Context-free grammars

A **context-free grammar** (CFG) is a four-tuple $\langle \Sigma, V, S, P \rangle$, where:

- Σ is a finite, non-empty set of **terminals**, the alphabet;
- V is a finite, non-empty set of **grammar variables** (categories, or non-terminal symbols), such that $\Sigma \cap V = \emptyset$;
- $S \in V$ is the **start symbol**;
- P is a finite set of **production rules**, each of the form $A \rightarrow \alpha$, where $A \in V$ and $\alpha \in (V \cup \Sigma)^*$.

For a rule $A \rightarrow \alpha$, A is the rule's **head** and α is its **body**.

Context-free grammars

Example: CFG example

$\Sigma = \{the,\ cat,\ in,\ hat\}$

$V = \{D,\ N,\ P,\ NP,\ PP\}$

The start symbol is NP

The rules:

$$\begin{array}{ll} D \rightarrow the & NP \rightarrow D\ N \\ N \rightarrow cat & PP \rightarrow P\ NP \\ N \rightarrow hat & NP \rightarrow NP\ PP \\ P \rightarrow in & \end{array}$$

Context-free grammars: language

Each non-terminal symbol in a grammar denotes a language.
A rule such as $N \rightarrow cat$ implies that the language denoted by the non-terminal N includes the alphabet symbol *cat*.
The symbol *cat* here is a single, atomic alphabet symbol, and not a string of symbols: the alphabet of this example consists of natural language words, not of natural language letters.
For a more complex rule such as $NP \rightarrow D\ N$, the language denoted by NP contains the concatenation of the language denoted by D with that denoted by N: $L(NP) \supseteq L(D) \cdot L(N)$.
Matters become more complicate when we consider recursive rules such as $NP \rightarrow NP\ PP$.

Context-free grammars: derivation

Given a grammar $G = \langle V, \Sigma, P, S \rangle$, we define the set of *forms* to be $(V \cup \Sigma)^*$: the set of all sequences of terminal and non-terminal symbols.

Derivation is a relation that holds between two forms, each a sequence of grammar symbols.

A form α *derives* a form β, denoted by $\alpha \Rightarrow \beta$, if and only if $\alpha = \gamma_l A \gamma_r$ and $\beta = \gamma_l \gamma_c \gamma_r$ and $A \rightarrow \gamma_c$ is a rule in P.

A is called the *selected symbol*. The rule $A \rightarrow \gamma$ is said to be **applicable** to α.

Derivation

Example: Forms

The set of non-terminals of G is $V = \{D, N, P, NP, PP\}$ and the set of terminals is $\Sigma = \{the, cat, in, hat\}$.
The set of forms therefore contains all the (infinitely many) sequences of elements from V and Σ, such as $\langle\rangle$, $\langle NP\rangle$, $\langle D\ cat\ P\ D\ hat\rangle$, $\langle D\ N\rangle$, $\langle the\ cat\ in\ the\ hat\rangle$, etc.

Derivation

Example: Derivation

Let us start with a simple form, $\langle NP \rangle$. Observe that it can be written as $\gamma_l NP \gamma_r$, where both γ_l and γ_r are empty. Observe also that NP is the head of some grammar rule: the rule $NP \longrightarrow D\ N$. Therefore, the form is a good candidate for derivation: if we replace the selected symbol NP with the body of the rule, while preserving its environment, we get $\gamma_l D\ N \gamma_r = D\ N$. Therefore, $\langle NP \rangle \Rightarrow \langle D\ N \rangle$.

Derivation

Example: Derivation

We now apply the same process to $\langle D\ N \rangle$. This time the selected symbol is D (we could have selected N, of course). The left context is again empty, while the right context is $\gamma_r = N$. As there exists a grammar rule whose head is D, namely $D \rightarrow the$, we can replace the rule's head by its body, preserving the context, and obtain the form $\langle the\ N \rangle$. Hence $\langle D\ N \rangle \Rightarrow \langle the\ N \rangle$.

