Section 1

One-Way Functions
A one-way function (OWF) is:

- Easy to compute, everywhere
- Hard to invert, on the average
Informal discussion

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Why should we care about OWFs?
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- Hidden in (almost) any cryptographic primitive: necessary for “cryptography"
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- Easy to compute, everywhere
- Hard to invert, on the average

Why should we care about OWFs?
- Hidden in (almost) any cryptographic primitive: necessary for “cryptography"
- Sufficient for many cryptographic primitives
“Application”: Authentication where server doesn’t store the user’s password.
Formal definition

Definition 1 (one-way functions (OWFs))
A polynomial-time computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ is one-way, if
\[
\Pr_{x \leftarrow \{0,1\}^n} \left[ A(1^n, f(x)) \in f^{-1}(f(x)) \right] = \text{neg}(n)
\]
for any PPT $A$. 
Formal definition

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- polynomial-time computable: there exists polynomial-time algorithm \( F \), such that \( F(x) = f(x) \) for every \( x \in \{0, 1\}^* \).

- neg: a function \( \mu : \mathbb{N} \mapsto [0, 1] \) is a negligible function of \( n \), denoted \( \mu(n) = \text{neg}(n) \), if for any \( p \in \text{poly} \) there exists \( n' \in \mathbb{N} \) such that \( \mu(n) < 1/p(n) \) for all \( n > n' \).
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- **\( x \leftarrow \{0, 1\}^n \)**: \( x \) is uniformly drawn from \( \{0, 1\}^n \)

- **PPT**: probabilistic polynomial-time algorithm.

We typically omit \( 1^n \) from the input list of \( A \).
Formal definition cont.

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   ▶ Asymptotic
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Formal definition cont.

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   - Asymptotic
   - Efficiently computable
   - On the average
   - Only against PPT’s

2. OWF = \[ P \neq NP \]

3. Does \[ P \neq NP = \Rightarrow \text{OWF} \]?

4. (most) Crypto implies OWFs

5. Do OWFs imply Crypto?

6. Where do we find them?

7. Non uniform OWFs

Definition 2 (Non-uniform OWF)

A polynomial-time computable function \( f : \{0,1\}^* \rightarrow \{0,1\}^* \) is non-uniformly one-way, if \( \Pr_{x \leftarrow \{0,1\}^n}[C_n(f(x)) \in f^{-1}(f(x))] = \neg(n) \) for any polynomial-size family of circuits \( \{C_n\} \).
Formal definition cont.

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   - Asymptotic
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2. OWF $\implies \mathcal{P} \neq \mathcal{NP}$
Formal definition cont.

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3. Does $\mathcal{P} \neq \mathcal{NP} \implies \text{OWF}$?
Formal definition cont.

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\[
\Pr_{x \leftarrow \{0,1\}^n} \left[ C_n(f(x)) \in f^{-1}(f(x)) \right] = \text{neg}(n)
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for any polynomial-size family of circuits \( \{C_n\}_{n \in \mathbb{N}} \).
Definition 3 (length preserving functions)

A function $f : \{0, 1\}^* \rightarrow f : \{0, 1\}^*$ is length preserving, if $|f(x)| = |x|$ for every $x \in \{0, 1\}^*$.

Theorem 4
Assume that OWFs exist, then there exist length-preserving OWFs.

Proof idea: use the assumed OWF to create a length preserving one.
Length-preserving functions

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Partial domain functions

**Definition 5 (Partial domain functions)**

Let $m, \ell : \mathbb{N} \mapsto \mathbb{N}$ be polynomials. Let $f : \{0, 1\}^{\ell(n)} \mapsto \{0, 1\}^{m(n)}$ denote a function defined over input lengths in $\{m(n)\}_{n \in \mathbb{N}}$, and maps strings of length $\ell(n)$ to strings of length $m(n)$. Such function is efficient, if it is poly-time computable. The definition of one-wayness naturally extends to such (efficient) functions.
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The definition of one-wayness naturally extends to such (efficient) functions.
OWFs imply length-preserving OWFs cont.

Let $f : \{0, 1\}^* \mapsto \{0, 1\}^*$ be a OWF, let $p \in \text{poly}$ be a bound on its computing-time, and assume w.l.g. that $p$ is monotony increasing (can we?). Note that $|f(x)| \leq p(|x|)$.

**Construction 6 (the length preserving function)**

Define $g : \{0, 1\} \overset{p(n)+1}{\mapsto} \{0, 1\} \overset{p(n)+1}{\mapsto}$ as $g(x) = f(x_1, \ldots, n, 1, 0)_{p(n)-|f(x_1, \ldots, n)|}$.

Note that $g$ is well defined, length preserving and efficient.

**Claim 7** $g$ is one-way.

How can we prove that $g$ is one-way?

Answer: using reduction.
OWFs imply length-preserving OWFs cont.

Let \( f : \{0, 1\}^* \mapsto \{0, 1\}^* \) be a OWF, let \( p \in \text{poly} \) be a bound on its computing-time, and assume wlg. that \( p \) is monotony increasing (can we?). Note that \(|f(x)| \leq p(|x|)\).

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Define $g : \{0, 1\}^{p(n)+1} \mapsto \{0, 1\}^{p(n)+1}$ as

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Answer: using reduction.
**Proving that** \( g \) **is one-way**

**Proof:** Assume that \( g \) **is not** one-way. Namely, there exists PPT \( A \), \( q \in \text{poly} \) and infinite set \( I \subseteq \{ p(n) + 1 : n \in \mathbb{N} \} \), with

\[
\Pr_{x \leftarrow \{0,1\}^{n'}} \left[ A(1^{n'}, y) \in g^{-1}(g(x)) \right] > 1/q(n') \tag{1}
\]

for every \( n' \in I \).
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for every \( n' \in \mathcal{I} \).

We show how to use \( A \) for inverting \( f \).
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for every $n' \in I$.

We show how to use $A$ for inverting $f$.

\begin{claim}
\begin{equation*}
w \in g^{-1}(y, 1, 0^{p(n) - |y|}) \implies w_1, ..., n \in f^{-1}(y)
\end{equation*}
\end{claim}
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$w \in g^{-1}(y, 1, 0^{p(n) - |y|}) \implies w_1, \ldots, n \in f^{-1}(y)$

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\( w \in g^{-1}(y, 1, \, 0^{p(n)} - |y|) \implies w_{1, \ldots, n} \in f^{-1}(y) \)

Proof: Since \( g(w) = f(w_{1, \ldots, n}), 1, \, 0^{p(n)} - |f(w_{1, \ldots, n})| = y, 1, \, 0^{p(n)} - |y| \),
Proving that \( g \) is one-way

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We show how to use \( A \) for inverting \( f \).

