Introduction to
Wavelets

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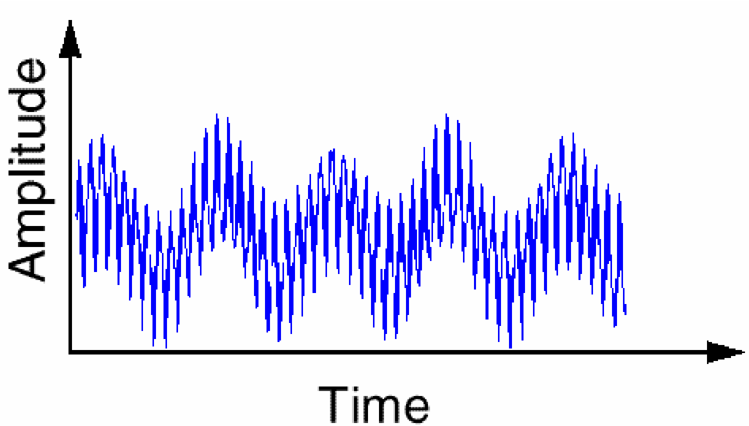
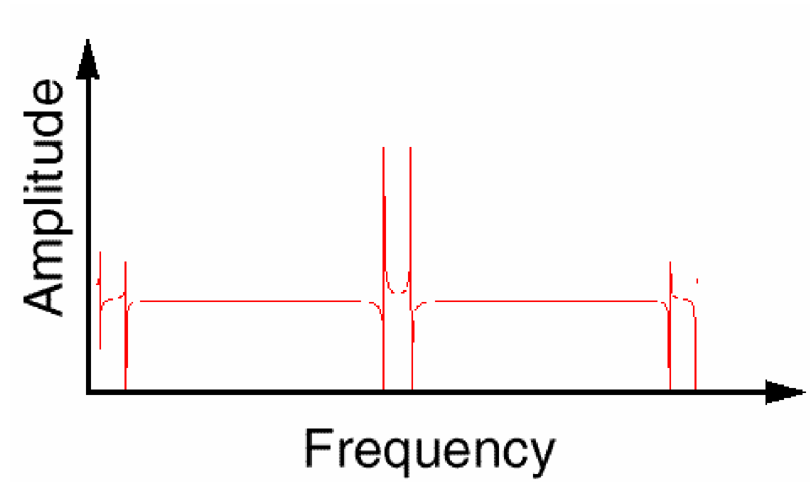
Update: Dec. 2000

Lets start with…Fourier Analysis

- Breaks down a signal into constituent sinusoids of different frequencies





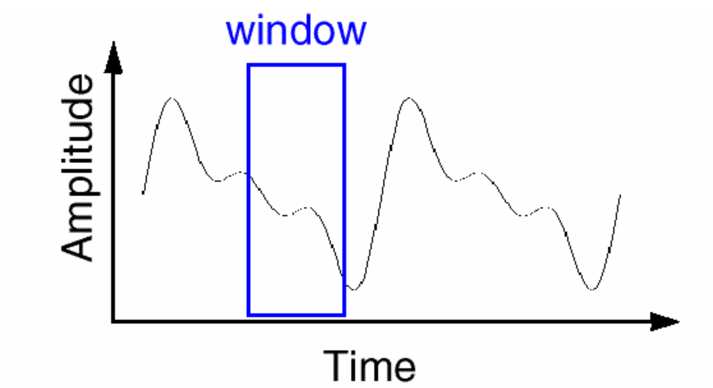
In other words: Transform the view of the signal from time-base to frequency-base.

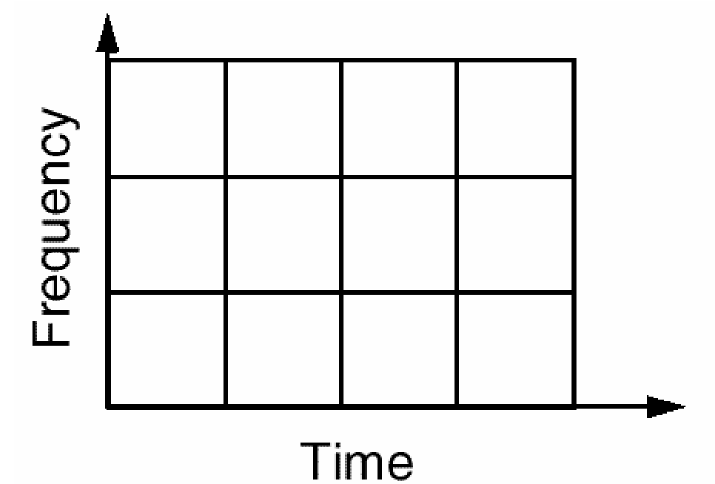
So,…What's wrong with Fourier?