Derivation

Example: Derivation

Given the form $\langle \textit{the } N \rangle$, there is exactly one non-terminal that we can select, namely N. However, there are two rules that are headed by N: $N \longrightarrow \textit{cat}$ and $N \longrightarrow \textit{hat}$. We can select either of these rules to show that both $\langle \textit{the } N \rangle \Rightarrow \langle \textit{the cat} \rangle$ and $\langle \textit{the } N \rangle \Rightarrow \langle \textit{the hat} \rangle$. Since the form $\langle \textit{the cat} \rangle$ consists of terminal symbols only, no non-terminal can be selected and hence it derives no form.

Extended derivation

$\alpha \overset{k}{\Rightarrow}_G \beta$ if α derives β in k steps:
$\alpha \Rightarrow_G \alpha_1 \Rightarrow_G \alpha_2 \Rightarrow_G \ldots \Rightarrow_G \alpha_k$ and $\alpha_k = \beta$.
The reflexive-transitive closure of '$\Rightarrow_G$' is '$\overset{*}{\Rightarrow}_G$': $\alpha \overset{*}{\Rightarrow}_G \beta$ if $\alpha \overset{k}{\Rightarrow}_G \beta$ for some $k \geq 0$.
A G-**derivation** is a sequence of forms $\alpha_1, \ldots, \alpha_n$, such that for every $i, 1 \leq i < n$, $\alpha_i \Rightarrow_G \alpha_{i+1}$.

Extended derivation: example

Example: Derivation

$$
\begin{array}{llcl}
(1) & \langle NP \rangle & \Rightarrow & \langle D\ N \rangle \\
(2) & \langle D\ N \rangle & \Rightarrow & \langle \mathit{the}\ N \rangle \\
(3) & \langle \mathit{the}\ N \rangle & \Rightarrow & \langle \mathit{the\ cat} \rangle
\end{array}
$$

Extended derivation: example

Example: Derivation

Therefore, we trivially have:

$$
\begin{array}{llcl}
(4) & \langle NP \rangle & \overset{*}{\Rightarrow} & \langle D\ N \rangle \\
(5) & \langle D\ N \rangle & \overset{*}{\Rightarrow} & \langle \mathit{the}\ N \rangle \\
(6) & \langle \mathit{the}\ N \rangle & \overset{*}{\Rightarrow} & \langle \mathit{the\ cat} \rangle
\end{array}
$$

From (2) and (6) we get

$$(7) \quad \langle D\ N \rangle \overset{*}{\Rightarrow} \langle \mathit{the\ cat} \rangle$$

and from (1) and (7) we get

$$(7) \quad \langle NP \rangle \overset{*}{\Rightarrow} \langle \mathit{the\ cat} \rangle$$

Languages

A form α is a **sentential form** of a grammar G iff $S \stackrel{*}{\Rightarrow}_G \alpha$, i.e., it can be derived in G from the start symbol.

The (formal) **language** generated by a grammar G with respect to a category name (non-terminal) A is $L_A(G) = \{w \mid A \stackrel{*}{\Rightarrow} w\}$. The language generated by the grammar is $L(G) = L_S(G)$.

A language that can be generated by some CFG is a *context-free language* and the class of context-free languages is the set of languages every member of which can be generated by some CFG. If no CFG can generate a language L, L is said to be *trans-context-free*.

Language of a grammar

Example: Language

For the example grammar (with NP the start symbol):

$$\begin{array}{ll} D \rightarrow the & NP \rightarrow D\ N \\ N \rightarrow cat & PP \rightarrow P\ NP \\ N \rightarrow hat & NP \rightarrow NP\ PP \\ P \rightarrow in & \end{array}$$

it is fairly easy to see that $L(D) = \{the\}$.
Similarly, $L(P) = \{in\}$ and $L(N) = \{cat, hat\}$.

Language of a grammar

Example: Language

It is more difficult to define the languages denoted by the non-terminals *NP* and *PP*, although is should be straight-forward that the latter is obtained by concatenating $\{in\}$ with the former.

Proposition: $L(NP)$ is the denotation of the regular expression

$$the \cdot (cat + hat) \cdot (in \cdot\ the \cdot (cat + hat))^*$$

Language: a formal example G_e

Example: Language

$$
\begin{array}{lcl}
S & \longrightarrow & V_a \;\; S \;\; V_b \\
S & \longrightarrow & \epsilon \\
V_a & \longrightarrow & a \\
V_b & \longrightarrow & b
\end{array}
$$

$L(G_e) = \{a^n b^n \mid n \geq 0\}$.