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\( w \in g^{-1}(y, 1, 0^{p(n) - |y|}) \implies w_1, \ldots, n \in f^{-1}(y) \)

Proof: Since \( g(w) = f(w_1, \ldots, n), 1, 0^{p(n) - |f(w_1, \ldots, n)|} = y, 1, 0^{p(n) - |y|} \), it follows that \( f(w_1, \ldots, n) = y \). \( \square \)
Algorithm 9 (Inverter B for f)

Input: $1^n$ and $y \in \{0, 1\}^*$

1. Let $x = A(1^{p(n)+1}, y, 1, 0^{p(n)} - |y|)$

2. Return $x_1, \ldots, n$
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### Claim 10

Let $\mathcal{I}' := \{n \in \mathbb{N} : p(n) + 1 \in \mathcal{I}\}$. Then

1. $\mathcal{I}'$ is infinite
2. $\Pr_{x \leftarrow \{0,1\}^n}[B(1^n, f(x)) \in f^{-1}(f(x))] > 1/q(p(n) + 1)$ for every $n \in \mathcal{I}'$
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This contradicts the assumed one-wayness of $f$. □
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Proof: (1) is clear, (2)
Algorithm 9 (Inverter B for \( f \))

Input: \( 1^n \) and \( y \in \{0, 1\}^* \)

1. Let \( x = A(1^{p(n)+1}, y, 1, 0^{p(n)}-|y|) \)

2. Return \( x_1, \ldots, n \)

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Let \( \mathcal{I}' := \{ n \in \mathbb{N} : p(n) + 1 \in \mathcal{I} \} \). Then

1. \( \mathcal{I}' \) is infinite

2. \( \Pr_{x \leftarrow \{0,1\}^n}[B(1^n, f(x)) \in f^{-1}(f(x))] > 1/q(p(n) + 1) \) for every \( n \in \mathcal{I}' \)

This contradicts the assumed one-wayness of \( f \). \( \square \)

Proof: (1) is clear, (2)

\[
\Pr_{x \leftarrow \{0,1\}^n} \left[ B(1^n, f(x)) \in f^{-1}(f(x)) \right]
\]
Algorithm 9 (Inverter B for f)

Input: $1^n$ and $y \in \{0, 1\}^*$

1. Let $x = A(1^p(n) + 1, y, 1, 0^p(n) - |y|)$

2. Return $x_1, ..., n$

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Let $\mathcal{I}' := \{n \in \mathbb{N}: p(n) + 1 \in \mathcal{I}\}$. Then

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Proof: (1) is clear, (2)

$$\Pr_{x \leftarrow \{0, 1\}^n} [B(1^n, f(x)) \in f^{-1}(f(x))]$$

$$= \Pr_{x \leftarrow \{0, 1\}^n} [A(1^p(n) + 1, f(x), 1, 0^p(n) - |f(x)|)_1, ..., n \in f^{-1}(f(x)))]$$
Algorithm 9 (Inverter B for $f$)

Input: $1^n$ and $y \in \{0, 1\}^*$

1. Let $x = A(1^{p(n)+1}, y, 1, 0^{p(n)} - |y|)$
2. Return $x_1, ..., n$

Claim 10

Let $I := \{ n \in \mathbb{N} : p(n) + 1 \in I \}$. Then

1. $I$ is infinite
2. $\Pr_{x \leftarrow \{0,1\}^n}[B(1^n, f(x)) \in f^{-1}(f(x))] > 1/q(p(n) + 1)$ for every $n \in I$

This contradicts the assumed one-wayness of $f$. □

Proof: (1) is clear, (2)

$$\Pr_{x \leftarrow \{0,1\}^n}[B(1^n, f(x)) \in f^{-1}(f(x))]$$

$$= \Pr_{x \leftarrow \{0,1\}^n}[A(1^{p(n)+1}, f(x), 1, 0^{p(n)} - |f(x)|)_{1,...,n} \in f^{-1}(f(x))]$$

$$= \Pr_{x' \leftarrow \{0,1\}^{p(n)+1}}[A(1^{p(n)+1}, g(x'))_{1,...,n} \in f^{-1}(f(x')_{1,...,n})]$$
**Algorithm 9 (Inverter B for f)**

**Input:** $1^n$ and $y \in \{0, 1\}^*$

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Let $\mathcal{I}' := \{n \in \mathbb{N}: p(n) + 1 \in \mathcal{I}\}$. Then

1. $\mathcal{I}'$ is infinite
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\]
From partial-domain length-preserving OWFs to length-preserving OWFs

**Construction 11**

Given a function $f : \{0, 1\}^{\ell(n)} \mapsto \{0, 1\}^{\ell(n)}$, define $f_{\text{all}} : \{0, 1\}^n \mapsto \{0, 1\}^n$ as

$$f_{\text{all}}(x) = f(x_1, \ldots, k), 0^{n-k}$$

where $n = |x|$ and $k := \max\{\ell(n') \leq n : n' \in [n]\}$.
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Clearly, $f_{\text{all}}$ is length preserving, defined for every input length, and efficient if $f$ is.
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**Claim 12**

Assume $f$ is efficient, $f$ is one-way, and $\ell$ satisfies $1 \leq \frac{\ell(n+1)}{\ell(n)} \leq p(n)$ for some $p \in \text{poly}$, then $f_{\text{all}}$ is one-way function.
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Proof: ?
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Assume \( f \) is efficient, \( f \) is one-way, and \( \ell \) satisfies \( 1 \leq \frac{\ell(n+1)}{\ell(n)} \leq p(n) \) for some \( p \in \text{poly} \), then \( f_{\text{all}} \) is one-way function.

Proof: ?

We conclude that the existence of OWF implies the existence of length-preserving OWF that is defined over all input lengths.
Few remarks

More “security-preserving" reductions exits.
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Convention for rest of the talk

Let $f : \{0, 1\}^n \leftrightarrow \{0, 1\}^n$ be a one-way function.
Weak one-way functions

Definition 13 (weak one-way functions)

A poly-time computable function $f : \{0, 1\}^* \rightarrow f : \{0, 1\}^*$ is $\alpha$-one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n} \left[ A(1^n, f(x)) \in f^{-1}(f(x)) \right] \leq \alpha(n)$$

for any PPT $A$ and large enough $n \in \mathbb{N}$. 

