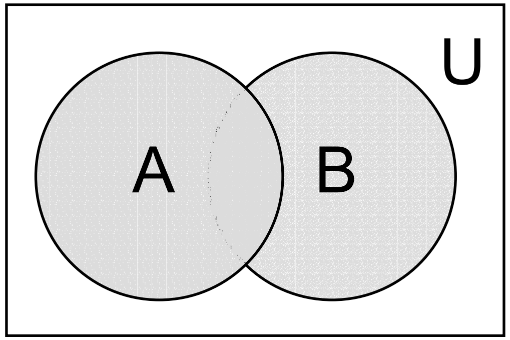
Lesson 2:

Set Operations

- Union
- Intersection
- Difference
- Compliment
- Set Identities
- Membership tables
- Computer Sets



Chapter 1.7

Set Operations

Definition:
Let A, B be sets. The *Union* (איחוד) of A and B is the set that includes elements that are either in A or in B or in both.

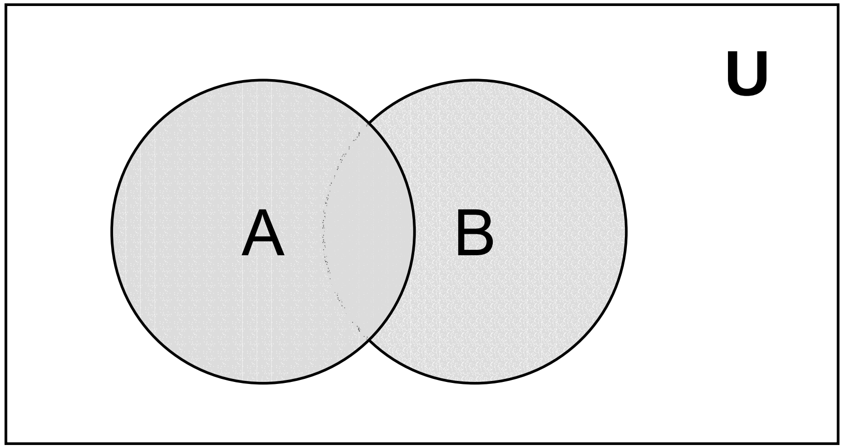
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Examples:

$\{1, 2, 3\} \cup \{1, 3, 5\} = \{1, 2, 3, 5\}$

{a, 2, Δ, ♥, משה} ∪ ∅ = {a, 2, Δ, ♥, משה}

Union Operation



$A \cup B$

$$\left\{\begin{array}{c}\text{Algebra}\\ \text{Students}\end{array}\right\} \cup \left\{\begin{array}{c}\text{Calculus}\\ \text{Students}\end{array}\right\} = \left\{\begin{array}{c}\text{Students studying}\\ \text{Algebra OR calculus}\\ \text{(or both)}\end{array}\right\}$$

Intersection Operations

<u>Definition</u>:

Let A, B be sets. The *Intersection* (חיתוך) of A and B is the set that includes elements that are in A and in B.

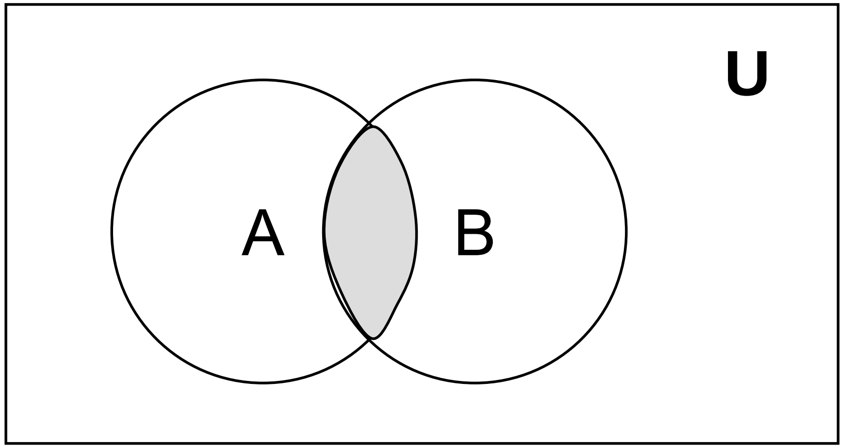
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Examples:

$\{1, 2, 3\} \cap \{1, 3, 5\} = \{1, 3\}$

$\{a, 2, \Delta, \heartsuit,$ משה$\} \cap \varnothing = \varnothing$

Intersection Operation



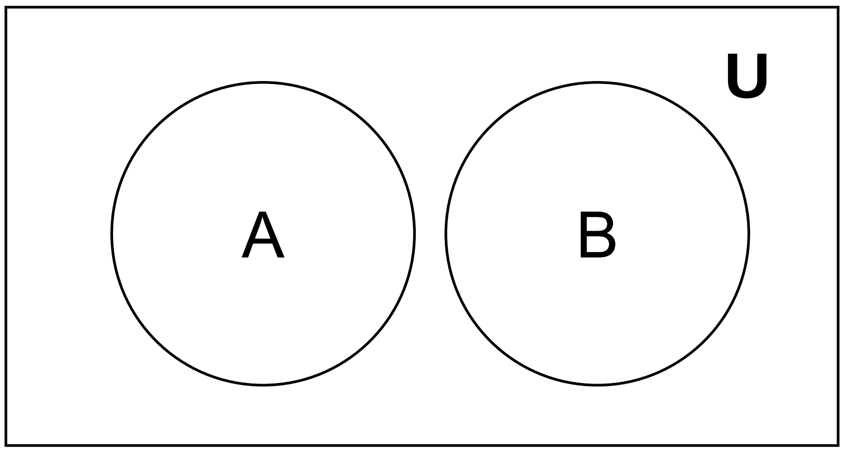
$A \cap B$

$$\begin{Bmatrix} \text{Algebra} \\ \text{Students} \end{Bmatrix} \cap \begin{Bmatrix} \text{Calculus} \\ \text{Students} \end{Bmatrix} = \begin{Bmatrix} \text{Students studying} \\ \text{Algebra AND calculus} \end{Bmatrix}$$

Intersection Operation

$$\{1, 2, 3\} \cap \{4, 6\} = \emptyset$$

<u>Definition</u>:
Two sets are *Disjoint* (זרות) if their intersection is empty. $A \cap B = \emptyset$



$A \cap B$

Difference Operations

<u>Definition</u>:

Let A, B be sets. The *Difference* (הפרש) of A and B is the set that includes elements that are in A and not in B.

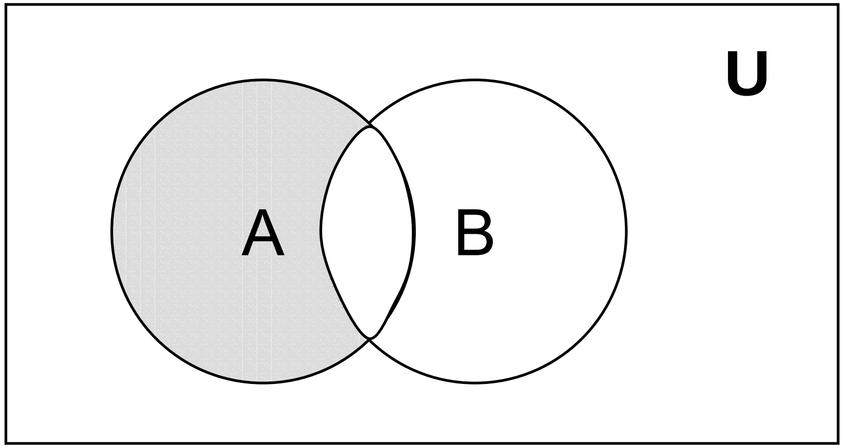
$$A-B = \{x \mid x \in A \text{ and } x \notin B \}$$

Examples:

$\{1, 2, 3\} - \{1, 3, 5\} = \{2\}$

{a, 2, Δ, ♥, משה} - ∅ = {a, 2, Δ, ♥, משה}

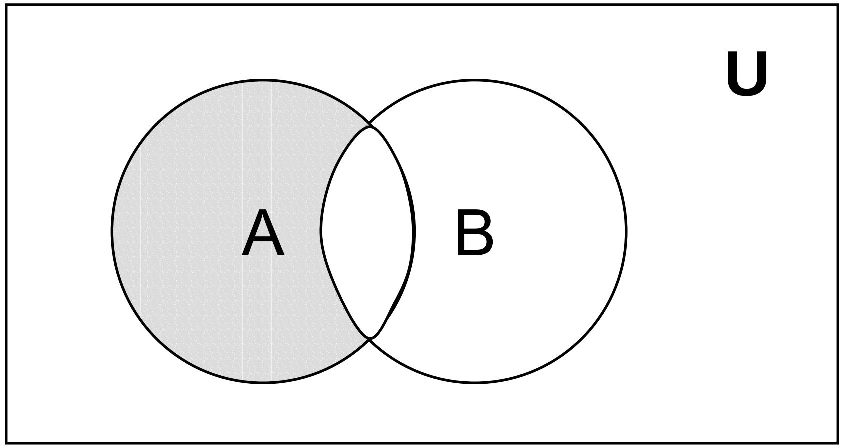
Difference Operation



A-B

$$\left\{\begin{array}{c}\text{Algebra}\\ \text{Students}\end{array}\right\} - \left\{\begin{array}{c}\text{Calculus}\\ \text{Students}\end{array}\right\} = \left\{\begin{array}{c}\text{Students studying}\\ \text{Algebra AND NOT}\\ \text{calculus}\end{array}\right\}$$

Difference Operation

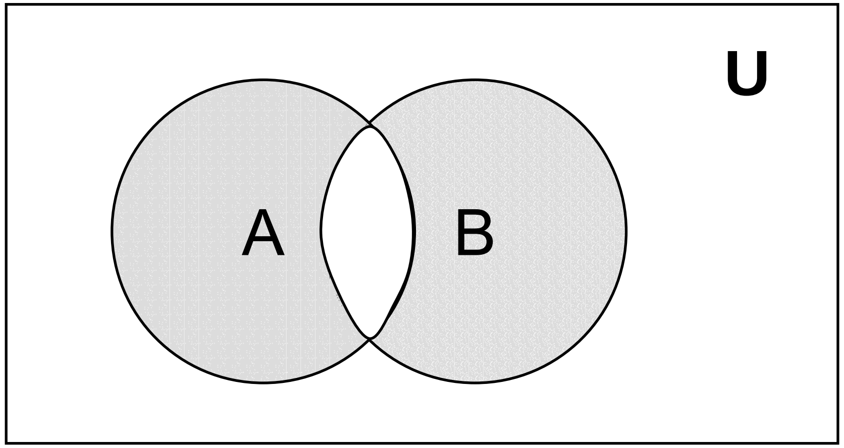


A-B

$$\left\{\begin{array}{c}\text{CS}\\ \text{Students}\end{array}\right\} - \left\{\begin{array}{c}\text{CS minor}\\ \text{Students}\end{array}\right\} = \left\{\begin{array}{c}\text{CS Major}\\ \text{Students}\end{array}\right\}$$

Difference Operation

Q: $A \cup B = (A - B) \cup (B - A) \cup ?$



$(A - B) \cup (B - A)$

Complement Operations

<u>Definition</u>:
Let U be the universal set. The *Complement* (משלים) of set A is the set that includes all elements that are not in A (but are in the universe).

$$\overline{A} = \{x \mid x \notin A\}$$

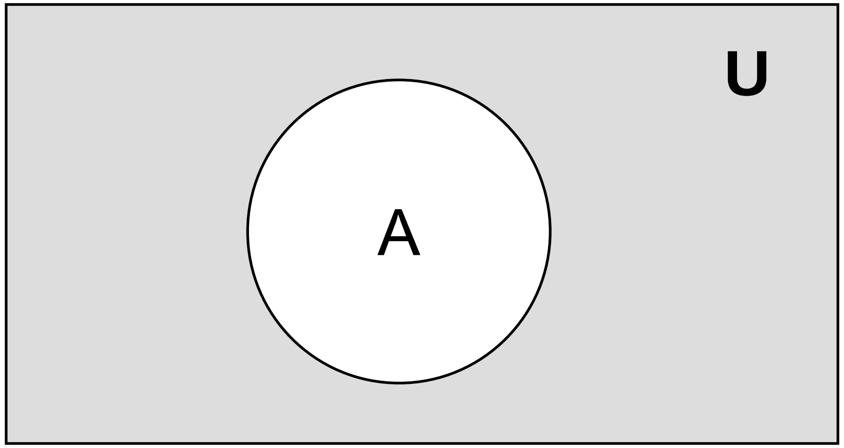
$$\overline{A} = U - A$$

Examples:

$$A = \{ x \mid x \in \mathcal{N} \text{ and } x > 3\}$$

$$\overline{A} = \{ 0, 1, 2, 3\}$$

Difference Operation

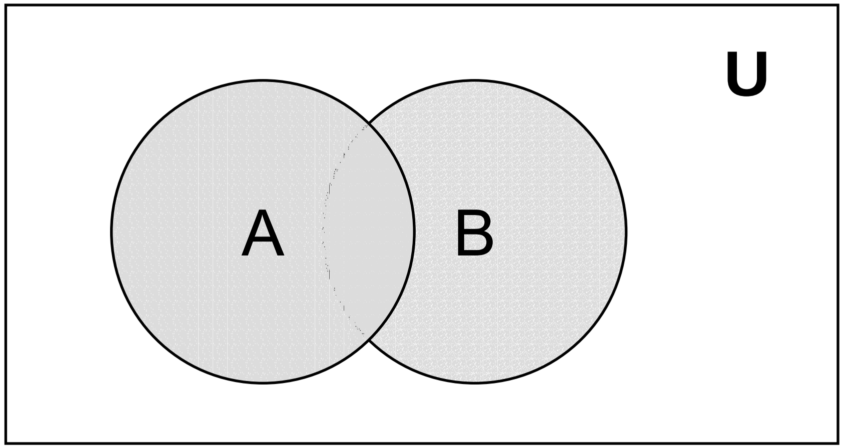


$\overline{A}$

Cardinality

Q: $|A \cup B| = ?$

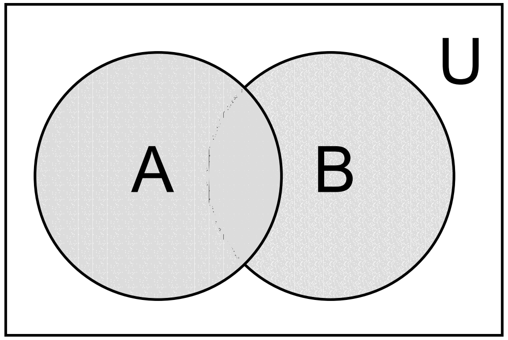
Q: $|A \cup B| = |A| + |B|$?



$|A \cup B| = |A| + |B|$ Only if A and B are Disjoint

$|A \cup B| = |A| + |B| - |A \cap B|$

Cardinality



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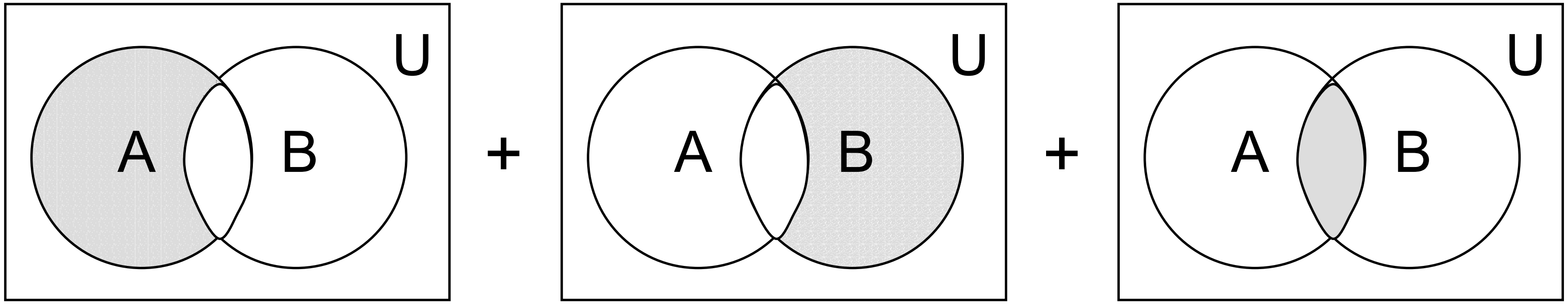
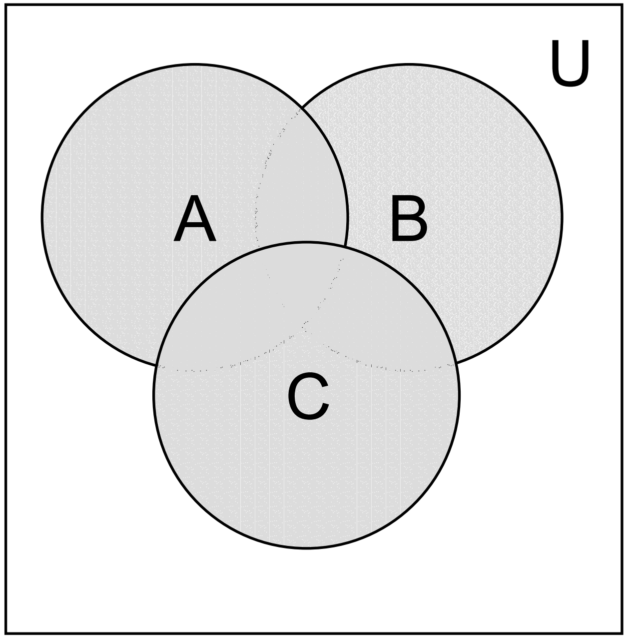