- By using Fourier Transform (FT), we loose the time information : WHEN did a particular event take place ?
- For stationary signals - this doesn't matter, but what about non-stationary or transients? E.g. drift, trends, abrupt changes, beginning and ends of events, etc.

Short Time Fourier Analysis

- In order to analyze small section of a signal, Denis Gabor (1946), developed a technique, based on the FT and using *windowing*: STFT





The STFT maps a signal into a two-dimensional function of time and frequency.

STFT (or: Gabor Transform)

- A compromise between time-based and frequency-based views of a signal.
- both time and frequency are represented in limited precision.
- The precision is determined by the size of the window.
- Once you choose a particular size for the time window - it will be the same for all frequencies.

So,…What's wrong with Gabor?

- Many signals require a more flexible approach - so we can vary the window size to determine more accurately either time or frequency.

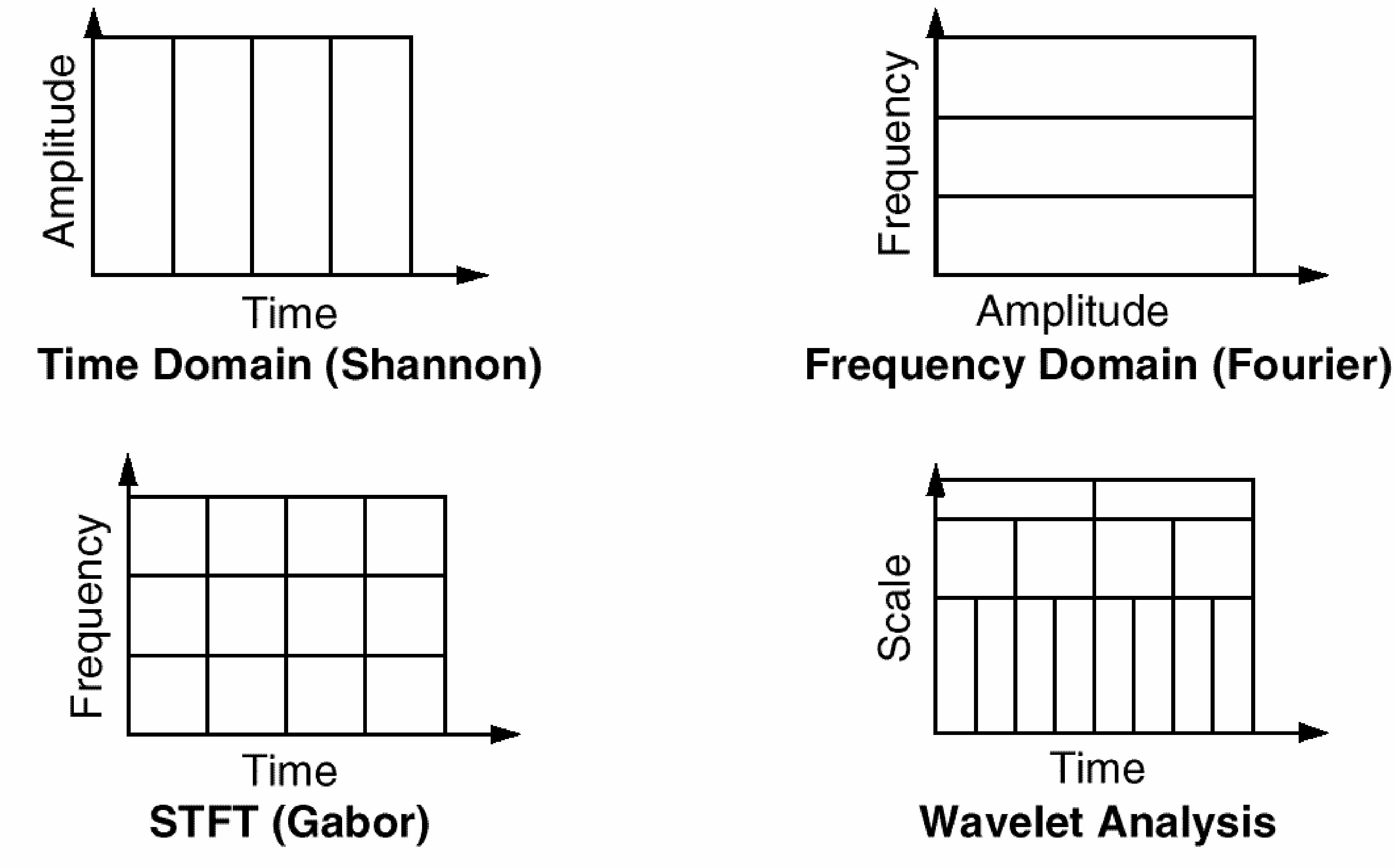
The next step: Wavelet Analysis

- Windowing technique with variable size window:

long time intervals when a more precise low frequency information is needed, and shorter intervals when high frequency is needed

- So, we have 4 steps:
  - Time Domain (Shannon - Nyquist)
  - Frequency Domain (Fourier)
  - STFT (Gabor)
  - Wavelet Analysis

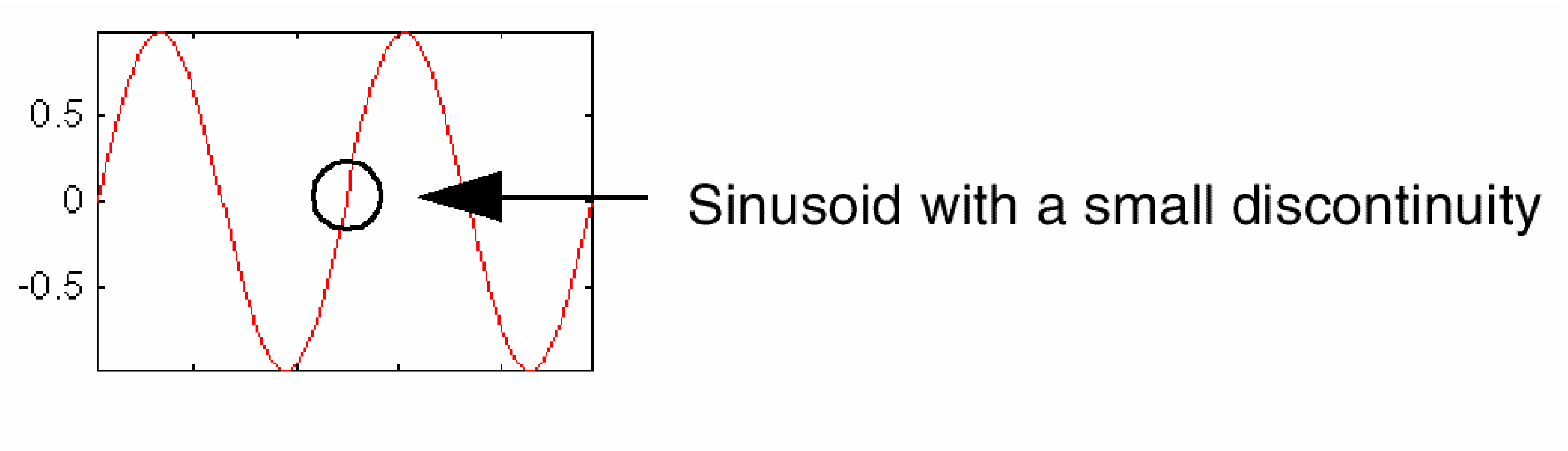
Here's what it looks like:



The main advantage: Local Analysis

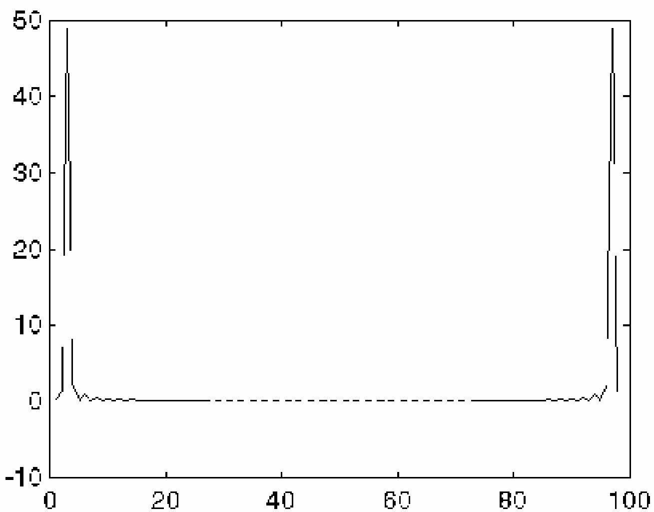
- Local analysis: To analyze a localized area of a larger signal