1. For example consider $\alpha(n) = 0.1$, or $\alpha(n) = 0.99$, or maybe even $\alpha(n) = 1 - 1/n$.
2. (strong) OWF according to Definition 1, are neg-one-way according to the above definition.
3. Can we "amplify" weak OWF to strong ones?
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Assume there exists OWFs, then there exist functions that are $\frac{2}{3}$-one-way, but not (strong) one-way.
Strong to weak OWFs

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Proof:
**Strong to weak OWFs**

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**Proof:** For a OWF $f$, let

$$g(x, b) = \begin{cases} (1, f(x)), & b = 1; \\ (0, x), & \text{otherwise } (b = 0). \end{cases}$$
Weak to strong OWFs

**Theorem 15 (weak to strong OWFs (Yao))**

Assume there exist \((1 - \delta)\)-weak OWFs with \(\delta(n) \geq 1/q(n)\) for some \(q \in \text{poly}\), then there exist (strong) one-way functions.
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Assume there exist \((1 - \delta)\)-weak OWFs with \(\delta(n) \geq 1/q(n)\) for some \(q \in \text{poly}\), then there exist (strong) one-way functions.

- Idea: parallel repetition (i.e., direct product): Consider \(g(x_1, \ldots, x_t) = f(x_1), \ldots, f(x_t)\) for large enough \(t\)
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- Motivation: if something is somewhat hard, than doing it many times is (very) hard
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  Consider matrix multiplication: Let \(A \in \mathbb{R}^{n \times n}\) and \(x \in \mathbb{R}^n\)

  Computing \(Ax\) takes \(\Theta(n^2)\) times, but computing \(A(x_1, x_2, \ldots, x_n)\) takes ...

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  Computing $Ax$ takes $\Theta(n^2)$ times, but computing $A(x_1, x_2, \ldots, x_n)$ takes ... only $O(n^{2.3\ldots}) < \Theta(n^3)$

- Fortunately, parallel repetition does amplify weak OWFs :-)

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  Foundation of Cryptography
  2018 16/27
Amplification via parallel repetition

Theorem 16

Let $f : \{0, 1\}^n \mapsto \{0, 1\}^n$ be a $(1 - \delta)$-weak OWF for $\delta(n) = 1/q(n)$ for some (positive) $q \in \text{poly}$, and let $t(n) = \lceil \log^2 n / \delta(n) \rceil$. Then $g : (\{0, 1\}^n)^{t(n)} \mapsto (\{0, 1\}^n)^{t(n)}$ defined by $g(x_1, \ldots, x_{t(n)}) = f(x_1), \ldots, f(x_{t(n)})$, is a one-way function.
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Difficultly: We need to use an inverter for $g$ with low success probability, e.g., $1/n$, to get an inverter for $f$ with high success probability, e.g., $1/2$ or even $1 - 1/n$.
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In the following we fix (an assumed) PPT \( A, p \in \text{poly} \) and infinite set \( I \subseteq \mathbb{N} \) s.t.

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\Pr_{w \leftarrow \{0,1\}^t(n) \cdot n} [A(g(w)) \in g^{-1}(g(w))] \geq 1/p(n)
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$$\Pr_{w \leftarrow \{0, 1\}^{t(n)} \cdot n}[A(g(w)) \in g^{-1}(g(w))] \geq 1/p(n)$$

for every $n \in \mathcal{I}$. We also “fix" $n \in \mathcal{I}$ and omit it from the notation.
Proving that \( g \) is One-Way – the Naive approach

Assume \( A \) attacks each of the \( t \) outputs of \( g \) independently: \( \exists \text{ PPT } A' \) such that \( A(z_1, \ldots, z_t) = A'(z_1) \ldots, A'(z_t) \)
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A less naive approach would be to assume that $A$ goes over the inputs sequentially.
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Unfortunately, we can assume none of the above.
Proving that $g$ is One-Way – the Naive approach

Assume $A$ attacks each of the $t$ outputs of $g$ independently: $\exists$ PPT $A'$ such that $A(z_1, \ldots, z_t) = A'(z_1) \ldots, A'(z_t)$

It follows that $A'$ inverts $f$ with probability greater than $(1 - \delta)$. Otherwise

$$\Pr_{w \leftarrow \{0,1\}^t \cdot n} [A(g(w)) \in g^{-1}(g(w))] = \prod_{i=1}^{t} \Pr_{x \leftarrow \{0,1\}^n} [A'(f(x)) \in f^{-1}(f(x))]$$

$$\leq (1 - \delta)^t \leq e^{-\log^2 n} \leq n^{-\log n}$$

Hence $A'$ violates the weak hardness of $f$

A less naive approach would be to assume that $A$ goes over the inputs sequentially.

Unfortunately, we can assume none of the above.

Any idea?
Hardcore sets

Assume $f$ is of the form

$\Pr[x \leftarrow \{0,1\}^n [f(x) \in S] \geq \delta(n)$ for large enough $n$, and

$\Pr[A(y) \in f^{-1}(y)] \leq \frac{1}{q(n)}$ for every $y \in S$.

Assuming $f$ has such a $\delta$-HC set seems like a good starting point :-)

Unfortunately, we do not know how to prove that $f$ has hardcore set :-<
Hardcore sets

Assume $f$ is of the form

Definition 17 (hardcore sets)

$S = \{S_n \subseteq \{0, 1\}^n\}$ is a $\delta$-hardcore set for $f: \{0, 1\}^n \mapsto \{0, 1\}^n$, if:

1. $\Pr_{x \leftarrow \{0, 1\}^n} [f(x) \in S] \geq \delta(n)$ for large enough $n$, and

2. For any PPT $A$ and $q \in \text{poly}$: for large enough $n$, it holds that $\Pr[A(y) \in f^{-1}(y)] \leq \frac{1}{q(n)}$ for every $y \in S_n$. 
Hardcore sets

Assume \( f \) is of the form

\[
\begin{align*}
\mathcal{S} = \{ S_n \subseteq \{0, 1\}^n \} & \text{ is a } \delta \text{-hardcore set for } f : \{0, 1\}^n \mapsto \{0, 1\}^n, \text{ if:} \\
1. \quad \Pr_{x \sim\{0,1\}^n} [f(x) \in S] \geq \delta(n) \text{ for large enough } n, \text{ and} \\
2. \quad \text{For any PPT } A \text{ and } q \in \text{poly}: \text{ for large enough } n, \text{ it holds that} \\
\Pr [A(y) \in f^{-1}(y)] \leq \frac{1}{q(n)} \text{ for every } y \in S_n.
\end{align*}
\]

Assuming \( f \) has such a \( \delta \)-HC set seems like a good starting point :-)

Definition 17 (hardcore sets)
Hardcore sets

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Assuming $f$ has such a $\delta$-HC set seems like a good starting point :-) 

Unfortunately, we do not know how to prove that $f$ has hardcore set :-<
Failing sets

Definition 18 (failing sets)

\( f : \{0,1\}^n \rightarrow \{0,1\}^n \) has a \( \delta \)-failing set for a pair \((A, q)\) of algorithm and polynomial, if exists \( S = \{ S_n \subseteq \{0,1\}^n \} \), such that the following holds for large enough \( n \):

1. \( \Pr_{x \leftarrow \{0,1\}^n}[f(x) \in S_n] \geq \delta(n) \), and
2. \( \Pr[A(y) \in f^{-1}(y)] \leq 1/q(n) \), for every \( y \in S_n \).