$$| A \cup B| = |A - B| + |B - A| + | A \cap B|$$

Cardinality

Q: $|A \cup B \cup C| = ?$



Set Identities

• Identity:	$A \cup \varnothing = A = A \cap U$
• Domination:	$A \cup U = U$, $A \cap \varnothing = \varnothing$
• Idempotent:	$A \cup A = A = A \cap A$
• Double Complement:	$\overline{(\overline{A})} = A$
• Commutative:	$A \cup B = B \cup A$, $A \cap B = B \cap A$
• Associative:	$A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

Set Identities - De Morgan's Law

- De Morgan

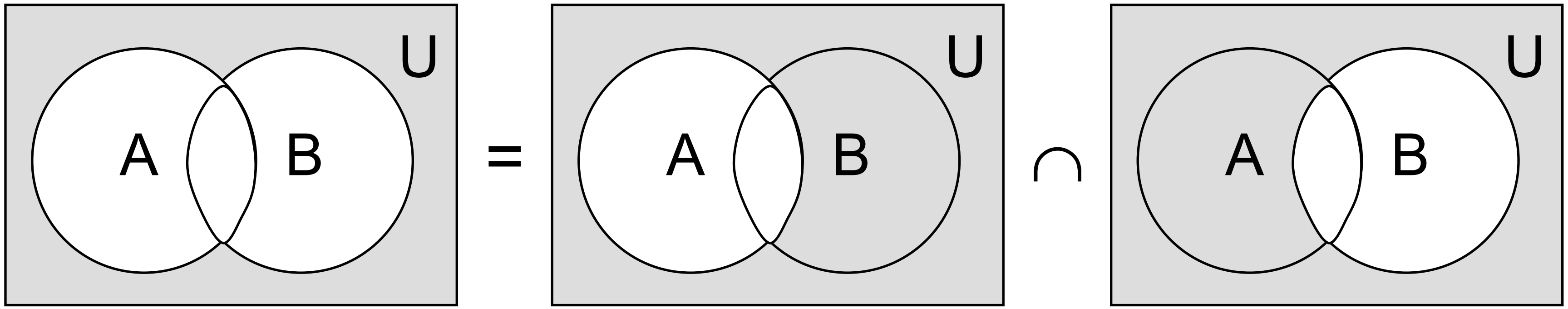
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

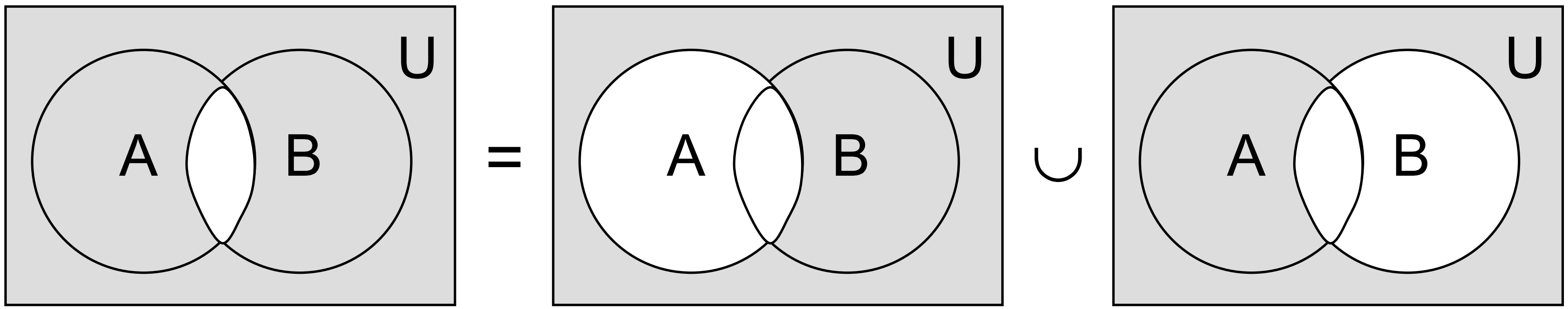
Augustus De Morgan (1806-1871)

