8,3) = \frac{8!}{5!} = 336$$

Permutations

<u>Example:</u>

The traveling Salesman problem with n nodes requires testing of $P(n,n) = n!$ tours.



<u>Example:</u>

How many words of length 8 with no repeated letters of the set: A,B,C,D,E,F,G,H are there that contain the sub string ABC ?

Answer:

Count the 6-permutations of the elements D,E,F,G,H,[ABC]

$P(6,6) = 6! = 720$

Combinations

Definition:

An *r-Combination* r - בחירה of elements of a set is an **unordered** selection of r elements of the set.

An *r-Combination* is a subset of S containing r elements.

The number of r-combinations of a set with n elements is denoted

$$C(n,r) = \binom{n}{r}$$

Combinations

<u>Example:</u>

S = { 3, 1, 2}

{3,1,2} = a 3-combination of S

{3,2} = a 2-combination of S

The 2-combinations of S :

{1,2} {1,3} {2,3}

C(3,2) = 3

Combinations

Theorem: The number of r-combinations of a set with n distinct elements ($0 \le r \le n$) equals:

$$C(n,r) = \frac{n!}{(n-r)!\ r!}$$

Proof:

The r-permutations of a n elements can be found by:

1) choosing r elements - C(n,r) possibilities

2) ordering the r elements - P(r,r) possibilities

From the product rule we have $P(n,r) = C(n,r)*P(r,r)$

and $$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!(r-r)!} = \frac{n!}{(n-r)!\,r!}$$

Combinations

<u>Example:</u>

$C(5,2) = 10$

$S = \{1,2,3,4,5\}$

The 2-combinations of S :

$\{1,2\}$ $\{1,3\}$ $\{1,4\}$ $\{1,5\}$

$\{2,3\}$ $\{2,4\}$ $\{2,5\}$

$\{3,4\}$ $\{3,5\}$

$\{4,5\}$

Question: $C(5,3) = ?$

Combinations

Theorem: For r,n non-negative integers $r \leq n$

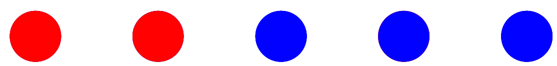
$$C(n,r) = C(n,n-r)$$

Proof:

$$C(n,n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = C(n,r)$$

Combinatorial Proof:

For every subset A of S with r elements there is subset of n-r elements $\overline{A}$.



Combinations

Examples:

How many ways to choose 5 members from a team of 10?

Answer:

$$C(10,5) = \binom{10}{5} = \frac{10!}{5!\,5!} = 252$$

How many bit strings of length 6 contain exactly 4 ones?

Answer:

Select 4 positions among the 6 bits in the 6-string in which to place 1s. So there are $C(6,4) = 15$.

Combinations

<u>Examples:</u>

There are : 9 MA-students in year 1
11 MA-students in year 2

How many ways to choose a committee of MA-students if 3 should be from year 1 and 4 from year 2?

Answer:

From the Product Rule we choose 3 students from year 1 and 4 students from year 2:

$$C(9,3)*C(11,4) = \frac{9!}{3!\ 6!} * \frac{11!}{4!\ 7!} = 84*330 = 27{,}720$$

Combinations

<u>Examples:</u>

How many ways can a race with 3 runners end if ties are allowed?

Answer:

The possibilities are (sum rule):

No ties, 2 tied 1^{st} place, 2 tied 2^{nd} place, 3 tied 1^{st} place.

$$P(3,3) + C(3,2) + C(3,2) + 1 = 3! + 3 + 3 + 1 = 13$$

Combinations

<u>Examples:</u>

How many bit strings contain exactly eight 0s and ten 1s if every 0 must be followed by 1 ?

Answer:

Eight of the 0s and 1s are paired leaving 2 ones.
The 2 ones can be in any of 9 places:

01 01 01 01 01 01 01 01

But both 1s can be in same place.