Recursion

The language $L(G_e)$ is *infinite*: it includes an infinite number of words; G_e is a finite grammar.
To be able to produce infinitely many words with a finite number of rules, a grammar must be *recursive*: there must be at least one rule whose body contains a symbol, from which the head of the rule can be derived.
Put formally, a grammar $\langle \Sigma, V, S, P \rangle$ is recursive if there exists a chain of rules, $p_1, \ldots, p_n \in P$, such that for every $1 < i \leq n$, the head of p_{i+1} occurs in the body of p_i, and the head of p_1 occurs in the body of p_n.
In G_e, the recursion is simple: the chain of rules is of length 0, namely the rule $S \rightarrow V_a \;\; S \;\; V_b$ is in itself recursive.

Derivation tree

Sometimes derivations provide more information than is actually needed. In particular, sometimes two derivations of the same string differ not in the rules that were applied but only in the order in which they were applied.

Starting with the form $\langle NP \rangle$ it is possible to derive the string *the cat* in two ways:

$$
\begin{array}{ll}
(1) & \langle NP \rangle \Rightarrow \langle D\ N \rangle \Rightarrow \langle D\ cat \rangle \Rightarrow \langle the\ cat \rangle \\
(2) & \langle NP \rangle \Rightarrow \langle D\ N \rangle \Rightarrow \langle the\ N \rangle \Rightarrow \langle the\ cat \rangle
\end{array}
$$

Since both derivations use the same rules to derive the same string, it is sometimes useful to collapse such "equivalent" derivations into one. To this end the notion of *derivation trees* is introduced.

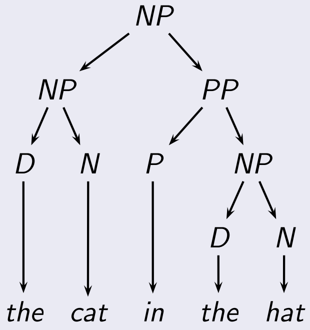
Derivation tree

A derivation tree (sometimes called *parse* tree, or simply tree) is a visual aid in depicting derivations, and a means for imposing structure on a grammatical string.
Trees consist of vertices and branches; a designated vertex, the *root* of the tree, is depicted on the top. Then, branches are simply connections between two vertices.
Intuitively, trees are depicted "upside down", since their root is at the top and their leaves are at the bottom.

Derivation tree

Example: Derivation tree

An example for a derivation tree for the string *the cat in the hat*:



Derivation tree

Formally, a tree consists of a finite set of vertices and a finite set of branches (or arcs), each of which is an ordered pair of vertices.
In addition, a tree has a designated vertex, the *root*, which has two properties: it is not the target of any arc, and every other vertex is accessible from it (by following one or more branches).
When talking about trees we sometimes use family notation: if a vertex v has a branch leaving it which leads to some vertex u, then we say that v is the *mother* of u and u is the *daughter*, or *child*, of v. If u has two daughters, we refer to them as *sisters*.

Derivation trees

Derivation trees are defined with respect to some grammar G, and must obey the following conditions:

1. every vertex has a *label*, which is either a terminal symbol, a non-terminal symbol or ϵ;
2. the label of the root is the start symbol;
3. if a vertex v has an outgoing branch, its label must be a non-terminal symbol, the head of some grammar rule; and the elements in body of the same rule must be the labels of the children of v, in the same order;
4. if a vertex is labeled ϵ, it is the only child of its mother.

Derivation trees

A *leaf* is a vertex with no outgoing branches.
A tree induces a natural "left-to-right" order on its leaves; when read from left to right, the sequence of leaves is called the *frontier*, or *yield* of the tree.

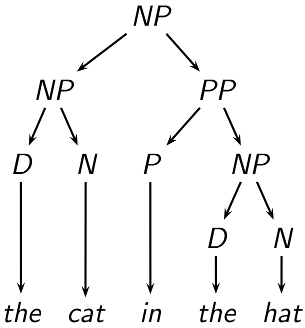
Correspondence between trees and derivations

Derivation trees correspond very closely to derivations.

For a form α, a non-terminal symbol A derives α if and only if α is the yield of some parse tree whose root is A.

Sometimes there exist different derivations of the same string that correspond to a single tree. In fact, the tree representation collapses exactly those derivations that differ from each other only in the order in which rules are applied.