Claim 19

Let \( f \) be a \((1 - \delta)\)-OWF, then \( f \) has a \( \delta/2 \)-failing set, for any pair of PPT \( A \) and \( q \in \text{poly} \).

High level idea:

Define \( S_n := \{ y \in \{0,1\}^n : \Pr[A(y) \in f^{-1}(y)] < 1/q(n) \} \).

1. If this set is small, show that \( A \) inverts \( f \) very well.
2. If this set is large, then it is by definition a fooling set.
Failing sets

**Definition 18 (failing sets)**

\( f : \{0, 1\}^n \mapsto \{0, 1\}^n \) has a \( \delta \)-failing set for a pair \((A, q)\) of algorithm and polynomial, if exists \( S = \{ S_n \subseteq \{0, 1\}^n \} \), such that the following holds for large enough \( n \):

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1. If this set is small, show that \( A \) inverts \( f \) very well.
2. If this set is large, then it is by definition a fooling set.
Proof:

Assume there exists a PPT algorithm $A$ and a polynomial $q$, such that for any $S = \{S_n \subseteq \{0, 1\}^n\}$, at least one of the following holds:

1. $\Pr_{x \leftarrow \{0, 1\}^n}[f(x) \in S] < \frac{\delta(n)}{2}$ for infinitely many $n$'s, or
2. For infinitely many $n$'s: $\exists y \in S_n$ with $\Pr[A(y) \in f^{-1}(y)] \geq \frac{1}{q(n)}$.

We'll use $A$ to contradict the hardness of $f$. 
Proof: Assume $\exists$ PPT $A$ and $q \in \text{poly}$, such that for any $S = \{S_n \subseteq \{0, 1\}^n\}$ at least one of the following holds:

1. $\Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] < \delta(n)/2$ for infinitely many $n$'s, or

2. For infinitely many $n$'s: $\exists y \in S_n$ with $\Pr [A(y) \in f^{-1}(y)] \geq 1/q(n)$. 
Proof: Assume \( \exists \) PPT \( A \) and \( q \in \text{poly} \), such that for any \( S = \{S_n \subseteq \{0, 1\}^n\} \) at least one of the following holds:

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Using $A$ to invert $f$

For $n \in \mathbb{N}$, let $S_n := \{y \in \{0, 1\}^n : \Pr[A(y) \in f^{-1}(y)] < 1/q(n)\}$. 
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For $n \in \mathbb{N}$, let $S_n := \{y \in \{0, 1\}^n : \Pr[A(y) \in f^{-1}(y)] < 1/q(n)\}$. The second item cannot hold, therefore the first item must hold, meaning that:

**Claim 20**

$\exists$ infinite $\mathcal{I} \subseteq \mathbb{N}$ with $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in S_n] < \delta(n)/2$ for every $n \in \mathcal{I}$. 

Proof: ?

Hence, for large enough $n \in \mathcal{I}$:

$\Pr_{x \leftarrow \{0,1\}^n}[B(f(x)) \in f^{-1}(f(x))] > 1 - \delta(n)/2$.

Namely, $f$ is not $(1 - \delta(n))/2$-one-way.
Using \( A \) to invert \( f \)

For \( n \in \mathbb{N} \), let \( S_n := \{ y \in \{0, 1\}^n : \Pr[A(y) \in f^{-1}(y)] < 1/q(n) \} \). The second item cannot hold, therefore the first item must hold, meaning that:

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**Algorithm 21 (The inverter \( B \) on input \( y \in \{0, 1\}^n \))**

Do (with fresh randomness) for \( n \cdot q(n) \) times:
- If \( x = A(y) \in f^{-1}(y) \), return \( x \)

Clearly, \( B \) is a \( \mathsf{PPT} \).

**Claim 22**

For \( n \in \mathcal{I} \), it holds that \( \Pr_{x \leftarrow \{0, 1\}^n} [B(f(x)) \in f^{-1}(f(x))] > 1 - \delta(n)/2 \) for every \( n \in \mathcal{I} \).

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Hence, for large enough \( n \in \mathcal{I} \):

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\[ \exists \text{ infinite } I \subseteq \mathbb{N} \text{ with } \Pr_{x \leftarrow \{0,1\}^n}[f(x) \in S_n] < \delta(n)/2 \text{ for every } n \in I. \]

**Algorithm 21 (The inverter $B$ on input $y \in \{0,1\}^n$)**

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If $x = A(y) \in f^{-1}(y)$, return $x$

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For \( n \in \mathbb{N} \), let \( S_n := \{ y \in \{0, 1\}^n : \Pr[A(y) \in f^{-1}(y)] < 1/q(n) \} \).

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Algorithm 21 (The inverter B on input \( y \in \{0, 1\}^n \))

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Claim 22

For \( n \in \mathcal{I} \), it holds that \( \Pr_{x \leftarrow \{0,1\}^n}[B(f(x)) \in f^{-1}(f(x))] > 1 - \frac{\delta(n)}{2} - 2^{-n} \)
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Hence, for large enough \( n \in I \):
\[ \Pr_{x \leftarrow \{0,1\}^n} [B(f(x)) \in f^{-1}(f(x))] > 1 - \delta(n). \]
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**Claim 22**

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Proof: ?

Hence, for large enough $n \in \mathcal{I}$: $\Pr_{x \leftarrow \{0,1\}^n} [B(f(x)) \in f^{-1}(f(x))] > 1 - \delta(n)$.

Namely, $f$ is not $(1 - \delta)$-one-way □
$g$ is not one-way $\implies f$ has no $\delta/2$ failing set

We show: $g$ is not one way $\implies f$ has no $\delta/2$ failing-set for some PPT $B$ and $q \in \text{poly}$. 

Claim 23

Assume $\exists$ PPT $A$, $p \in \text{poly}$ and an infinite set $I \subseteq \mathbb{N}$ such that $Pr_{w \leftarrow \{0,1\}^t(n)}[A(g(x)) \in g^{-1}(g(w))] \geq 1/p(n)$ for every $n \in I$.

