The History and Families of Wavelets

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Reminder: CWT

$$\gamma(s,\tau) = \int f(t)\psi^{*}_{s,\tau}(t)dt$$

S : Scale

τ : time shift, translation

* : complex conjugation

and the inverse transform:

$$f(t) = \int\int \gamma(s,\tau)\psi_{s,\tau}(t)d\tau ds$$

The scaled and translated *Mother Wavelet:*

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

$S^{-1/2}$: energy normalization for different scales

The required properties

The most important: *Admissibility* and *regularity*

Admissibility condition

$$\int \frac{|\psi(\omega)|^2}{|\omega|} d\omega < \infty$$

$\psi(\omega)$: Fourier Transform of $\psi(t)$

A square integrable function that satisfies this condition can be us
to decompose and reconstruct a signal without information loss.

- As a result we obtain: $|\psi(\omega)|^2 = 0 \quad @ \quad \omega = 0$

Zero at zero frequency means: A band-pass “behavior”, and also means zero average:

$$\int \psi(t) dt = 0$$ ➡ Wave

The *Regularity* Condition

- A wavelet transform of 1D function is 2D function, and the transform of 2D function (image) is 4D function: the time-bandwidth product of the output is square of the input !
- To avoid it, we make the wavelet transform decrease quickly with decreasing scale (s), using the *regularity condition*:

The wavelet function should be quite smooth and
concentrated in both frequency and time domains

Regularity (2)

- The regularity condition is shown using the expansion of the wavelet transform into a Taylor series at $t=0$
- It can be shown* that for a smooth signal $f(t)$ the wavelet transform coefficients decay “fast” : *Vanishing Moments* (also called *Approximation Order*)
  - The moments do not have to be zero, and a small value is good enough for most applications.

Where *Wavelet* came from ?

- *Admissibility* condition gave us the "Wave"
- Regularity (vanishing moments) gave us fast decay - the "let"

- more details and references in:

  C.Valens,

  A Really Friendly guide to Wavelets,

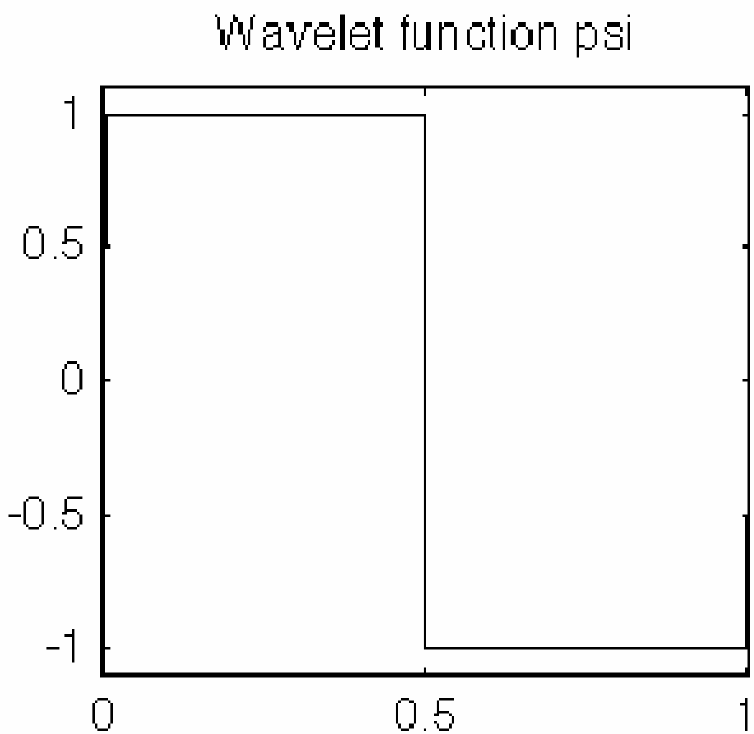
  http://perso.wanadoo.fr/polyvalens/clemens/wavelets/wavelets.html

Wavelets Analysis History

- "Wavelets" were first in 1909, in a thesis by Alfred Haar.
- The present theoretical form was first proposed by Jean Morlet (et al.) in the Marseille Theoretical Physics Center.
- Wavelet analysis have been developed mainly by Y.Meyer.
- The main algorithm developed by Stephane Mallat in 1988.