$C(9,2) + 9 = 36 + 9 = 45$

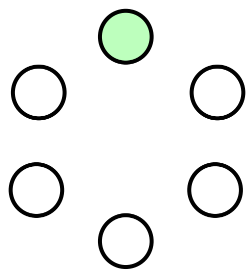
Combinations

Examples:

How many ways are there to sit 6 people around a round table invariant to rotation?

Answer:

Sit #1 in 1 seat (forming the marker invariant to rotation). The rest of the 5 people can sit in any 5-permutation in the remaining 5 seats: $P(5,5) = 5! = 120$



The Binomial Theorem משפט הבינום

What is the polynomial expansion of $(x+y)^3$?

What are the coefficients of the terms in the expansion?

$$(x+y)^3 = (x+y)*(x+y)*(x+y)$$

The terms in the expansion are obtained by choosing a term in each of the sums.

The terms in the expansion are: x^3 , y^3 , x^2y , xy^2

The Binomial Theorem משפט הבינום

The terms are obtained as follows:

x^3 by choosing x in each sum $\rightarrow$ C(3,3)=1 way.

x^2y by choosing x in 2 terms and y in one $\rightarrow$ C(3,2)=3 ways.

xy^2 by choosing x in 1 term and y in two $\rightarrow$ C(3,1)=3 ways.

y^3 by choosing x in none of the sums $\rightarrow$ C(3,0)=1 way.

$$(x+y)^3 = C(3,3)\ x^3 + C(3,2)\ x^2y + C(3,1)\ xy^2 + C(3,0)\ y^3$$

$$(x+y)^3 = C(3,0)\ x^3 + C(3,1)\ x^2y + C(3,2)\ xy^2 + C(3,3)\ y^3$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

The Binomial Theorem משפט הבינום

Theorem: *The Binomial theorem* משפט הבינום

Let x,y be variables and n a non-negative integer. Then

$$(x + y)^n = \sum_{j=0}^{n} C(n,j)\, x^{n-j}\, y^j$$

$$(x + y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n$$

Proof:

The coefficient of the term $x^{n-j}y^j$ counts the number of times it appears in the expansion and equals the number of ways to choose x in n-j of the sums (or y in j of the sums) and equals C(n,n-j) = C(n,j)

The Binomial Theorem משפט הבינום

Example:

What is the expansion of $(x + y)^4$?

Answer:

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

The Binomial Theorem משפט הבינום

Definition:

The number $\binom{n}{r}$ is called a *Binomial Coefficient* מקדם בינומילי

Examples:

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Answer:

According to the Binomial Theorem :

$$C(25,13) = \binom{25}{13} = \frac{25!}{13!\ 12!} = 5{,}200{,}300$$

The Binomial Theorem משפט הבינום

Example:

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Answer:

$(2x - 3y)^{25} = ((2x) + (-3y))^{25}$

According to the Binomial Theorem :

$((2x) + (-3y))^{25} = \sum C(25,j)\,(2x)^{n-j}\,(-3y)^j$

For j = 13, the coefficient of $x^{12}y^{13}$ is

$$C(25,13)*2^{12}*(-3)^{13} = \frac{25!}{13!\,12!} * 2^{12} * (-3)^{13}$$

Using The Binomial Theorem

Theorem: For every non-negative integer n :

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

Using the Binomial Theorem :

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}\, 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

Using The Binomial Theorem

Combinatorial Proof:

A set with n elements has 2^n subsets.

Counting them, there are:

$C(n,0)$ subsets with 0 elements

$C(n,1)$ subsets with 1 element

$C(n,2)$ subsets with 2 elements

$\vdots$

$C(n,n)$ subsets with n elements

Using the Sum Rule:

The number of subsets = $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Using The Binomial Theorem

Theorem: For every positive integer n :

$$\sum_{k=0}^{n} (-1)^k C(n,k) = 0$$

Proof:

Using the Binomial Theorem :

$$0 = (1 + (-1))^n = \sum_{k=0}^{n} C(n,k)\, 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k C(n,k)$$