Then $\exists$ PPT $B$ such that $Pr_{x \leftarrow \{0,1\}^n}[y = f(x) \in S_n|B(y) \in f^{-1}(y)] \geq 1/t(n)p(n) - n - \log n$ for every $n \in I$ and every $S_n \subseteq \{0,1\}^n$ with $Pr_{x \leftarrow \{0,1\}^n}[f(x) \in S_n] \geq \delta(n)/2$.

Thm follows: Fix $S = \{S_n \subseteq \{0,1\}^n\}$.

By Claim 23, for every $n \in I$, either $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in S_n] < \delta(n)/2$, or $\Pr_{x \leftarrow \{0,1\}^n}[y = f(x) \in S_n|B(y) \in f^{-1}(y)] \geq 1/t(n)p(n) - n - \log n$ (for large enough $n$).

Namely, $f$ has no $\delta/2$ failing set for $(B,q = 2t(n)p(n))$. 

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Foundation of Cryptography  
2018 23/27
**Claim 23**

Assume $\exists$ PPT $A$, $p \in \text{poly}$ and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that

$$\Pr_{w \leftarrow \{0,1\}^{t(n) \cdot n}} \left[ A(g(x)) \in g^{-1}(g(w)) \right] \geq \frac{1}{p(n)}$$

for every $n \in \mathcal{I}$. 

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**$g$ is not one-way $\implies f$ has no $\delta/2$ failing set**

We show: $g$ is not one way $\implies f$ has no $\delta/2$ failing-set for some PPT $B$ and $q \in \text{poly}$. 

**g is not one-way** $\implies f$ has no $\delta/2$ failing set

We show: **g is not one way** $\implies f$ has no $\delta/2$ failing-set for some PPT $B$ and $q \in \text{poly}$.

**Claim 23**

Assume $\exists$ PPT $A$, $p \in \text{poly}$ and an infinite set $I \subseteq \mathbb{N}$ such that

$$\Pr_{w \leftarrow \{0,1\}^n} \left[ A(g(x)) \in g^{-1}(g(w)) \right] \geq \frac{1}{p(n)}$$

for every $n \in I$. Then $\exists$ PPT $B$ such that

$$\Pr_{x \leftarrow \{0,1\}^n|y=f(x)\in S_n} \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$$

for every $n \in I$ and every $S_n \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S] \geq \delta(n)/2$. 

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**Thm follows:** Fix $S = \{S_n \subseteq \{0,1\}^n\}$. By Claim 23, for every $n \in I$, either

$\Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] < \delta(n)/2$, or

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Namely, $f$ has no $\delta/2$ failing set for $(B, q = 2t(n)p(n))$. 

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$g$ is not one-way $\implies f$ has no $\delta/2$ failing set

**Claim 23**

Assume $\exists$ PPT $A$, $p \in \text{poly}$ and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that

$$\Pr_{w \leftarrow \{0,1\}^{t(n) \cdot n}} \left[ A(g(x)) \in g^{-1}(g(w)) \right] \geq \frac{1}{p(n)}$$

for every $n \in \mathcal{I}$. Then $\exists$ PPT $B$ such that

$$\Pr_{x \leftarrow \{0,1\}^{n}} \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$$

for every $n \in \mathcal{I}$ and every $S_n \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] \geq \delta(n)/2$. 
Claim 23

Assume \( \exists \) PPT \( A, p \in \text{poly} \) and an infinite set \( \mathcal{I} \subseteq \mathbb{N} \) such that

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for every \( n \in \mathcal{I} \). Then \( \exists \) PPT \( B \) such that

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for every \( n \in \mathcal{I} \) and every \( S_n \subseteq \{0,1\}^{n} \) with \( \Pr_{x \leftarrow \{0,1\}^{n}} [f(x) \in S_n] \geq \delta(n)/2 \).

Thm follows: Fix \( S = \{S_n \subseteq \{0,1\}^{n}\} \).
Claim 23

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for every $n \in \mathcal{I}$ and every $S_n \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^{n}} [f(x) \in S_n] \geq \delta(n)/2$.

Thm follows: Fix $S = \{S_n \subseteq \{0,1\}^n\}$. By Claim 23, for every $n \in \mathcal{I}$, either

- $\Pr_{x \leftarrow \{0,1\}^{n}} [f(x) \in S_n] < \delta(n)/2$, or
- $\Pr_{x \leftarrow \{0,1\}^{n} \mid y = f(x) \in S_n} \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$
\( g \) is not one-way \( \implies f \) has no \( \delta/2 \) failing set

**Claim 23**

Assume \( \exists \) PPT \( A, p \in \text{poly} \) and an infinite set \( \mathcal{I} \subseteq \mathbb{N} \) such that

\[
\Pr_{w \leftarrow \{0,1\}^{t(n) \cdot n}} \left[ A(g(x)) \in g^{-1}(g(w)) \right] \geq \frac{1}{p(n)}
\]

for every \( n \in \mathcal{I} \). Then \( \exists \) PPT \( B \) such that

\[
\Pr_{x \leftarrow \{0,1\}^n} \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n)p(n)} - n^{-\log n}
\]

for every \( n \in \mathcal{I} \) and every \( S_n \subseteq \{0,1\}^n \) with \( \Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] \geq \delta(n)/2 \).

Thm follows: Fix \( S = \{S_n \subseteq \{0,1\}^n\} \). By Claim 23, for every \( n \in \mathcal{I} \), either

- \( \Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] < \delta(n)/2 \), or
- \( \Pr_{x \leftarrow \{0,1\}^n} [y = f(x) \in S_n] \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n)p(n)} - n^{-\log n} 
\)
  (for large enough \( n \))
  \[
  \geq \frac{1}{2t(n)p(n)}
  \]
$g$ is not one-way $\implies f$ has no $\delta/2$ failing set

**Claim 23**

Assume $\exists$ PPT $A$, $p \in \text{poly}$ and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that

$$\Pr_{w \leftarrow \{0,1\}^{t(n) \cdot n}} [A(g(x)) \in g^{-1}(g(w))] \geq \frac{1}{p(n)}$$

for every $n \in \mathcal{I}$. Then $\exists$ PPT $B$ such that

$$\Pr_{x \leftarrow \{0,1\}^{n} \mid y = f(x) \in S_n} [B(y) \in f^{-1}(y)] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$$

for every $n \in \mathcal{I}$ and every $S_n \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] \geq \delta(n)/2$.