Correspondence between trees and derivations



Each non-leaf vertex in the tree corresponds to some grammar rule (since it must be labeled by the head of some rule, and its children must be labeled by the body of the same rule).

Correspondence between trees and derivations

This tree represents the following derivations (among others):

$$
\begin{array}{ll}
(1) & NP \Rightarrow NP\ PP \Rightarrow D\ N\ PP \Rightarrow D\ N\ P\ NP \\
 & \Rightarrow D\ N\ P\ D\ N \Rightarrow the\ N\ P\ D\ N \\
 & \Rightarrow the\ cat\ P\ D\ N \Rightarrow the\ cat\ in\ D\ N \\
 & \Rightarrow the\ cat\ in\ the\ N \Rightarrow the\ cat\ in\ the\ hat \\
(2) & NP \Rightarrow NP\ PP \Rightarrow D\ N\ PP \Rightarrow the\ N\ PP \\
 & \Rightarrow the\ cat\ PP \Rightarrow the\ cat\ P\ NP \\
 & \Rightarrow the\ cat\ in\ NP \Rightarrow the\ cat\ in\ D\ N \\
 & \Rightarrow the\ cat\ in\ the\ N \Rightarrow the\ cat\ in\ the\ hat \\
(3) & NP \Rightarrow NP\ PP \Rightarrow NP\ P\ NP \Rightarrow NP\ P\ D\ N \\
 & \Rightarrow NP\ P\ D\ hat \Rightarrow NP\ P\ the\ hat \\
 & \Rightarrow NP\ in\ the\ hat \Rightarrow D\ N\ in\ the\ hat \\
 & \Rightarrow D\ cat\ in\ the\ hat \Rightarrow the\ cat\ in\ the\ hat
\end{array}
$$

Correspondence between trees and derivations

While exactly the same rules are applied in each derivation (the rules are uniquely determined by the tree), they are applied in different orders. In particular, derivation (2) is a *leftmost* derivation: in every step the leftmost non-terminal symbol of a derivation is expanded. Similarly, derivation (3) is *rightmost*.

Ambiguity

Sometimes, however, different derivations (of the same string!) correspond to different trees.

This can happen only when the derivations differ in the rules which they apply.

When more than one tree exists for some string, we say that the string is *ambiguous*.

Ambiguity is a major problem when grammars are used for certain formal languages, in particular programming languages. But for natural languages, ambiguity is unavoidable as it corresponds to properties of the natural language itself.

Ambiguity: example

Consider again the example grammar and the following string:

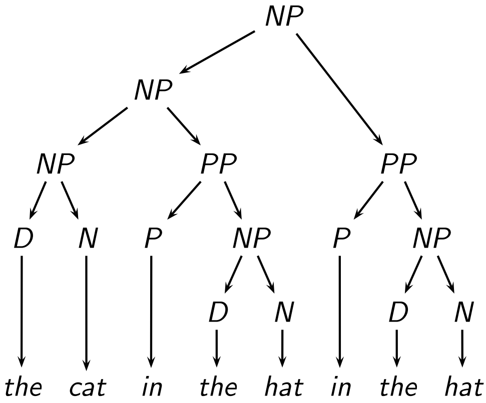
the cat in the hat in the hat

Intuitively, there can be (at least) two readings for this string: one in which a certain cat wears a hat-in-a-hat, and one in which a certain cat-in-a-hat is inside a hat:

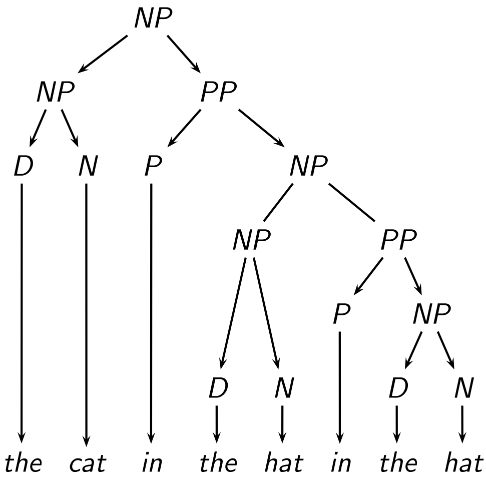
((the cat in the hat) in the hat)
(the cat in (the hat in the hat))

This distinction in intuitive meaning is reflected in the grammar, and hence two different derivation trees, corresponding to the two readings, are available for this string:

Ambiguity: example



Ambiguity: example



Ambiguity: example

Using linguistic terminology, in the left tree the second occurrence of the prepositional phrase *in the hat* modifies the noun phrase *the cat in the hat*, whereas in the right tree it only modifies the (first occurrence of) the noun phrase *the hat*. This situation is known as *syntactic* or *structural* ambiguity.