Using The Binomial Theorem

Theorem: *The General Principle of Inclusion-Exclusion*

עקרון ההכלה וההדחה

Let $A_1, A_2, \ldots, A_n$ be finite sets then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum|A_i| - \sum_{1\le i<j\le n}|A_i \cap A_j| + \sum_{1\le i<j<k\le n}|A_i \cap A_j \cap A_k| - \ldots\ldots (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$$

Using The Binomial Theorem

Proof:

Using The Binomial Theorem

Theorem: *Vandermonde's identity* הזהות של ונדרמונדה

Let m,n,r be non-negative integers ($r \le n,m$). Then

$$\binom{m+n}{r} = \sum_{k=0}^{n} \binom{m}{r-k}\binom{n}{k}$$

Alexandre-Theophile Vandermonde (1735 - 1796)

Using The Binomial Theorem

Proof:

Assume n elements in a set and m elements in another set then the number of ways to choose r elements from the union of the 2 sets equals: C(n+m,r).

To choose r from the union, choose k from n (set1) and choose r-k from m (set2). with $0 \leq k \leq r$.

For each k: C(m,r-k)*C(n,k)

Sum over all k:

$$\binom{m+n}{r} = \sum_{k=0}^{n} \binom{m}{r-k} \binom{n}{k}$$

Pascal's Identity

Theorem: *Pascal's Identity* הזהות של פסקל

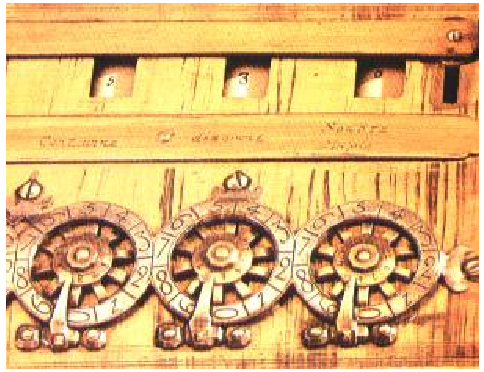
Let n, k be positive integers ($k \le n$). Then

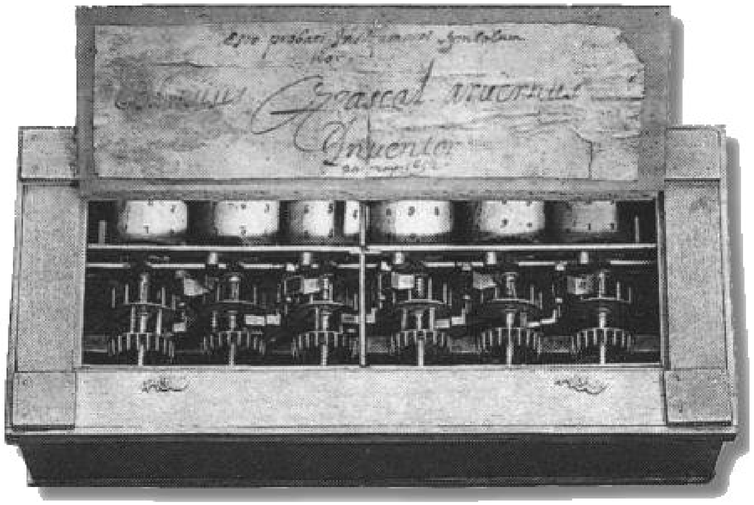
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Blaise Pascal (1623 - 1662)

The Pascaline

1641 - The first Calculating machine invented by Pascal.





Pascal's Identity

Proof:

Assume T is a set with n+1 elements. There exists $a \in T$.

Define $S = T - \{a\}$ (or $T = S \cup \{a\}$).

The number of subsets of size k in T is C(n+1,k).

Each k-subset of T either

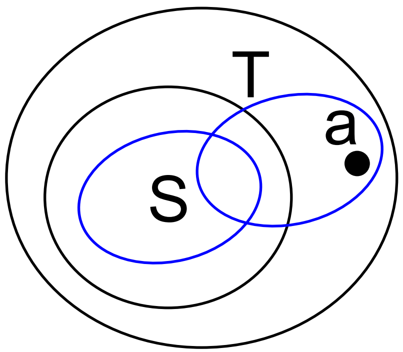
1) includes a and k-1 additional elements from S.

