Complexity Theory : Exercise 1

Submit until 30/3

March 17, 2009

Please write clear and precise answers. A 10 point bonus is given for printed solutions. Questions 1-3 are on the material of the first two lectures. For questions 4,5 you need material from the 3rd lecture that will be given on 23/3.

1. (Reminder of NP and NP completeness) **No need to submit a solution to this question**). Consider the languages:
$$Clique = \{(G, k) : \text{G is a graph that has a clique of size } k\}$$
$$Half - Clique = \{G : \text{G is a graph that has a clique of size } |V|/2\}$$
   (a) Show that $Half - Clique \in NP$.
   (b) Show that $Clique \leq_L Half - Clique$.

2. (Composition of logspace functions).
   (a) Let M_1, M_2 be machines that run in space $O(\log n)$ and compute functions f_1 and f_2 respectively. Consider the function $f(x) = f_2(f_1(x))$. Show that this function is computable by a machine that runs in space $O(\log n)$.
   (b) Show that if $L_1 \leq_L L_2$ and $L_2 \leq_L L_3$ then $L_1 \leq_L L_3$.
   (c) Show that if $L_1 \leq_L L_2$ and $L_2 \in L$ then $L_1 \in L$.
   (d) Show that if $L_1 \leq_L L_2$ and $L_2 \in NL$ then $L_1 \in NL$.

3. Let $dPATH = \{G, s, t : G$ is a directed graph, s, t are vertices and there is a path from s to $t\}$. In class we showed that $dPATH \in NL$. Show that $dPATH$ is NL complete. (Hint: we explained this proof informally in class and the purpose of this exercise is for you to go over it and make sure you understand the details).

4. Show that $TQBF \in PSPACE$. (Recall that $TQBF$ is the set of all true quantified boolean formulae). (Hint: we explained this proof informally in class and the purpose of this exercise is for you to go over it and make sure you understand the details.

5. ($2-SAT$ is NL complete) Consider the language $2-SAT = \{\phi : \phi$ is a satisfiable 2-CNF formula$\}$. (Reminder: a 2-CNF formula is a conjunction of clauses where each clause is the disjunction of two literals).

   In class we showed that $2 - SAT \in NL$. (Recall that we actually showed that $2 - SAT \in co - NL$ and used the fact that $NL = co - NL$). Our goal is to show that $2 - SAT$ is NL complete.

   (a) Show that $co{-}dPATH = \{G, s, t : G$ is a directed graph, s, t are vertices and there is no path from s to $t\}$ is NL complete.

(b) Show that $co - dPATH \leq_L 2 - SAT$. (Hint: use the ideas we used in class to relate $2 - SAT$ to connectivity in graphs).

(c) Conclude that $2 - SAT$ is NL complete.