Grammar equivalence

It is common in formal language theory to relate different grammars that generate the same language by an equivalence relation:

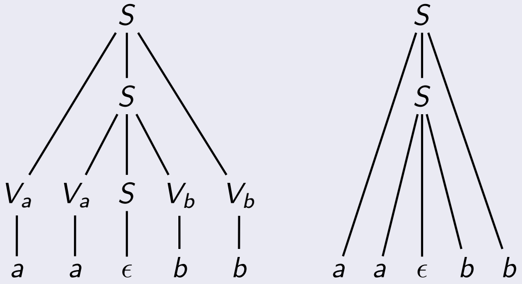
Two grammars G_1 and G_2 (over the same alphabet Σ) are **equivalent** (denoted $G_1 \equiv G_2$) iff $L(G_1) = L(G_2)$.

We refer to this relation as *weak equivalence*, as it only relates the generated languages. Equivalent grammars may attribute totally different syntactic structures to members of their (common) languages.

Grammar equivalence

Example: Equivalent grammars, different trees

Following are two different tree structures that are attributed to the string *aabb* by the grammars G_e and G_f, respectively.



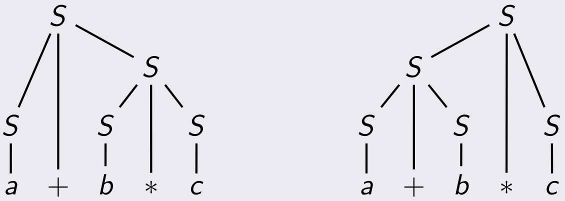
Grammar equivalence

Example: Structural ambiguity

A grammar, G_{arith}, for simple arithmetic expressions:

$$S \quad \rightarrow \quad a \mid b \mid c \mid S + S \mid S * S$$

Two different trees can be associated by G_{arith} with the string $a + b * c$:



Grammar equivalence

Weak equivalence relation is stated in terms of the generated language. Consequently, equivalent grammars do not have to be described in the same formalism for them to be equivalent. We will later see how grammars, specified in different formalisms, can be compared.

Normal form

It is convenient to divide grammar rules into two classes: one that contains only *phrasal rules* of the form $A \longrightarrow \alpha$, where $\alpha \in V^*$, and another that contains only *terminal rules* of the form $B \longrightarrow \sigma$ where $\sigma \in \Sigma$. It turns out that every CFG is equivalent to some CFG of this form.

Normal form

A grammar G is in **phrasal/terminal normal form** iff for every production $A \to \alpha$ of G, either $\alpha \in V^*$ or $\alpha \in \Sigma$. Productions of the form $A \to \sigma$ are called **terminal rules**, and A is said to be a **pre-terminal category**, the **lexical entry** of σ. Productions of the form $A \to \alpha$, where $\alpha \in V^*$, are called **phrasal rules**. Furthermore, every category is either pre-terminal or phrasal, but not both. For a phrasal rule with $\alpha = A_1 \cdots A_n, w = w_1 \cdots w_n, w \in L_A(G)$ and $w_i \in L_{A_i}(G)$ for $i = 1, \ldots, n$, we say that w is a phrase of category A, and each w_i is a **sub-phrase** (of w) of category A_i. A sub-phrase w_i of w is also called a **constituent** of w.