- e.g. : discontinuity caused by a noisy switch

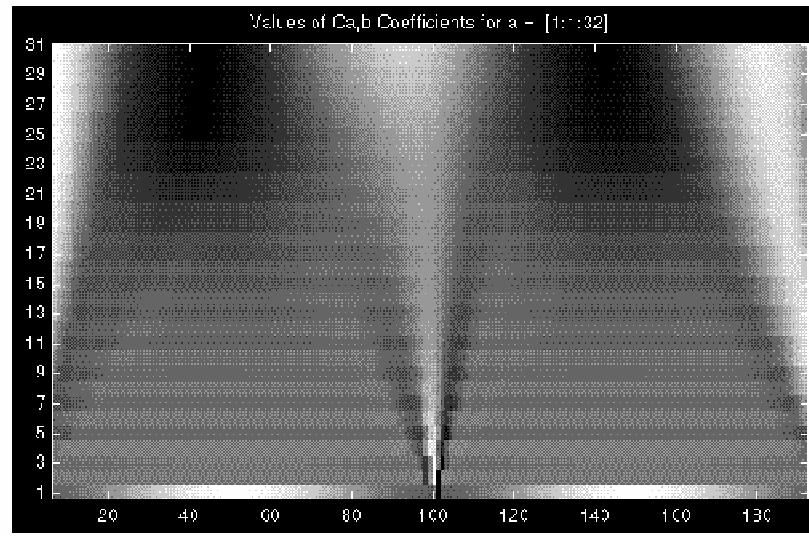


Local Analysis (Cont'd)

- Fourier analysis Vs. Wavelet analysis:



Fourier Coefficients

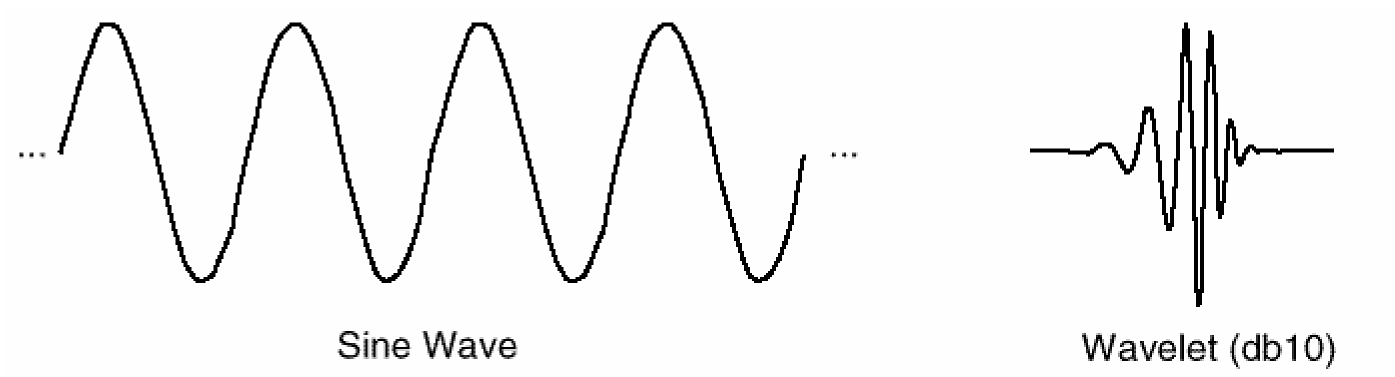


Wavelet Coefficients

In the FT we can only see the sinus frequency.
In the Wavelet plot we can clearly see the exact location in time of the discontinuity.

What is Wavelet Analysis ?

- And…what is a wavelet…?



- A wavelet is a waveform of effectively limited duration that has an average value of zero.

Wavelets Vs. Sine Waves

Sine waves	Wavelets
Average value of zero	Average value of zero
Infinite in time	Limited time duration
Extend from minus to plus	Asymmetric
Smooth	Irregular

Wavelet analysis Vs. Fourier analysis

- Fourier analysis:

consists of breaking up a signal into sine waves of various frequencies.