   There are C(n,k-1) such subsets

OR

2) includes k elements of S alone.

   There are C(n,k)



From the Sum Rule we have C(n+1,k) = C(n,k-1)+C(n,k)

Pascal's Triangle

Pascal's identity is used to recursively define the Binomial Coefficients. Basis of recursion is $\binom{n}{0} = \binom{n}{n} = 1$

This defines a geometric arrangement of the Coefficients as a triangle called *Pascal's triangle* משולש פסקל

The n-th row of Pascal's triangle consists of the coeffs:

$$\binom{n}{k} \quad \text{for } k = 0,1, \ldots, n$$

Pascal's Triangle

$$\binom{0}{0}$$ n = 0

$$\binom{1}{0} \quad \binom{1}{1}$$ n = 1

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$ n = 2

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$ n = 3

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$ n = 4

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$ n = 5

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$ n = 6

Pascal's Triangle

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

							1							n = 0
						1		1						n = 1
					1		2		1					n = 2
				1		3		3		1				n = 3
			1		4		6		4		1			n = 4
		1		5		10		10		5		1		n = 5
	1		6		15		20		15		6		1	n = 6

Counting with Repetitions

- Choose k elements from n with ‘repetitions’.
- Choose k times from a ‘bag’ of n but return the chosen element to the bag.
- Choose k elements and order them with repetition.

- Choose k different elements from n non-distinct elements.
- Choose k times from a ‘bag’ without returning the chosen element to the bag, but bag contains repeated elements.
- Order k different elements from n non-distinct elements.

Permutations with Repetitions

<u>Example:</u>

How many 'words' of length 4 can be created from the English Alphabet?

Answer:

There are 26 letters in the alphabet. The letters can be used repeatedly. from the product rule we have:
$26*26*26*26 = 26^4 = 456,976$ words.

26 26 26 26

Permutations with Repetitions

Theorem: The number of r-permutations of a set of n objects with repetition is n^r.

Proof:

There are n possibilities to choose an element for each of the r positions of the permutation.
From the product rule the number of r-permutations is n^r.

Note: r may be greater than n!

Permutations with Repetitions

Example:

A box contains 5 red balls (numbered 1..5) and 7 blue balls (numbered 6..12). How many possibilities of drawing 3 balls from the box if after each draw the ball is returned?

Answer:

There are 12 balls in the box. Each draw is independent so there are 12^3 possibilities.

How many possibilities of drawing 3 red balls from the box? (with returns)

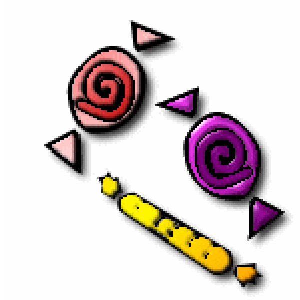
Combinations with Repetitions

<u>Example:</u>

How many possibilities are there of choosing 4 candies of 3 kinds: chocolate (C) gum (G) lollipops (L)?

Answer:

CCCC	GGGG	LLLL
CCCG	GGGL	LLLG
CCCL	GGGC	LLLC
CCGG	GGLL	LLCG
CCLL	GGLC	
CCGL		



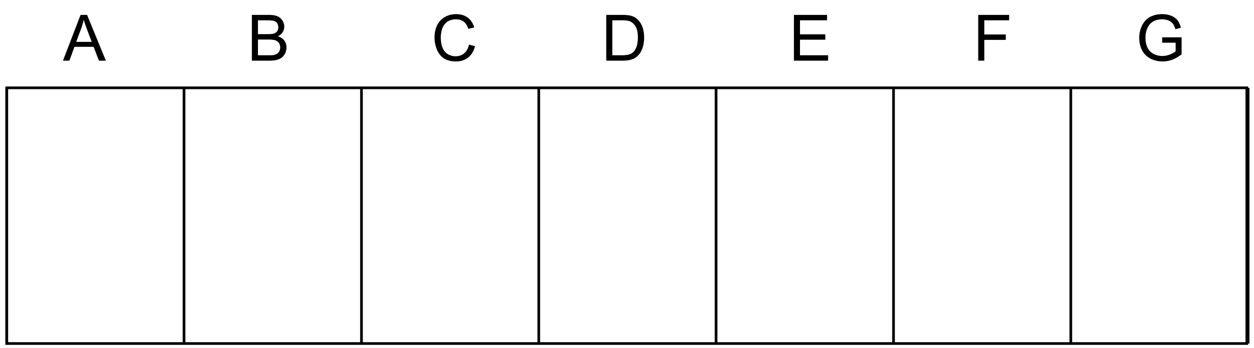
Total: 15 possibilities

Combinations with Repetitions

<u>Example:</u>

How many possibilities are there of choosing 5 candies of 7 kinds: A,B,C,D,E,F,G ?

Answer:



Markers: ★ ★ ★ ★ ★

Place a marker in the drawer of the chosen candies

Combinations with Repetitions

<u>Example cont.:</u>

A	B	C	D	E	F	G
★ ★		★		★ ★		

There is a 1:1 and onto mapping between the possibilities of choosing 5 candies out of 7 and the possibilities of ordering 5 ★ and 6 | .

AACDF ★ ★ | | ★ | ★ | | ★ |

BBCEE | ★ ★ | ★ | ★ ★ | | |

BDDDG | ★ | | ★ ★ ★ | | | ★

Combinations with Repetitions

<u>Example cont.:</u>

There are 6+5 = 11 'places' for ★ and | .

Choose 5 places for the ★ and in the others place | .

Thus the number of possibilities to choose 5 of 11 equals the number of possibilities to choose 5 candies of 7 kinds:

$$C(11,5) = \binom{11}{5} = \frac{11!}{5!\,6!} = 462$$

Combinations with Repetitions

Theorem: The number of r-combinations of a set of n elements with repetition is C(n+r-1,r).

Proof:

Every r-combination of a set of n elements can be represented as a list of n-1 bars and r stars. Where the bars represent the division between cells and a star is placed in the cell whose element is chosen.
There are C(n-1+r,r) possibilities to order r stars and n-1 bars.

Combinations with Repetitions

<u>Example:</u>

How many possibilities of choosing 6 cookies out of 4 kinds?

Answer:

From the Theorem the number of possible choices of 6 from 4 with repetitions is:

$$C(9,6) = C(9,3) = \frac{9!}{3!\,6!} = \frac{9 * 8 * 7}{3 * 2 * 1} = 84$$

Combinations with Repetitions

<u>Example:</u>

How many solutions does the following equation have:

$$x_1 + x_2 + x_3 = 11$$

where $x_1, x_2, x_3 \geq 0$ are integers.

Answer:

Consider a solution as a choice of 11 elements of 3 types (x_1 of type 1, x_2 of type 2 and x_3 of type 3).
Answer is the number of 11-combinations with repetition from a set with 3 elements:

$$C(11+3-1,11) = C(13,11) = \frac{13!}{11!\ 2!} = \frac{13 * 12}{2 * 1} = 78$$

Combinations with Repetitions

<u>Example:</u>

How many solutions does the following equation have:

$$x_1 + x_2 + x_3 = 11$$

where $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$ are integers.

Answer:

Consider a solution as a choice of 11 elements of 3 types where 6 have already been chosen (1,2,3 from types 1,2,3 respectively).

There are 5 more elements to choose from 3 types:

$$C(5+3-1,5) = C(7,5) = \frac{7!}{5!\ 2!} = \frac{7 * 6}{2 * 1} = 21$$

Summary - Table

Type	Repetition	Formula
r-permutation	No	$\frac{n!}{(n-r)!}$
r-combination	No	$\frac{n!}{r!\,(n-r)!}$
r-permutation	Yes	n^r
r-combination	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Non-Distinct Elements

Example:

How many permutations are there of the letters of the word: SUCCESS ?

Answer:

There are 3 S's so number of possible positions: C(7,3).

This leaves 4 positions free.

There are 2 C's so number of possible positions: C(4,2).