Set Identities - De Morgan's Law

$$\overline{A\cup B}=\overline{A}\cap\overline{B}$$



$$\overline{A\cap B}=\overline{A}\cup\overline{B}$$



Proving Set Identities

Several ways to prove $A = B$
(where *A and B* are sets or set equations):

1. Prove $A \subseteq B$ and $B \subseteq A$ separately.
2. Use Prepositional Logic and equivalences.
3. Use a membership table.

Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Method 1:

- Part 1: Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
  - Assume $x \in A \cap (B \cup C)$, & show $x \in (A \cap B) \cup (A \cap C)$.
  - We know that $x \in A$, and either $x \in B$ or $x \in C$.
    - Case 1: $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.
    - Case 2: $x \in C$. Then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.
  - Therefore, $x \in (A \cap B) \cup (A \cap C)$.
  - Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
- Part 2: Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. …

Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Method 2*:

Translate set equation into propositional logic, reason within propositional logic, then translate back into set theory.

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

Suppose $x \in A \wedge (x \in B \vee x \in C).$

Prove $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C).$

* we will learn later in the course.

Membership Table

Denote by

1 the fact that an element is in the set.

0 the fact that an element is NOT in the set.

Build table that shows all possible combinations of memberships in the given sets.

Every row in table is 1 possible combination.

Compare columns of sets that are to be proven equal.

Membership Table - Example

A	B	$A \cup B$	$A \cap B$
0	0	0	0
1	0	1	0
0	1	1	0
1	1	1	1

Membership Table - Example

Prove $(A \cup B) - B = A - B$

A	B	$A \cup B$	$(A \cup B)-B$	$A-B$
0	0	0	0	0
1	0	1	1	1
0	1	1	0	0
1	1	1	0	0

Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

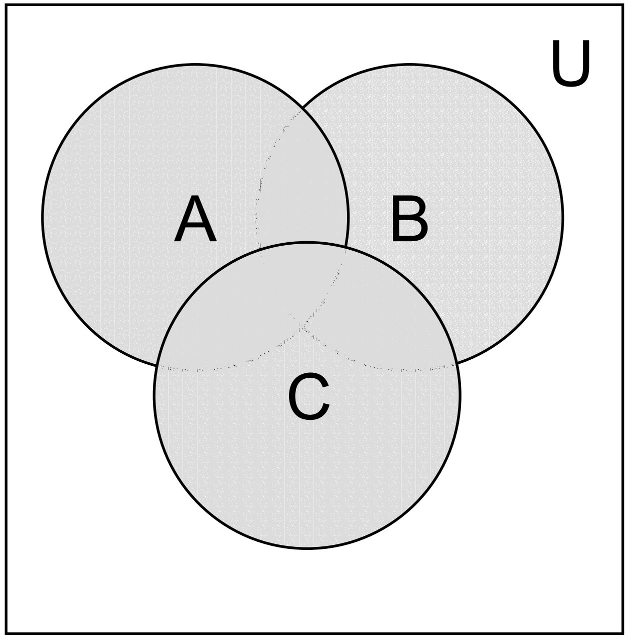
Method 3: Membership table.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

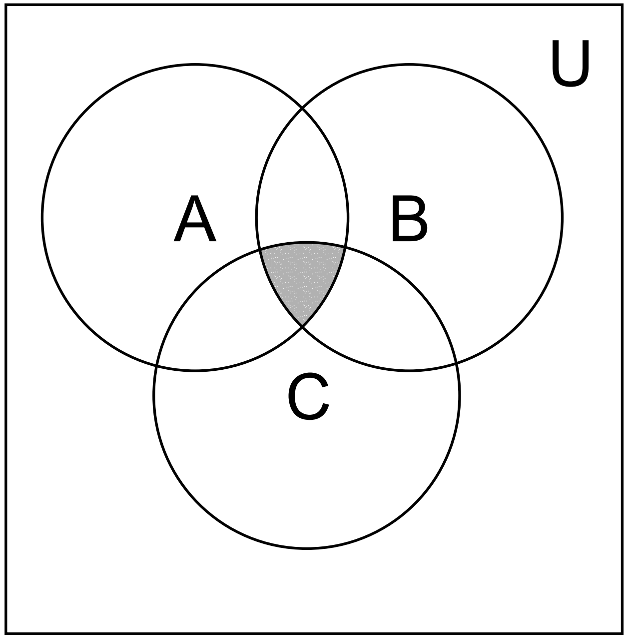
Multiple set Operations

Generalization to more than 2 sets is well defined due to the associative rule.