Introduction to Wavelets Families

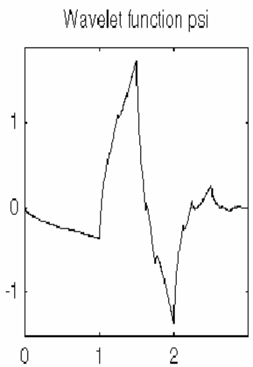
- Haar: the first and simplest : A step function



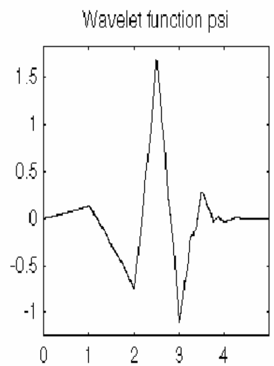
Daubechies

- Ingrid Daubechies invented the *compactly-supported* orthonormal wavelets, making DWT practicable…
- The names of Daubechies family wavelets are signed dbN (N is the order).

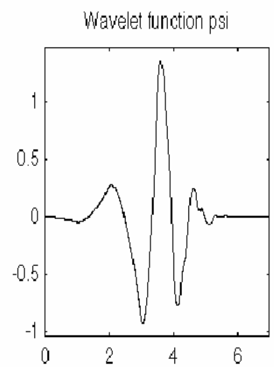
db family (N=2-10)