This leaves 2 positions free.

There is 1 U so number of possible positions: C(2,1).

This leaves 1 positions free for E : C(1,1).

Permutations with Non-Distinct Elements

Answer Cont.:

From the Product Rule, the number of permutations equals:

$$C(7,3)*C(4,2)*C(2,1)*C(1,1) \ =$$

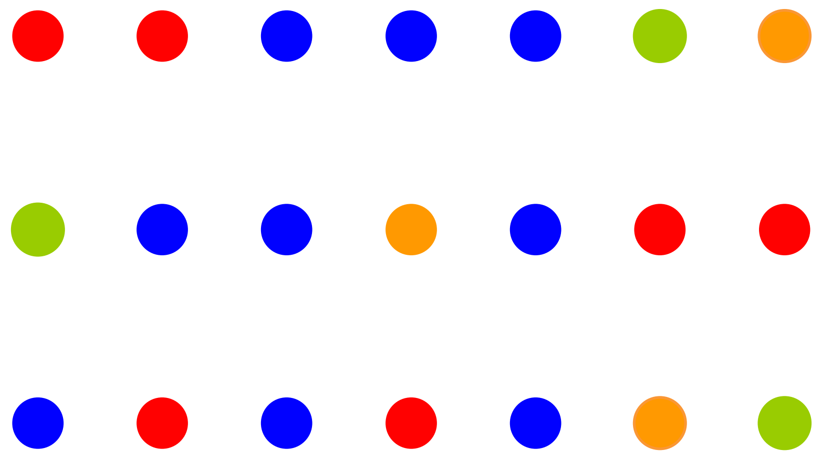
$$\frac{7!}{3!\,\cancel{4}!} * \frac{\cancel{4}!}{2!\cancel{2}!} * \frac{\cancel{2}!}{1!\cancel{1}!} * \frac{\cancel{1}!}{1!0!} \ =$$

$$\frac{7!}{3!\,2!\,1!\,1!} \ = \ 420$$

Permutations with Non-Distinct Elements

Theorem: The number of permutations of a set of n elements of k types ($k \leq n$) with n_i elements of type i is:

$$\frac{n!}{n_1!n_2!...n_k!}$$



Permutations with Non-Distinct Elements

Proof:

For each type count possibilities of choosing their positions:

Positioning n_1 objects of type 1: $C(n,n_1)$

Positioning n_2 objects of type 2: $C(n-n_1,n_2)$

$\vdots$

Positioning n_k objects of type k: $C(n-n_1-n_2-\ldots-n_{k-1},n_k)$

By the product rule:

$$C(n,n_1)*C(n-n_1,n_2)*\ldots*C(n-n_1-\ldots n_{k-1},n_k) =$$

$$\frac{n!}{n_1!\cancel{(n-n_1)!}} * \frac{\cancel{(n-n_1)!}}{n_2!\cancel{(n-n_1-n_2)!}} * \cdots * \frac{\cancel{(n-n_1-n_2-\ldots-n_{k-1})!}}{n_k!\underbrace{(n-n_1-n_2-\ldots-n_k)!}_{0!}} =$$

$$= \frac{n!}{n_1!n_2!\ldots n_k!}$$

Permutations with Non-Distinct Elements

<u>Example:</u>

How many ways are there to travel in xyz-space ($\mathcal{Z}^3$) from (0,0,0) to (4,3,5) in unit steps in the x, y, and z directions?

Answer:

4 steps must be taken in x-direction

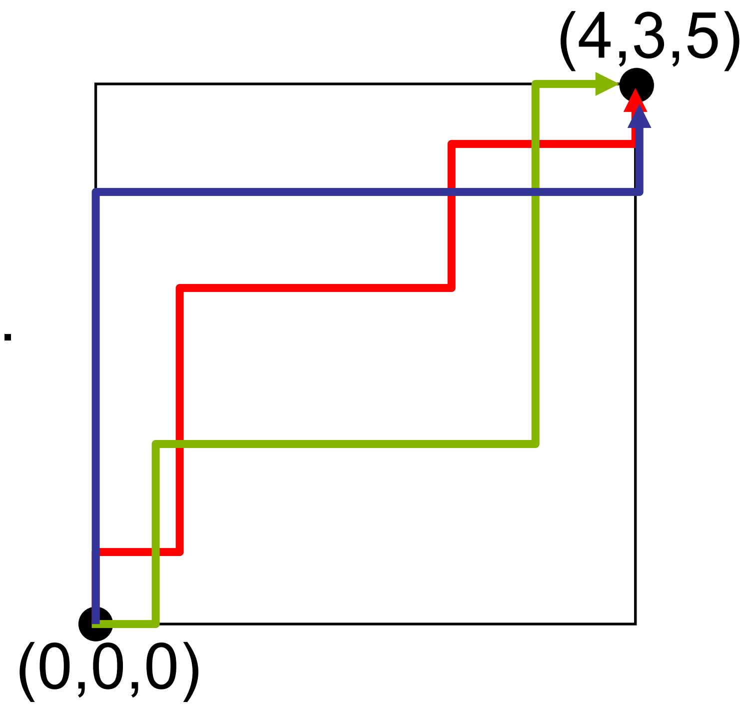
3 steps must be taken in y-direction

5 steps must be taken in z-direction.

Total of 4+3+5 steps of three types.

Count permutations of these steps:

$$\frac{12!}{4!3!5!} = 27{,}720$$

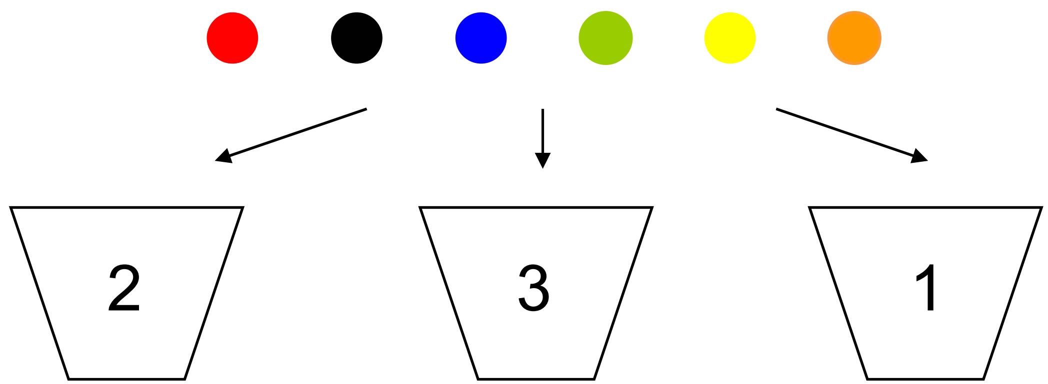


Multiple r-Combinations of n

Distributing distinguishable objects into several boxes.

Divide n objects into k boxes so that r_1 are in box 1, r_2 are in box 2, ... r_k are in box k. $r_1+r_2+...+r_k = n$.

If $r_1+r_2+...+r_k \le n$ then define another box that will have $n-r_1-r_2-...-r_k$ objects.



Multiple r-Combinations of n

<u>Example:</u>

How many possibilities of distributing 5 cards to each of 4 different players from a deck of 52?

Answer:

Number of possibilities of 5 cards to 1st player: C(52,5).

This leaves 47 cards.

Number of possibilities of 5 cards to 2nd player: C(47,5).

This leaves 42 cards.

Number of possibilities of 5 cards to 3rd player: C(42,5).

This leaves 37 cards.

Number of possibilities of 5 cards to 4th player: C(37,5).