$A \cup B \cup C$



$A \cap B \cap C$



Multiple set Operations

Definition:

The Union (איחוד) of a collection of sets is a set of elements each of which is in at least one of the sets in the collection.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

Definition:

The Intersection (חיתוך) of a collection of sets is a set of elements each of which is all of the sets in the collection.

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Multiple set Operations

Examples:

$$A_i = \{i, i+1, i+2, ...\}$$

$$\bigcup_{i=1}^{n} A_i = \cup\{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \cap\{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\}$$

Computer Representation of Sets

Option 1: Represent sets as lists of elements (numbers, variables, strings etc)

100	2	30	15	-17	....

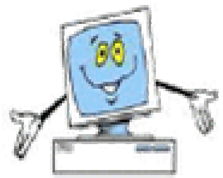
‘Dog’	‘Lion’	‘Horse’	....

Problem:

Memory consuming.

Operations are time consuming (e.g. intersection).

Computer Representation of Sets



Option 2: Represent sets as bit strings.

1. Order all elements of U (the Universal set):

   $(a_1, a_2, ..., a_n) \qquad a_i \in U$

2. Represent a set $A \subseteq U$ using an n-bit string such that position i contains 1 if $a_i \in A$ and contains 0 if $a_i \notin A$.

$$A \leftrightarrow 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 1$$

Computer Representation of Sets

Example:

Consider the Universal set of all integers between 1 and 10.

$$U = \{1, 2, \ldots, 10\}$$

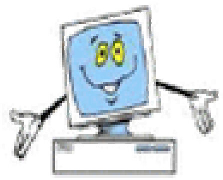
Define an order on U : $a_i = i$

(1 0 1 0 1 0 1 0 1 0) - all odd numbers {1, 3, 5, 7, 9}

(0 1 0 1 0 1 0 1 0 1) - all even numbers {2, 4, 6, 8, 10}

(1 1 1 1 0 0 0 0 0 0) - numbers smaller than 5

Computer Representation of Sets



Advantage: set operations are easy.

$\overline{A}$ (complement) - replace 0 and 1.

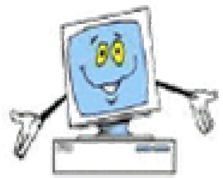
$A = (1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0)$ - odd numbers

$\overline{A} = (0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1)$ - even numbers

$A \cup B$ (union) - bitwise OR

$A \cap B$ (intersection) - bitwise AND

Computer Representation of Sets



Example:

$U = \{1, 2, 3, 4, 5\}$

$A = \{1, 5, 3\}$ $= (1\ 0\ 1\ 0\ 1)$

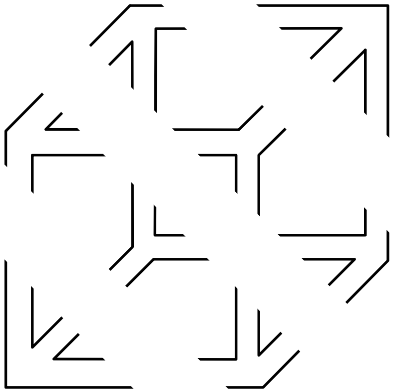
$B = \{2, 1, 3\}$ $= (1\ 1\ 1\ 0\ 0)$

$\overline{A} = (0\ 1\ 0\ 1\ 0)$

$\overline{B} = (0\ 0\ 0\ 1\ 1)$

$$A \cap B = \begin{array}{c} (1\ 0\ 1\ 0\ 1) \\ (1\ 1\ 1\ 0\ 0) \\ \hline (1\ 0\ 1\ 0\ 0) \end{array} = \{1,3\}$$

$$A \cup B = \begin{array}{c} (1\ 0\ 1\ 0\ 1) \\ (1\ 1\ 1\ 0\ 0) \\ \hline (1\ 1\ 1\ 0\ 1) \end{array} = \{1,2,3,5\}$$



Demos

1. Shape from motion slide
2. Hello slide red and wholes