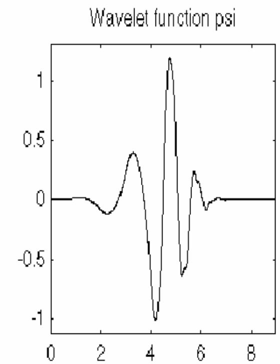
db2



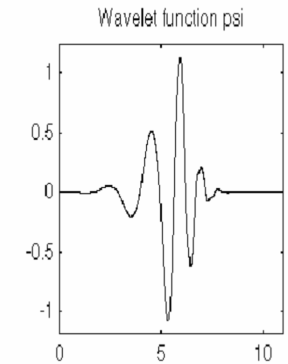
db3



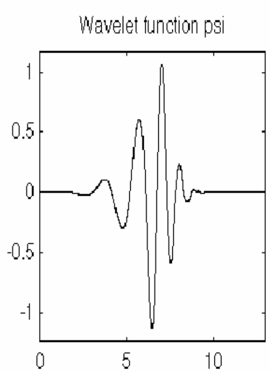
db4



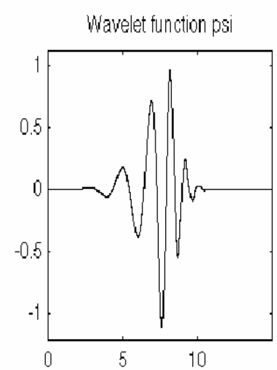
db5



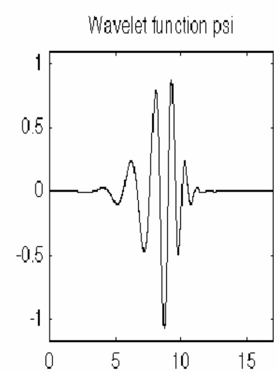
db6



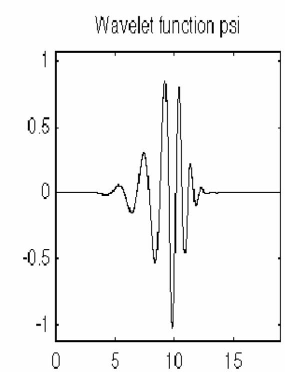
db7



db8



db9



db10

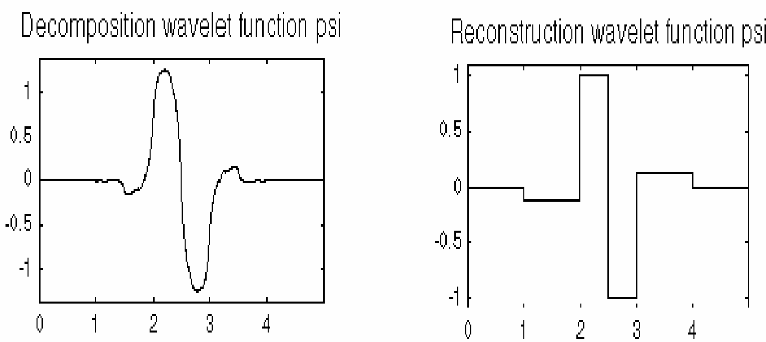
Biorthogonal

- Exhibits the *linear phase* property, needed for signal (and image) reconstruction
- Main features: (matlab: " waveinfo('bior') ")
- General characteristics: Compactly supported
- biorthogonal spline wavelets for which
- symmetry and exact reconstruction are possible
- with FIR filters (in orthogonal case it is impossible).
- Family Biorthogonals
- Short name bior
- Order Nr,Nd Nr = 1 , Nd = 1, 3, 5
- r for reconstruction Nr = 2 , Nd = 2, 4, 6, 8
- d for decomposition Nr = 3 , Nd = 1, 3, 5, 7, 9

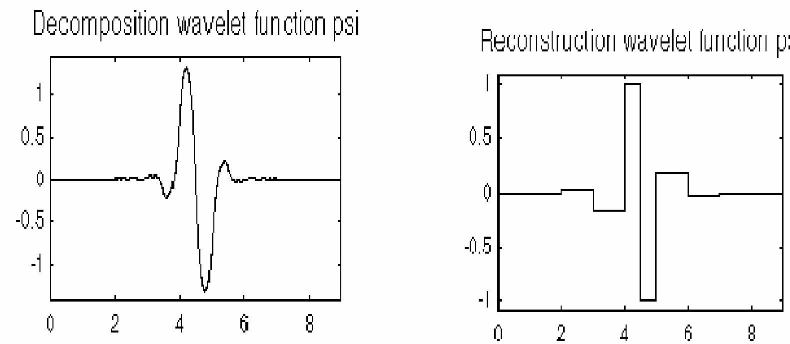
Biorthogonal main features (cont'd)

Orthogonal	no
Biorthogonal	yes
Compact support	yes
DWT	possible
CWT	possible
Regularity for	
psi rec.	Nr-1 and Nr-2 at the knots
Symmetry	yes
Number of vanishing moments for psi dec.	Nr-1