Thm follows: Fix $S = \{S_n \subseteq \{0,1\}^n\}$. By Claim 23, for every $n \in \mathcal{I}$, either

1. $\Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] < \delta(n)/2$, or

2. $\Pr_{x \leftarrow \{0,1\}^n \mid y = f(x) \in S_n} [B(y) \in f^{-1}(y)] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$

(for large enough $n$)

$$\geq \frac{1}{2t(n)p(n)}$$

(for large enough $n$)

$$\implies \exists y \in S_n: \Pr [B(y) \in f^{-1}(y)] \geq \frac{1}{2t(n)p(n)}.$$
Claim 23

Assume $\exists \text{PPT } A$, $p \in \text{poly}$ and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that

$$\Pr_{w \leftarrow \{0,1\}^{t(n) \cdot n}} \left[ A(g(x)) \in g^{-1}(g(w)) \right] \geq \frac{1}{p(n)}$$

for every $n \in \mathcal{I}$. Then $\exists \text{PPT } B$ such that

$$\Pr_{x \leftarrow \{0,1\}^n \mid y = f(x) \in S_n} \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$$

for every $n \in \mathcal{I}$ and every $S_n \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] \geq \delta(n)/2$.

Thm follows: Fix $S = \{S_n \subseteq \{0,1\}^n\}$. By Claim 23, for every $n \in \mathcal{I}$, either

- $\Pr_{x \leftarrow \{0,1\}^n} [f(x) \in S_n] < \delta(n)/2$, or
- $\Pr_{x \leftarrow \{0,1\}^n \mid y = f(x) \in S_n} \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$ (for large enough $n$)

$$\geq \frac{1}{2t(n)p(n)}$$

(both for large enough $n$)

$$\implies \exists y \in S_n: \Pr \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{2t(n)p(n)}.$$

Namely, $f$ has no $\delta/2$ failing set for $(B, q = 2t(n)p(n))$.
The no failing-set algorithm: Proof of main claim

Algorithm 24 (Inverter B on input $y \in \{0, 1\}^n$)

1. Choose $w \leftarrow (\{0, 1\}^n)^{t(n)}$, $z = (z_1, \ldots, z_t) = g(w)$ and $i \leftarrow [t]$.
2. Set $z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t)$.
3. Return $A(z')_i$. 

Fix $n \in \mathbb{N}$ and a set $S_n \subseteq \{0, 1\}^n$ with $\Pr_{x \leftarrow \{0, 1\}^n}[f(x) \in S_n] \geq \delta(n)/2$.

Claim 25

$$\Pr_{x \leftarrow \{0, 1\}^n}[y = f(x) \in S_n] \geq \frac{1}{t(n)} \cdot p(n) - \log n.$$
Algorithm 24 (Inverter B on input $y \in \{0, 1\}^n$)

1. Choose $w \leftarrow (\{0, 1\}^n)^{t(n)}$, $z = (z_1, \ldots, z_t) = g(w)$ and $i \leftarrow [t]$
2. Set $z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t)$
3. Return $A(z')_i$

Fix $n \in \mathcal{I}$ and a set $S_n \subseteq \{0, 1\}^n$ with $\Pr_{x \leftarrow \{0, 1\}^n} [f(x) \in S] \geq \delta(n)/2$.

Claim 25

$$\Pr_{x \leftarrow \{0, 1\}^n \mid y = f(x) \in S_n} [B(y) \in f^{-1}(y)] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}$$
Proving \( \Pr_{x \leftarrow \{0,1\}^n} | y = f(x) \in S_n \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n) \cdot p(n) - n^{-\log n}} \)

Algorithm 26 (Inverter B on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)}, z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)
Proving \[ \Pr_{x \leftarrow \{0,1\}^n} [y = f(x) \in S_n \mid B(y) \in f^{-1}(y)] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \]

Algorithm 26 (Inverter B on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)} \), \( z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z'_i) \)

For \( \text{Typ} = \{v \in \{0,1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n\} \), it holds \( \Pr_z [\text{Typ}] \geq 1 - n^{-\log n} \)
Algorithm 26 (Inverter B on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)} \), \( z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)

For \( \text{Typ} = \{ v \in \{0,1\}^{t\cdot n} : \exists i \in [t] : v_i \in S_n \} \), it holds \( \Pr_{z}[\text{Typ}] \geq 1 - n^{-\log n} \)

Proving \( \Pr_{x \leftarrow \{0,1\}^n | y = f(x) \in S_n} \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)
Proving \( \Pr_{x \leftarrow \{0,1\}^n} y = f(x) \in S_n \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)

Algorithm 26 (Inverter \( B \) on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)}, z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)

- For \( \text{Typ} = \{ v \in \{0,1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n \} \), it holds \( \Pr_{z} [\text{Typ}] \geq 1 - n^{-\log n} \)
- \( \forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n} : \)
  \[
  \Pr_z [\mathcal{L}' = \mathcal{L} \cap \text{Typ}] = \sum_{\ell \in \mathcal{L}'} \Pr[z = \ell] \leq \sum_{\ell \in \mathcal{L}'} \frac{\Pr[z' = \ell]}{t}
  \]
Proving \( \Pr_{x \leftarrow \{0,1\}^n} | y = f(x) \in S_n \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)

Algorithm 26 (Inverter \( B \) on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)_{t(n)}, z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)

- For \( \text{Typ} = \{ v \in \{0,1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n \} \), it holds \( \Pr_z [\text{Typ}] \geq 1 - n^{-\log n} \)
- \( \forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n} : \)
  \( \Pr_z [\mathcal{L}' = \mathcal{L} \cap \text{Typ}] = \sum_{\ell \in \mathcal{L}'} \Pr[z = \ell] \leq \sum_{\ell \in \mathcal{L}'} \frac{\Pr[z' = \ell]}{t} \)
Proving \[ \Pr_{x \leftarrow \{0, 1\}^n}|y = f(x) \in S_n \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \]

Algorithm 26 (Inverter B on input \( y \in \{0, 1\}^n \))

1. Choose \( w \leftarrow (\{0, 1\}^n)^{t(n)} \), \( z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)

- For \( \text{Typ} = \{v \in \{0, 1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n\} \), it holds \( \Pr_{z}[\text{Typ}] \geq 1 - n^{-\log n} \)
- For any \( \mathcal{L} \subseteq \{0, 1\}^{t(n) \cdot n} : \)
  \[
  \Pr_{z}[\mathcal{L}' = \mathcal{L} \cap \text{Typ}] = \sum_{\ell \in \mathcal{L}'} \Pr_{z}[z = \ell] \leq \sum_{\ell \in \mathcal{L}'} \frac{\Pr_{z'}[z' = \ell]}{t} = \frac{\Pr_{z'}[\mathcal{L}']}{t}.
  \]
Proving $\Pr_{x \leftarrow \{0,1\}^n \mid y=f(x) \in S_n} \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}$

Algorithm 26 (Inverter $B$ on input $y \in \{0,1\}^n$)