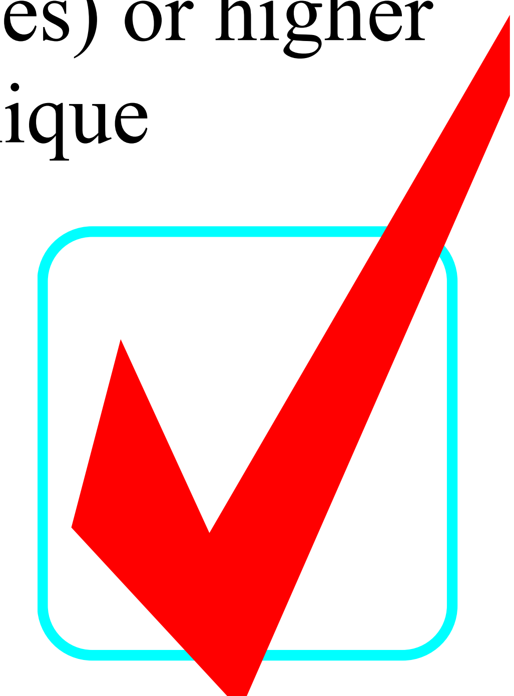
- Wavelet analysis:

Consists of breaking up a signal into *shifted* and *scaled* version of the original wavelet.

(called: *mother wavelet*)

Number of Dimensions

- Like the Fourier analysis, the Wavelet analysis can also be applied to two-dimensional data (such as images) or higher dimensions, and preserve its unique features.



The Continuous Wavelet Transform

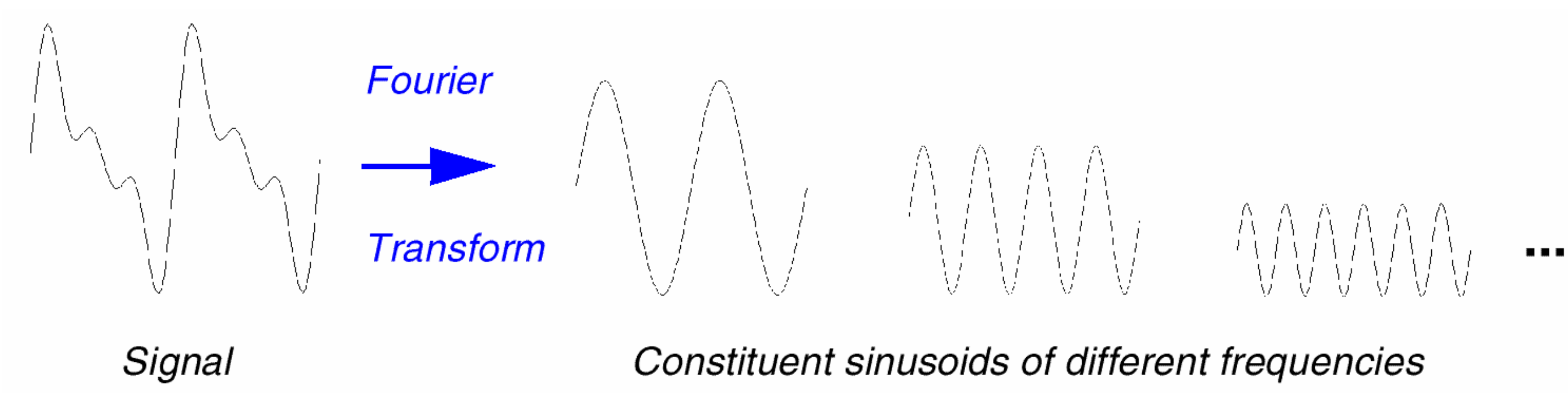
- A mathematical representation of the Fourier transform:

$$F_{(\omega)} = \int_{-\infty}^{\infty} f_{(t)} e^{-j\omega t} dt$$

- Meaning: the sum over all time of the signal *f(t)* multiplied by a complex exponential, and the result is the Fourier coefficients F(ω) .