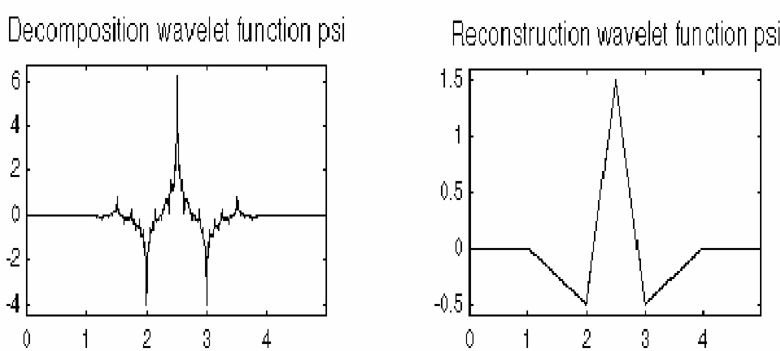
A few Examples



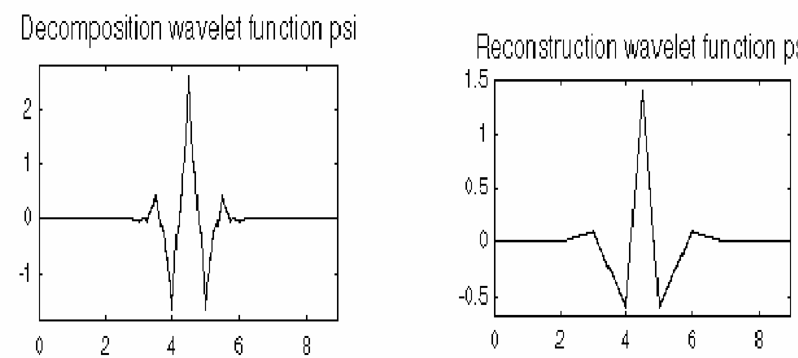
bior1.3



bior1.5



bior2.2

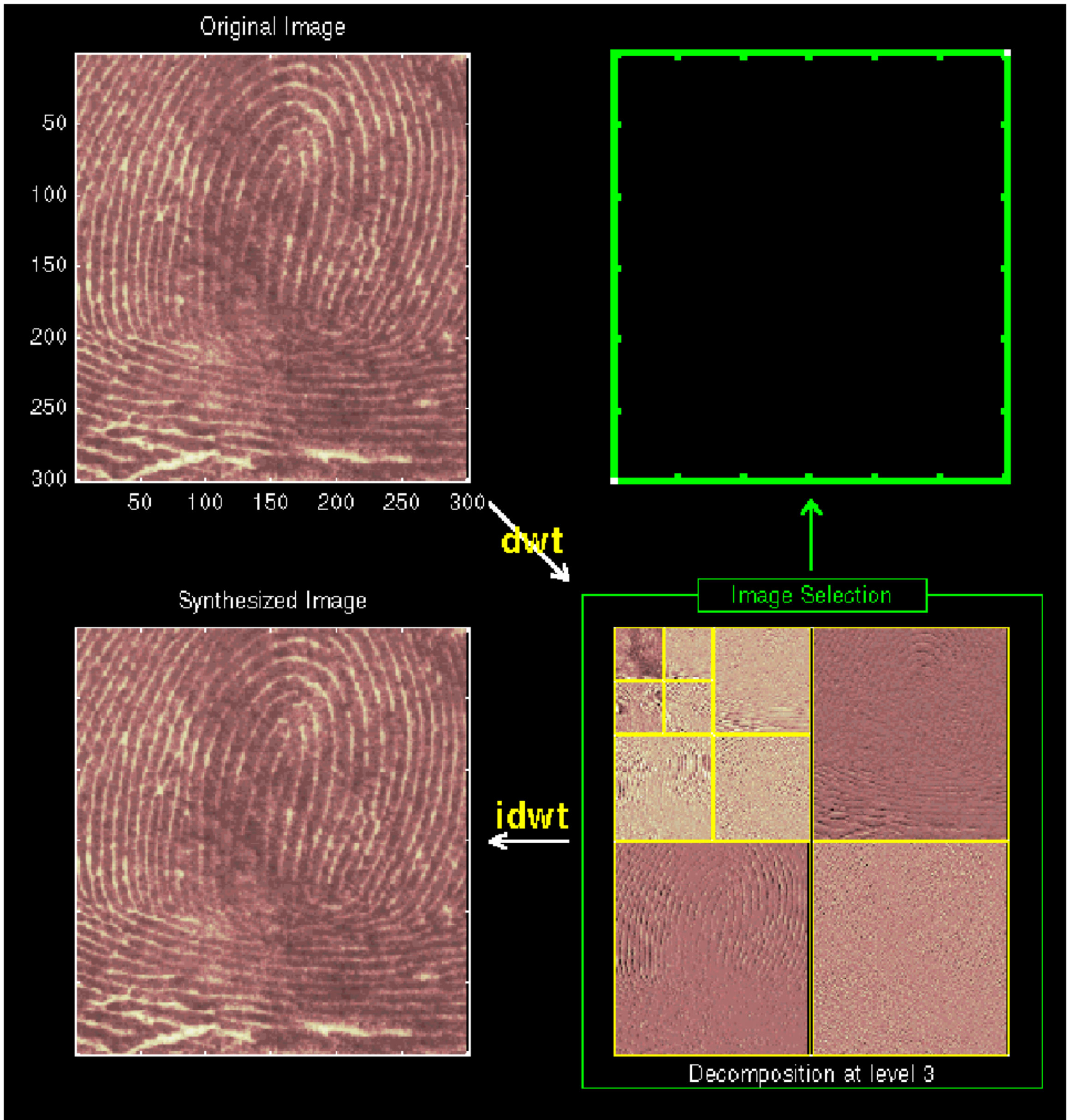


bior2.4

Compression Example

- A two dimensional (image) compression, using 2D wavelets analysis.
- The image is a Fingerprint.
- FBI uses a wavelet technique to compress its fingerprints database.