1. Choose $w \leftarrow (\{0,1\}^n)^{t(n)}$, $z = (z_1, \ldots, z_t) = g(w)$ and $i \leftarrow [t]$
2. Set $z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t)$
3. Return $A(z'_i)$

For $\text{Typ} = \{v \in \{0,1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n\}$, it holds $\Pr_{z} [\text{Typ}] \geq 1 - n^{-\log n}$

For $\forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n}$:

$$\Pr_{z} [\mathcal{L}' = \mathcal{L} \cap \text{Typ}] = \sum_{\ell \in \mathcal{L}'} \Pr_{z} [z = \ell] \leq \sum_{\ell \in \mathcal{L}'} \frac{\Pr_{z'} [z' = \ell]}{t} = \frac{\Pr_{z'} [\mathcal{L}']}{t}.$$
Proving \( \Pr_{x \leftarrow \{0,1\}^n} [y = f(x) \in S_n \mid B(y) \in f^{-1}(y)] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)

**Algorithm 26 (Inverter B on input \( y \in \{0,1\}^n \))**

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)}, z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z'_i) \)

- For \( \text{Typ} = \{ v \in \{0,1\}^{t \cdot n} : \exists i \in [t]: v_i \in S_n \} \), it holds \( \Pr_z [\text{Typ}] \geq 1 - n^{-\log n} \)
- \( \forall L \subseteq \{0,1\}^{t(n) \cdot n} : \)
  \( \Pr_{z'} [L' = L \cap \text{Typ}] = \sum_{\ell \in L'} \Pr[z = \ell] \leq \sum_{\ell \in L'} \frac{\Pr[z' = \ell]}{t} = \frac{\Pr_{z'} [L']}{t}. \)
- Hence \( \forall L \subseteq \{0,1\}^{t(n) \cdot n} : \Pr_{z'} [L] \geq \frac{\Pr_{z} [L \cap \text{Typ}]}{t(n)} \geq \frac{\Pr_{z} [L] - n^{-\log n}}{t(n)}. \)
Proving \( \Pr_{x \leftarrow \{0,1\}^n} [y = f(x) \in S_n \mid B(y) \in f^{-1}(y)] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)

Algorithm 26 (Inverter \( B \) on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)}, z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)

- For \( \text{Typ} = \{v \in \{0,1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n\} \), it holds \( \Pr_{z}[\text{Typ}] \geq 1 - n^{-\log n} \)
- \( \forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n} : \Pr_{z}[\mathcal{L}' = \mathcal{L} \cap \text{Typ}] = \sum_{\ell \in \mathcal{L}'} \Pr_{z}[z = \ell] \leq \sum_{\ell \in \mathcal{L}'} \frac{\Pr_{z}[z' = \ell]}{t} = \frac{\Pr_{z}[\mathcal{L}']}{t} \).
- Hence \( \forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n} : \Pr_{z'}[\mathcal{L}] \geq \frac{\Pr_{z}[\mathcal{L} \cap \text{Typ}]}{t(n)} \geq \frac{\Pr_{z}[\mathcal{L}] - n^{-\log n}}{t(n)} \).
- Assume \( A \) is deterministic and let \( \mathcal{L}_A = \{v \in \{0,1\}^{t \cdot n} : A(v) \in g^{-1}(v)\} \).
Proving \( \Pr_{x \leftarrow \{0, 1\}^n} y = f(x) \in S_n \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)

Algorithm 26 (Inverter \( B \) on input \( y \in \{0, 1\}^n \))

1. Choose \( w \leftarrow (\{0, 1\}^n)^{t(n)}, z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)

- For \( \text{Typ} = \{v \in \{0, 1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n\} \), it holds \( \Pr_z [\text{Typ}] \geq 1 - n^{-\log n} \)
- \( \forall \mathcal{L} \subseteq \{0, 1\}^{t(n) \cdot n} : \)
  \[ \Pr_z [\mathcal{L}' = \mathcal{L} \cap \text{Typ}] = \sum_{\ell \in \mathcal{L}'} \Pr_z [z = \ell] \leq \sum_{\ell \in \mathcal{L}'} \frac{\Pr_z [z' = \ell]}{t} = \frac{\Pr_z [\mathcal{L} \cap \text{Typ}]}{t} \]
- Hence \( \forall \mathcal{L} \subseteq \{0, 1\}^{t(n) \cdot n} : \Pr_z [\mathcal{L}] \geq \frac{\Pr_z [\mathcal{L} \cap \text{Typ}]}{t(n)} \geq \frac{\Pr_z [\mathcal{L}] - n^{-\log n}}{t(n)} \).
- Assume \( A \) is deterministic and let \( \mathcal{L}_A = \{v \in \{0, 1\}^{t \cdot n} : A(v) \in g^{-1}(v)\} \).
Proving \( \Pr_{x \leftarrow \{0,1\}^n} [y = f(x) \in S_n] \left[ B(y) \in f^{-1}(y) \right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)

Algorithm 26 (Inverter \( B \) on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)}, z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)

- For \( \text{Typ} = \{ \nu \in \{0,1\}^{t \cdot n} : \exists i \in [t] : \nu_i \in S_n \} \), it holds \( \Pr_z[\text{Typ}] \geq 1 - n^{-\log n} \)
- For all \( L \subseteq \{0,1\}^{t(n) \cdot n} : \Pr_z[L' = L \cap \text{Typ}] = \sum_{\ell \in L'} \Pr[z = \ell] \leq \sum_{\ell \in L'} \frac{\Pr[z' = \ell]}{t} = \frac{Pr[z' \in L']}{t} \).
- Hence \( \forall L \subseteq \{0,1\}^{t(n) \cdot n} : \Pr_{z'}[L] \geq \frac{\Pr_z[L \cap \text{Typ}]}{t(n)} \geq \frac{\Pr_z[L] - n^{-\log n}}{t(n)} \).
- Assume \( A \) is deterministic and let \( \mathcal{L}_A = \{ \nu \in \{0,1\}^{t \cdot n} : A(\nu) \in g^{-1}(\nu) \} \).

\[
\Pr_{x \leftarrow \{0,1\}^n} [B(y) \in f^{-1}(y)] \geq \Pr[z' \in \mathcal{L}_A]
\]
Proving \( \Pr_{x \leftarrow \{0,1\}^n} \{ y = f(x) \in S_n \mid \mathbb{B}(y) \in f^{-1}(y) \} \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)