Context-free grammars for natural languages

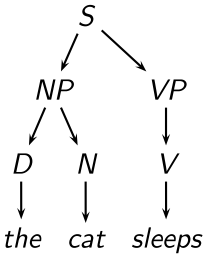
A context-free grammar for English sentences: $G = \langle V, \Sigma, P, S \rangle$ where $V = \{D,\ N,\ P,\ NP,\ PP,\ V,\ VP,\ S\}$, $\Sigma = \{the,\ cat,\ in,\ hat,\ sleeps,\ smile,\ loves,\ saw\}$, the start symbol is S and P is the following set of rules:

$$
\begin{array}{lcl@{\qquad\qquad}lcl}
S & \rightarrow & NP\ VP & D & \rightarrow & the \\
NP & \rightarrow & D\ N & N & \rightarrow & cat \\
NP & \rightarrow & NP\ PP & N & \rightarrow & hat \\
PP & \rightarrow & P\ NP & V & \rightarrow & sleeps \\
VP & \rightarrow & V & P & \rightarrow & in \\
VP & \rightarrow & VP\ NP & V & \rightarrow & smile \\
VP & \rightarrow & VP\ PP & V & \rightarrow & loves \\
 & & & V & \rightarrow & saw
\end{array}
$$

Context-free grammars for natural languages

The augmented grammar can derive strings such as *the cat sleeps* or *the cat in the hat saw the hat*.

A derivation tree for *the cat sleeps* is:



Context-free grammars for natural languages

There are two major problems with this grammar.

1. it ignores the valence of verbs: there is no distinction among subcategories of verbs, and an intransitive verb such as *sleep* might occur with a noun phrase complement, while a transitive verb such as *love* might occur without one. In such a case we say that the grammar *overgenerates*: it generates strings that are not in the intended language.
2. there is no treatment of subject–verb agreement, so that a singular subject such as *the cat* might be followed by a plural form of verb such as *smile*. This is another case of overgeneration.

Both problems are easy to solve.

Verb valence

To account for valence, we can replace the non-terminal symbol V by a set of symbols: *Vtrans, Vintrans, Vditrans* etc. We must also change the grammar rules accordingly:

$$
\begin{array}{ll}
VP \longrightarrow Vintrans & Vintrnas \longrightarrow sleeps \\
VP \longrightarrow Vtrans\ NP & Vintrans \longrightarrow smile \\
VP \longrightarrow Vditrans\ NP\ PP & Vtrans \longrightarrow loves \\
 & Vditrans \longrightarrow give
\end{array}
$$

Agreement

To account for agreement, we can again extend the set of non-terminal symbols such that categories that must agree reflect in the non-terminal that is assigned for them the features on which they agree. In the very simple case of English, it is sufficient to multiply the set of "nominal" and "verbal" categories, so that we get *Dsg, Dpl, Nsg, Npl, NPsg, NPpl, Vsg, Vlp, VPsg, VPpl* etc. We must also change the set of rules accordingly:

Agreement

$$\begin{array}{lll}
Nsg \longrightarrow cat & \qquad & Npl \longrightarrow cats \\
Nsg \longrightarrow hat & & Npl \longrightarrow hats \\
P \longrightarrow in & & \\
Vsg \longrightarrow sleeps & & Vpl \longrightarrow sleep \\
Vsg \longrightarrow smiles & & Vpl \longrightarrow smile \\
Vsg \longrightarrow loves & & Vpl \longrightarrow love \\
Vsg \longrightarrow saw & & Vpl \longrightarrow saw \\
Dsg \longrightarrow a & & Dpl \longrightarrow many
\end{array}$$

Agreement

$S \longrightarrow NPsg\ VPsg$
$NPsg \longrightarrow Dsg\ Nsg$
$NPsg \longrightarrow NPsg\ PP$
$PP \longrightarrow P\ NP$
$VPsg \longrightarrow Vsg$
$VPsg \longrightarrow VPsg\ NP$
$VPsg \longrightarrow VPsg\ PP$

$S \longrightarrow NPpl\ VPpl$
$NPpl \longrightarrow Dpl\ Npl$
$NPpl \longrightarrow NPpl\ PP$

$VPpl \longrightarrow Vpl$
$VPpl \longrightarrow VPpl\ NP$
$VPpl \longrightarrow VPpl\ PP$

Context-free grammars for natural languages

Context-free grammars can be used for a variety of syntactic constructions, including some non-trivial phenomena such as unbounded dependencies, extraction, extraposition etc.

However, some (formal) languages are not context-free, and therefore there are certain sets of strings that cannot be generated by context-free grammars.

The interesting question, of course, involves natural languages: are there natural languages that are not context-free? Are context-free grammars sufficient for generating every natural language?