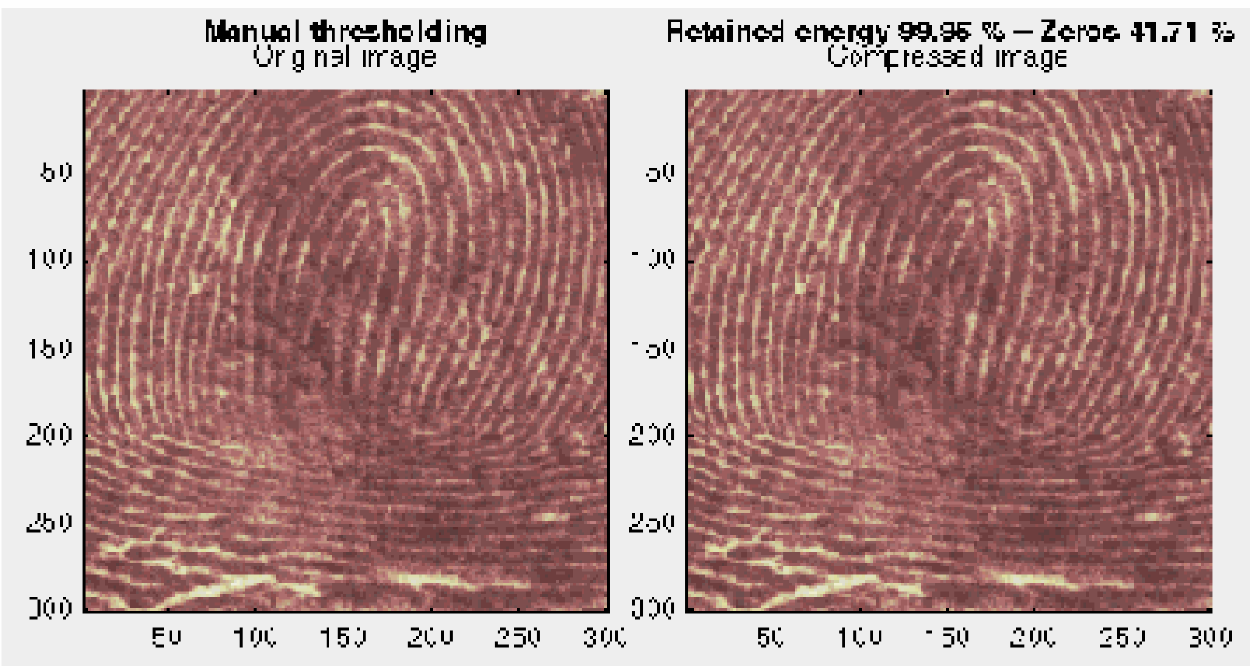
Fingerprint compression



Wavelet:
Haar
Level:3

Results (1)

