Complexity Theory : Exercise 1

Submit until 30/3
March 17, 2009

Please write clear and precise answers. A 10 point bonus is given for printed solutions. Questions 1-3 are on the material of the first two lectures. For questions 4,5 you need material from the 3rd lecture that will be given on 23/3.

1. (Reminder of NP and NP completeness) **No need to submit a solution to this question**. Consider the languages:

\( \text{Clique} = \{ (G, k) : G \text{ is a graph that has a clique of size } k \} \)

\( \text{Half} - \text{Clique} = \{ G : G \text{ is a graph that has a clique of size } |V|/2 \} \)

(a) Show that \( \text{Half} - \text{Clique} \in \text{NP} \).

(b) Show that \( \text{Clique} \leq L \text{Half} - \text{Clique} \).

2. (Composition of logspace functions).

(a) Let \( M_1, M_2 \) be machines that run in space \( O(\log n) \) and compute functions \( f_1 \) and \( f_2 \) respectively. Consider the function \( f(x) = f_2(f_1(x)) \). Show that this function is computable by a machine that runs in space \( O(\log n) \).

(b) Show that if \( L_1 \leq_L L_2 \) and \( L_2 \leq_L L_3 \) then \( L_1 \leq_L L_3 \).

(c) Show that if \( L_1 \leq_L L_2 \) and \( L_2 \in L \) then \( L_1 \in L \).

(d) Show that if \( L_1 \leq_L L_2 \) and \( L_2 \in \text{NL} \) then \( L_1 \in \text{NL} \).

3. Let \( dPATH = \{ G, s, t : G \text{ is a directed graph, } s, t \text{ are vertices and there is a path from } s \text{ to } t \} \). In class we showed that \( dPATH \in \text{NL} \). Show that \( dPATH \) is NL complete. (Hint: we explained this proof informally in class and the purpose of this exercise is for you to go over it and make sure you understand the details).

4. Show that \( TQBF \in \text{PSPACE} \). (Recall that \( TQBF \) is the set of all true quantified boolean formulae). (Hint: we explained this proof informally in class and the purpose of this exercise is for you to go over it and make sure you understand the details).

5. (2−SAT is NL complete) Consider the language \( 2−SAT = \{ \phi : \phi \text{ is a satisfiable 2-CNF formula} \} \).

(Reminder: a 2-CNF formula is a conjunction of clauses where each clause is the disjunction of two literals).

In class we showed that \( 2−SAT \in \text{NL} \). (Recall that we actually showed that \( 2−SAT \in \text{co}−\text{NL} \) and used the fact that \( \text{NL} = \text{co}−\text{NL} \)). Our goal is to show that \( 2−SAT \) is NL complete.

(a) Show that \( \text{co}−dPATH = \{ G, s, t : G \text{ is a directed graph, } s, t \text{ are vertices and there is no path from } s \text{ to } t \} \) is NL complete.
(b) Show that $co - dPATH \leq_L 2 - SAT$. (Hint: use the ideas we used in class to relate $2 - SAT$ to connectivity in graphs).

(c) Conclude that $2 - SAT$ is NL complete.