Wavelet Transform (Cont’d)

- Those coefficients, when multiplied by a sinusoid of appropriate frequency ω, yield the constituent sinusoidal component of the original signal:



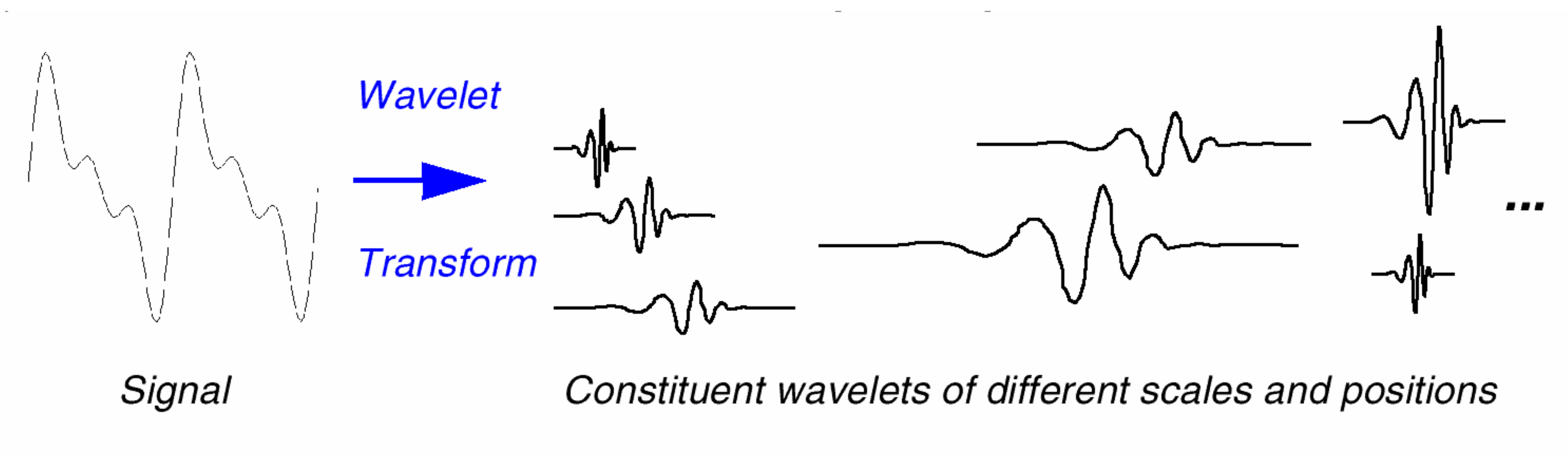
Wavelet Transform (Cont'd)

- Similarly, The *Continuous Wavelet Transform (CWT)* Is defined as the sum over all time of the signal, multiplied by scaled and shifted versions of the wavelet function Ψ:

$$C_{(scale,\ postion)} = \int_{-\infty}^{\infty} f_{(t)} \Psi_{(scale,\ position,\ t)} dt$$

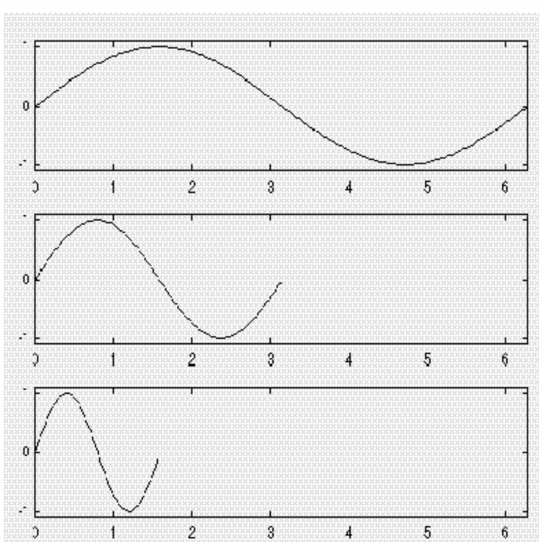
Wavelet Transform (Cont'd)

- And the result of the CWT are Wavelet coefficients .
- Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelet of the original signal:



Scaling

- Wavelet analysis produces a time-scale view of the signal.
- *Scaling* means stretching or compressing of the signal.
- scale factor (a) for sine waves:



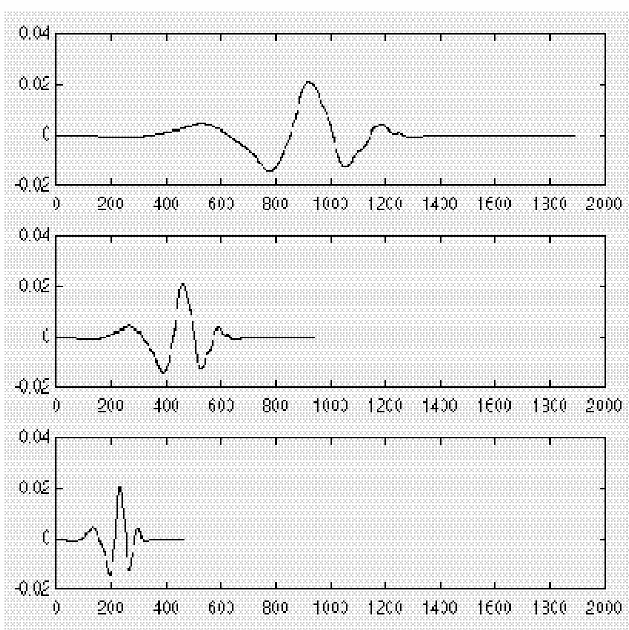
$$f(t) = \sin(t) \quad ; \quad a = 1$$

$$f(t) = \sin(2t) \quad ; \quad a = \tfrac{1}{2}$$

$$f(t) = \sin(4t) \quad ; \quad a = \tfrac{1}{4}$$

Scaling (Cont'd)

- Scale factor works exactly the same with wavelets:



$$f(t) = \Psi(t) \;;\; a = 1$$

$$f(t) = \Psi(2t) \;;\; a = 1/2$$

$$f(t) = \Psi(4t) \;;\; a = 1/4$$

Scaling Factor

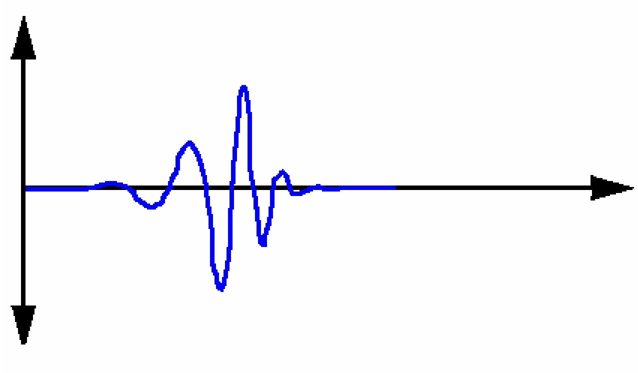
- For Sinusoid, $sin(\omega t)$ the scale factor a is inversely related to the radian frequency ω

- For Wavelets, the scale factor a is inversely related to the frequency f

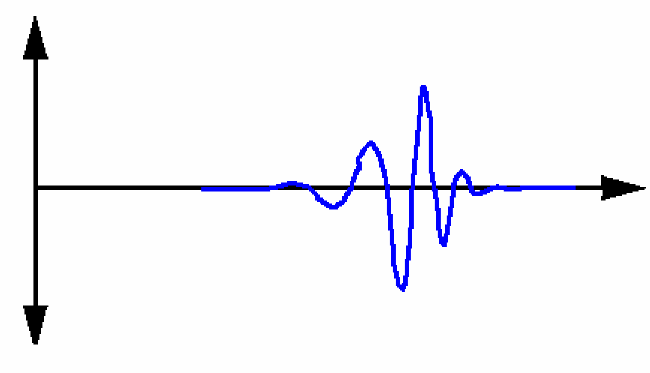
Shifting

- Shifting means to delaying or hastening its onset (starting point)

f(t-k) is *f(t)* delayed by *k* :



Wavelet function
$\Psi(t)$

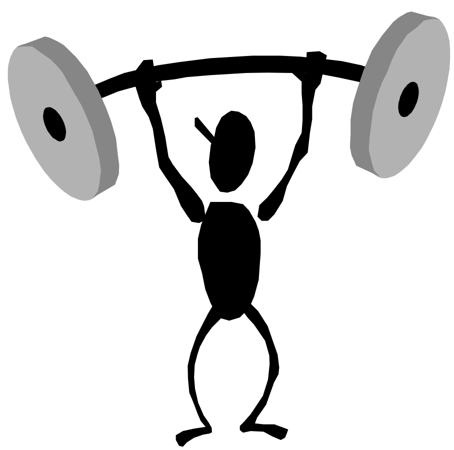


Shifted Wavelet function
$\Psi(t-k)$

CWT: The Process

- Reminder: The *CWT* Is the sum over all time of the signal, multiplied by scaled and shifted versions of the wavelet function Ψ