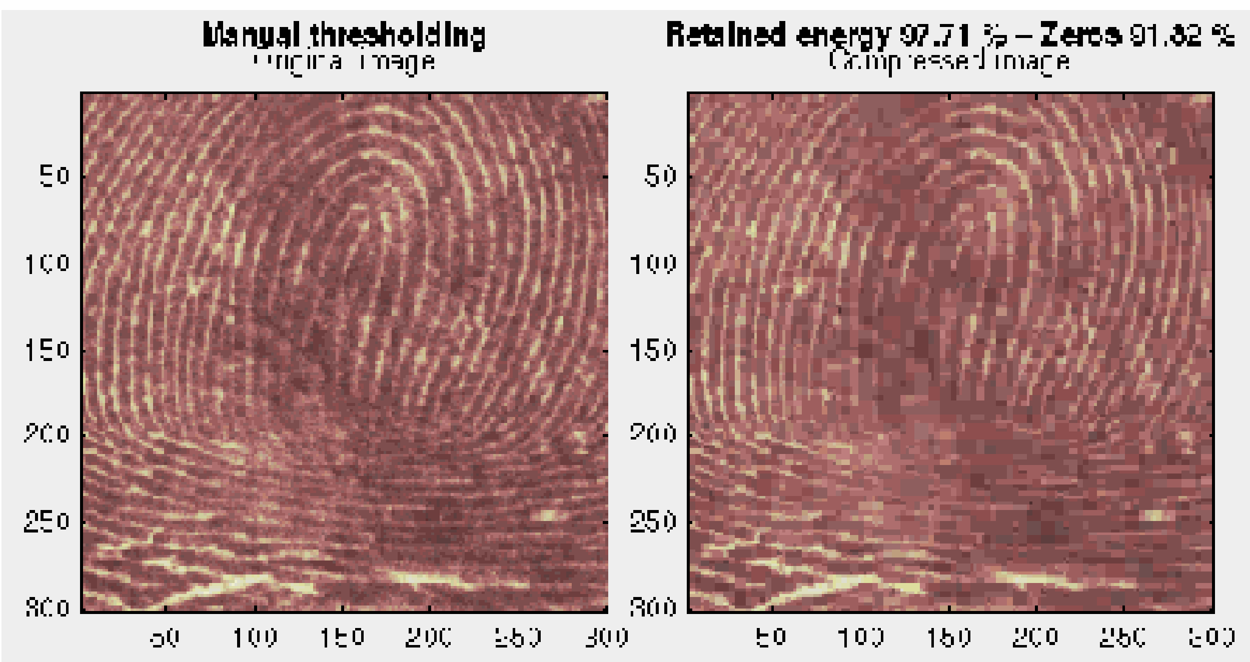
Original Image Compressed Image



Threshold: 3.5
Zeros: 42%
Retained energy: 99.95%

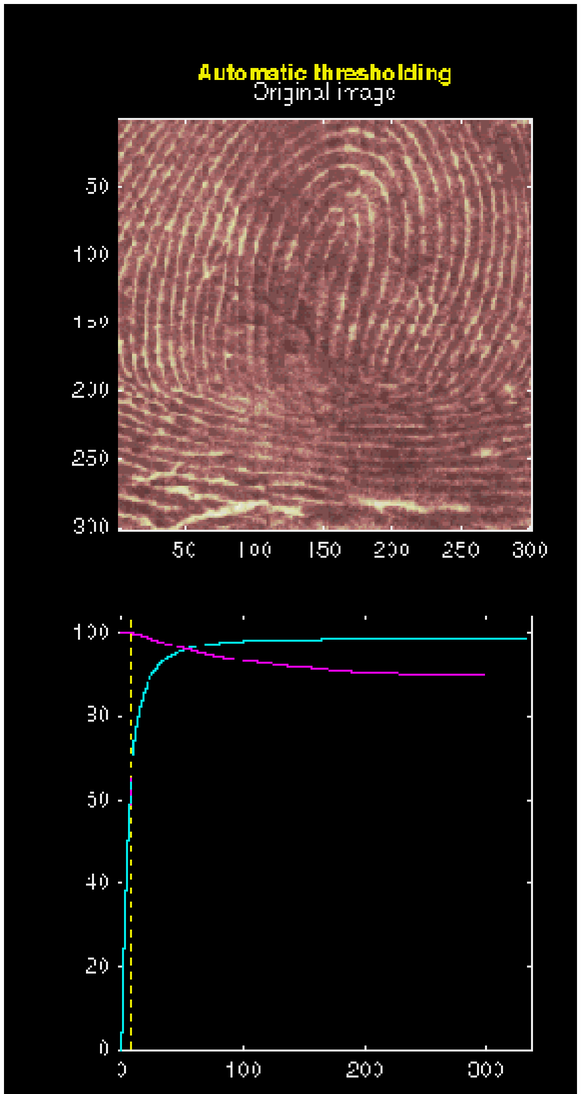
Results (2)