Multiple r-Combinations of n

Answer Cont.:

From the Product Rule, the number of possibilities equals:

$$C(52,5)*C(47,5)*C(42,5)*C(37,5) =$$

$$\frac{52!}{5!\,\cancel{47!}} * \frac{\cancel{47!}}{5!\,\cancel{42!}} * \frac{\cancel{42!}}{5!\,\cancel{37!}} * \frac{\cancel{37!}}{5!\,32!} =$$

$$\frac{52!}{5!\;5!\;5!\;5!\,32!}$$

Multiple r-Combinations of n

Theorem: The number of ways to distribute n distinguishable objects into k different boxes so that n_i objects are in box i, i = 1...k is:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Multiple r-Combinations of n

Proof:

For each box count possibilities of choosing their content:

Choosing n_1 objects for box 1: $C(n,n_1)$

Choosing n_2 objects for box 2: $C(n-n_1,n_2)$

$\vdots$

Choosing n_k objects for box k: $C(n-n_1-n_2-\ldots-n_{k-1},n_k)$

By the product rule:

$$C(n,n_1)*C(n-n_1,n_2)*\ldots*C(n-n_1-\ldots n_{k-1},n_k) =$$

$$\frac{n!}{n_1!\cancel{(n-n_1)!}} * \frac{\cancel{(n-n_1)!}}{n_2!\cancel{(n-n_1-n_2)!}} * \cdots * \frac{\cancel{(n-n_1-n_2-\ldots-n_{k-1})!}}{n_k!\underbrace{\cancel{(n-n_1-n_2-\ldots-n_k)!}}_{0!}} =$$

$$= \frac{n!}{n_1!n_2!\ldots n_k!}$$

Multiple r-Combinations of n

Proof II:

There is a 1:1 and onto mapping between the distribution of n elements into k boxes, and the permutation of n non-distinct elements.

Distribution of n distinguishable objects into k different boxes so that n_i objects are in box i.

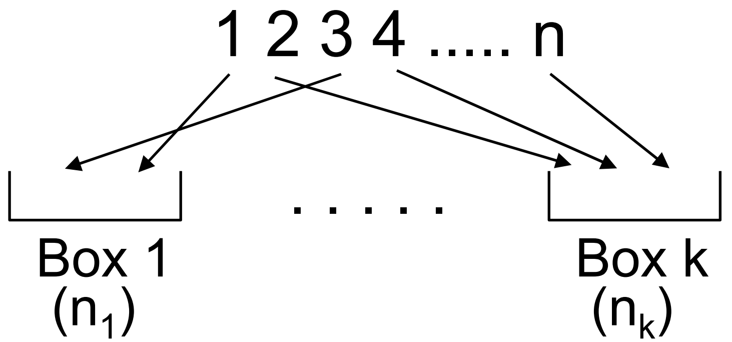
$$\Updownarrow$$

Permutation of a set of n elements of k types ($k \leq n$) with n_i elements of type i.

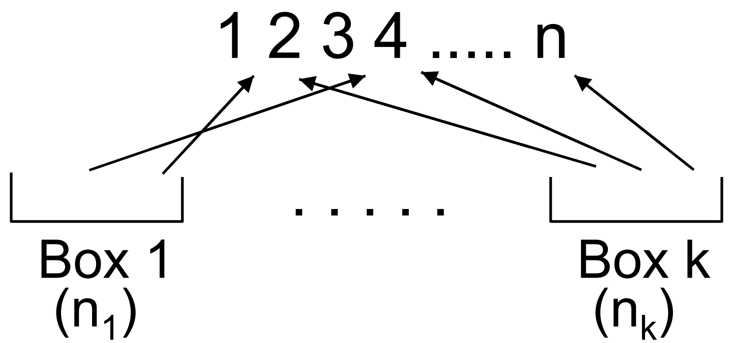
Multiple r-Combinations of n

Proof II cont:

There are n distinct elements - enumerate them.



Every distribution is mapped to a permutation:



Arrows indicate index at which object of type n_i is placed.

Multiple r-Combinations of n

<u>Example:</u>

How many possibilities of distributing all 52 cards equally between 4 different players?

Answer:

Each player gets 13 cards so: $\dfrac{52!}{13!13!13!13!}$

What is the probability that each player has one Ace?

$$\frac{48!}{12!12!12!12!} \Big/ \frac{52!}{13!13!13!13!} = \frac{13^4}{52*51*50*49}$$

Counting Fun!

Count 9 people/faces!