Algorithm 26 (Inverter \( B \) on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)} \), \( z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z')_i \)

\[ \Pr_{x \leftarrow \{0,1\}^n} \{ y = f(x) \in S_n \mid \mathbb{B}(y) \in f^{-1}(y) \} \geq \Pr_{z' \in \mathbb{L}'} \geq \Pr_{z \in \mathbb{L}} - n^{-\log n} \]

For \( \mathbb{Typ} = \{ v \in \{0,1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n \} \), it holds \( \Pr_{z} [\mathbb{Typ}] \geq 1 - n^{-\log n} \)

\[ \forall \mathbb{L} \subseteq \{0,1\}^{t(n) \cdot n} : \Pr_{z} [\mathbb{L}' = \mathbb{L} \cap \mathbb{Typ}] = \sum_{\ell \in \mathbb{L}'} \Pr_{z} [z = \ell] \leq \sum_{\ell \in \mathbb{L}'} \frac{\Pr_{z'} [\ell]}{t} = \frac{\Pr_{z'} [\mathbb{L}']}{t} \]

Hence \( \forall \mathbb{L} \subseteq \{0,1\}^{t(n) \cdot n} : \Pr_{z'} [\mathbb{L}] \geq \frac{\Pr_{z} [\mathbb{L} \cap \mathbb{Typ}]}{t(n)} \geq \frac{\Pr_{z} [\mathbb{L}] - n^{-\log n}}{t(n)} \)

Assume \( A \) is deterministic and let \( \mathbb{L}_A = \{ v \in \{0,1\}^{t \cdot n} : A(v) \in g^{-1}(v) \} \).

\[ \Pr_{x \leftarrow \{0,1\}^n} \{ y = f(x) \in S_n \mid \mathbb{B}(y) \in f^{-1}(y) \} \geq \Pr_{z'} \in \mathbb{L}_A \geq \frac{\Pr_{z} \in \mathbb{L}_A} {t(n)} \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \]
Proving \( \Pr_{x \leftarrow \{0,1\}^n} [y = f(x) \in S_n] \quad [B(y) \in f^{-1}(y)] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \)

### Algorithm 26 (Inverter \( B \) on input \( y \in \{0,1\}^n \))

1. Choose \( w \leftarrow (\{0,1\}^n)^{t(n)}, z = (z_1, \ldots, z_t) = g(w) \) and \( i \leftarrow [t] \)
2. Set \( z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t) \)
3. Return \( A(z'_i) \)

- For \( \text{Typ} = \{v \in \{0,1\}^{t \cdot n} : \exists i \in [t] : v_i \in S_n\} \), it holds \( \Pr_{z}[\text{Typ}] \geq 1 - n^{-\log n} \)
- \[ \forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n} : \quad \Pr_{z}[\mathcal{L}' = \mathcal{L} \cap \text{Typ}] = \sum_{\ell \in \mathcal{L}'} \Pr[z = \ell] \leq \sum_{\ell \in \mathcal{L}'} \frac{\Pr[z' = \ell]}{t} = \frac{\Pr_{z'}[\mathcal{L}']}{t}. \]
- Hence \( \forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n} : \Pr_{z'}[\mathcal{L}] \geq \frac{\Pr_{z}[\mathcal{L} \cap \text{Typ}]}{t(n)} \geq \frac{\Pr_{z}[\mathcal{L}] - n^{-\log n}}{t(n)}. \)
- Assume \( A \) is deterministic and let \( \mathcal{L}_A = \{v \in \{0,1\}^{t \cdot n} : A(v) \in g^{-1}(v)\}. \)

\[
\Pr_{x \leftarrow \{0,1\}^n} [B(y) \in f^{-1}(y)] \geq \Pr[z' \in \mathcal{L}_A] \geq \frac{\Pr[z \in \mathcal{L}_A] - n^{-\log n}}{t(n)} \\
\geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}
\]
Proving  \( \Pr_{x \leftarrow \{0,1\}^n} [y = f(x) \in S_n \mid B(y) \in f^{-1}(y)] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n} \), cont.

In the case that \( A \) is randomized, let

- \( A_r \) — \( A \) whose coins fixed to \( r \)
- \( \alpha_r(n) \) — the inversion probability of \( A_r \), for a uniform input for \( g \)
Proving \( \Pr_{x \leftarrow \{0,1\}^n} [y=f(x) \in S_n \ (B(y) \in f^{-1}(y))] \geq \frac{1}{t(n) \cdot p(n) - n^{-\log n}}, \) cont.

In the case that \( A \) is randomized, let

- \( A_r \) — \( A \) whose coins fixed to \( r \)
- \( \alpha_r(n) \) — the inversion probability of \( A_r \), for a uniform input for \( g \)

Note that \( E_r [\alpha_r(n)] \geq 1/p(n) \).
In the case that $A$ is randomized, let

1. $A_r$ — $A$ whose coins fixed to $r$
2. $\alpha_r(n)$ — the inversion probability of $A_r$, for a uniform input for $g$

Note that $E_r[\alpha_r(n)] \geq 1/p(n)$.

It follows that

$$\Pr_{x \leftarrow \{0,1\}^n | y = f(x) \in S_n} [B(y) \in f^{-1}(y)] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}$$
In the case that $A$ is randomized, let

- $A_r$ — $A$ whose coins fixed to $r$
- $\alpha_r(n)$ — the inversion probability of $A_r$, for a uniform input for $g$

Note that $E_r[\alpha_r(n)] \geq 1/p(n)$.

It follows that

$$\Pr_{x \leftarrow \{0,1\}^n, y=f(x) \in S_n} [B(y) \in f^{-1}(y)] \geq E_r \left[ \frac{\alpha_r(n)}{t(n)} - n^{-\log n} \right] = E_r[\alpha_r(n)] / t(n) - n^{-\log n}$$
In the case that $A$ is randomized, let

- $A_r$ — $A$ whose coins fixed to $r$
- $\alpha_r(n)$ — the inversion probability of $A_r$, for a uniform input for $g$

Note that $E_r[\alpha_r(n)] \geq 1/p(n)$.

It follows that

$$
\Pr_{x \leftarrow \{0,1\}^n \mid y = f(x) \in S_n} [B(y) \in f^{-1}(y)] \geq E_r \left[ \frac{\alpha_r(n)}{t(n)} - n^{-\log n} \right] = E_r [\alpha_r(n)] / t(n) - n^{-\log n} \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}.
$$
Closing remarks

- Weak OWFs can be amplified into strong one
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- Can we give a more security preserving amplification?
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- Can we give a more security preserving amplification?
- Similar hardness amplification theorems for other cryptographic primitives (e.g., Captchas, general protocols)?
Closing remarks

- Weak OWFs can be amplified into strong one
- Can we give a more security preserving amplification?
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- What properties of the weak OWFs have we used in the proof?