Original Image Compressed Image



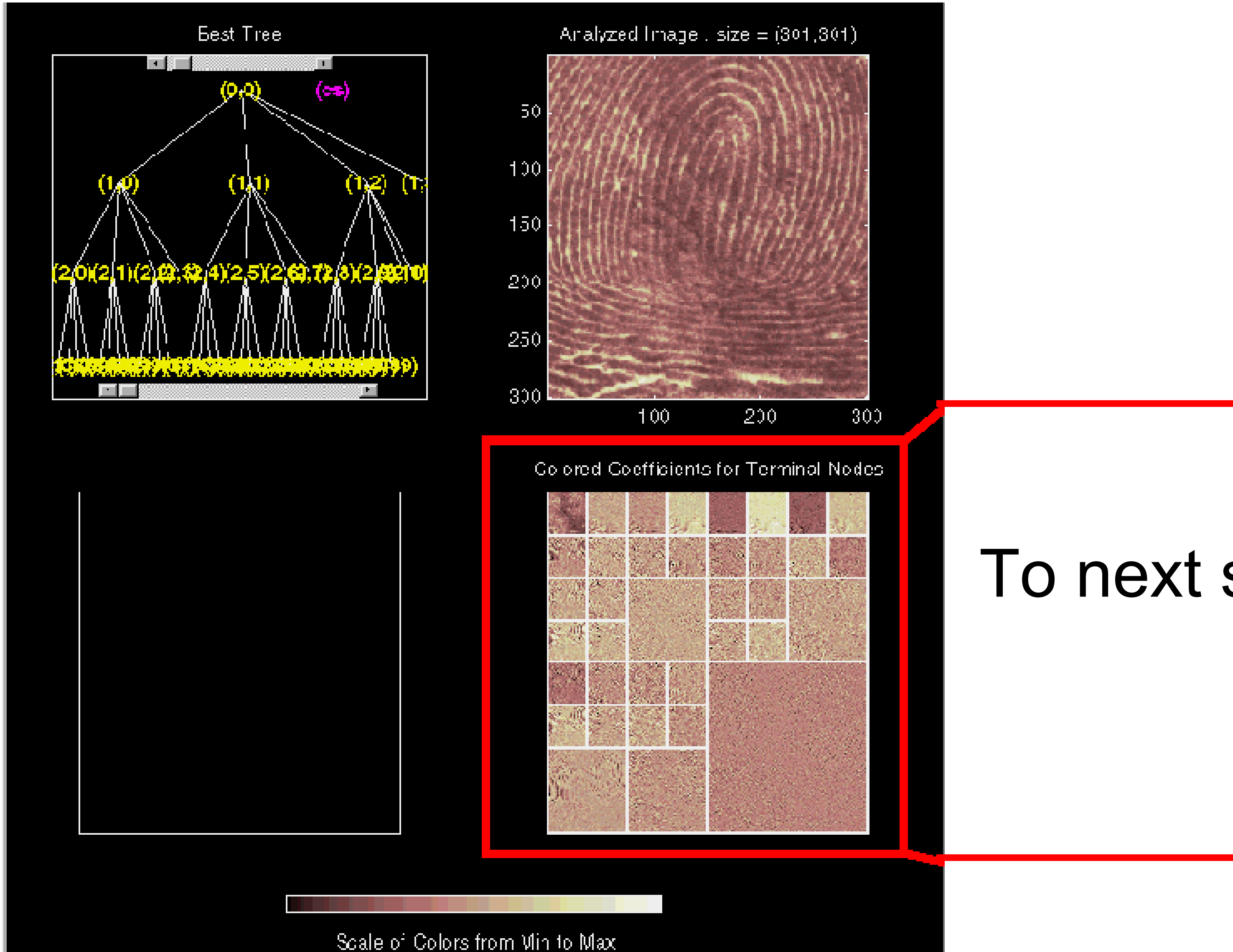
Threshold: 30
Zeros: 91.8%
Retained energy: 97.7%

Compression using 2D Wavelets Packet



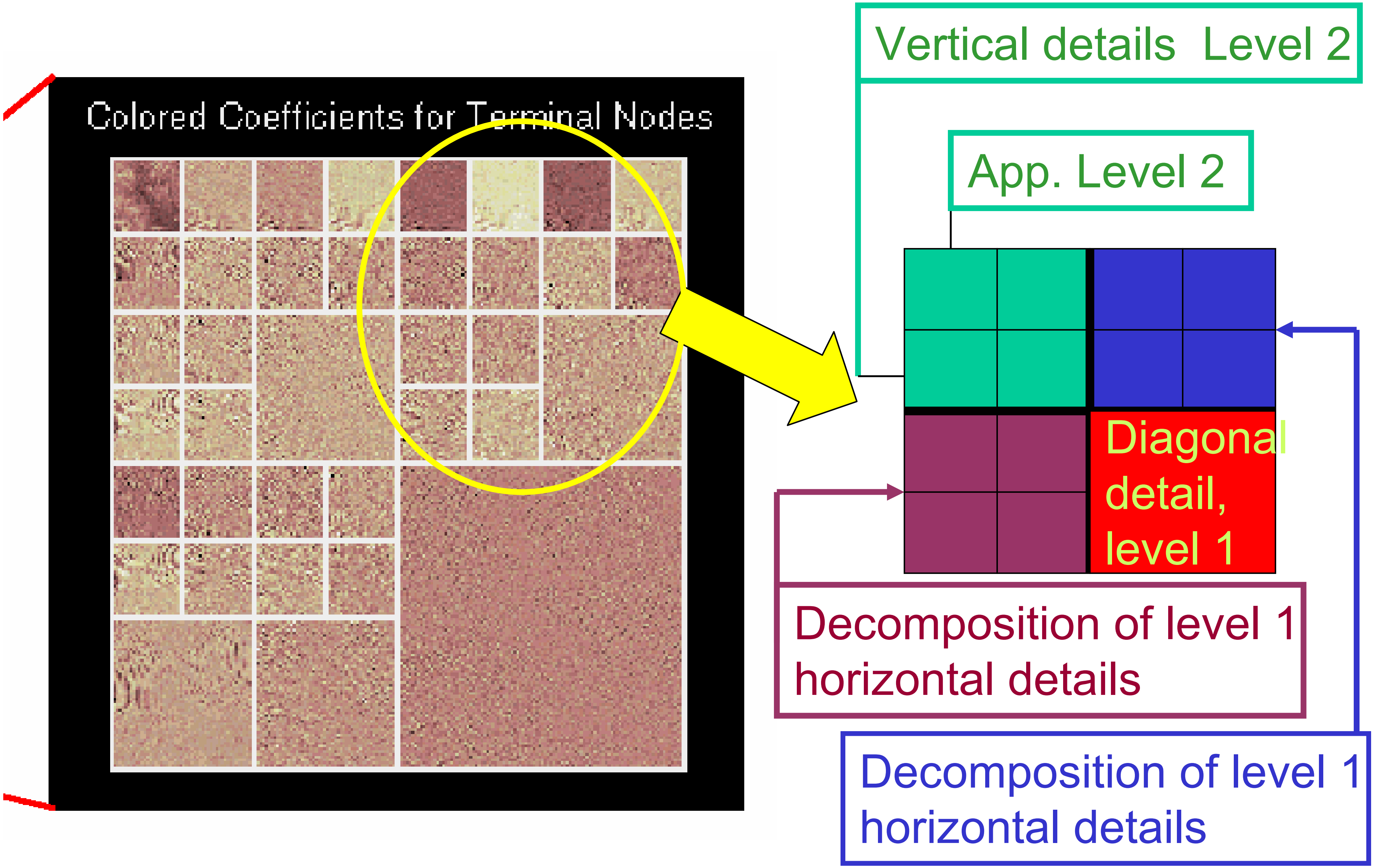
- Threshold: 7.25
- Zeros: 64%
- Retained energy: 99.77 %

The Tree Structure



To next slide...

Colored coefficients for terminal nodes



Why not to split the Diagonal details ?

- Note that the fingerprint pattern has sharp edges predominantly oriented horizontally and vertically - that's why the "best tree" algorithm has chosen not to decompose the diagonal details:

They do not provide much information….

Important references:

http://www.c3.lanl.gov/~brislawn/FBI/FBI.html

(A kind of home-page for FBI fingerprints format)

http://www.ora.com/centers/gff/formats/fbi/index.htm

(O'reilly Encyclopedia of GFF)