- A 5 steps process to be taken:



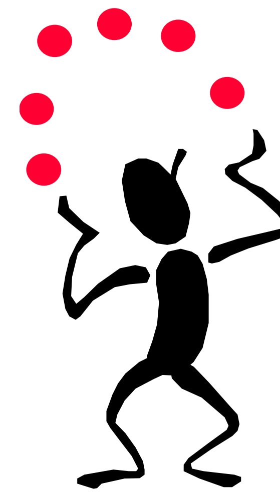
Step 1:

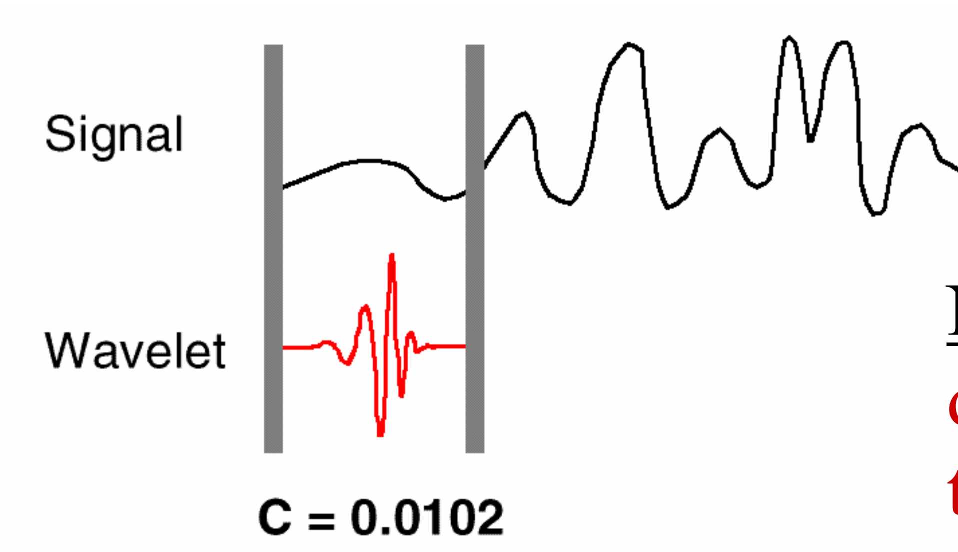
Take a Wavelet and compare it to a section at the start of the original signal

CWT: The Process (Cont'd)

Step 2:

Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity.

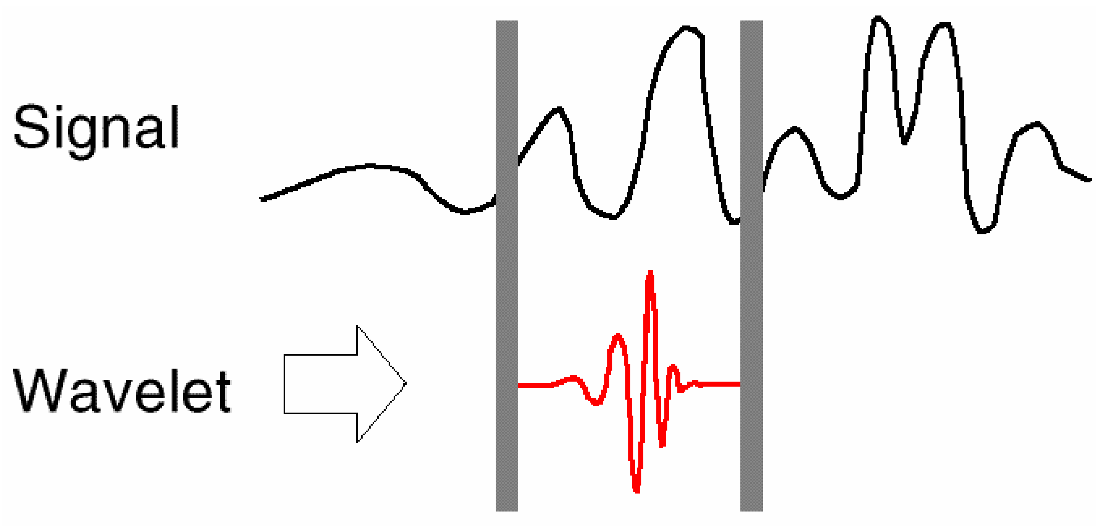




Note: The results will depend on the shape of the wavelet you choose !

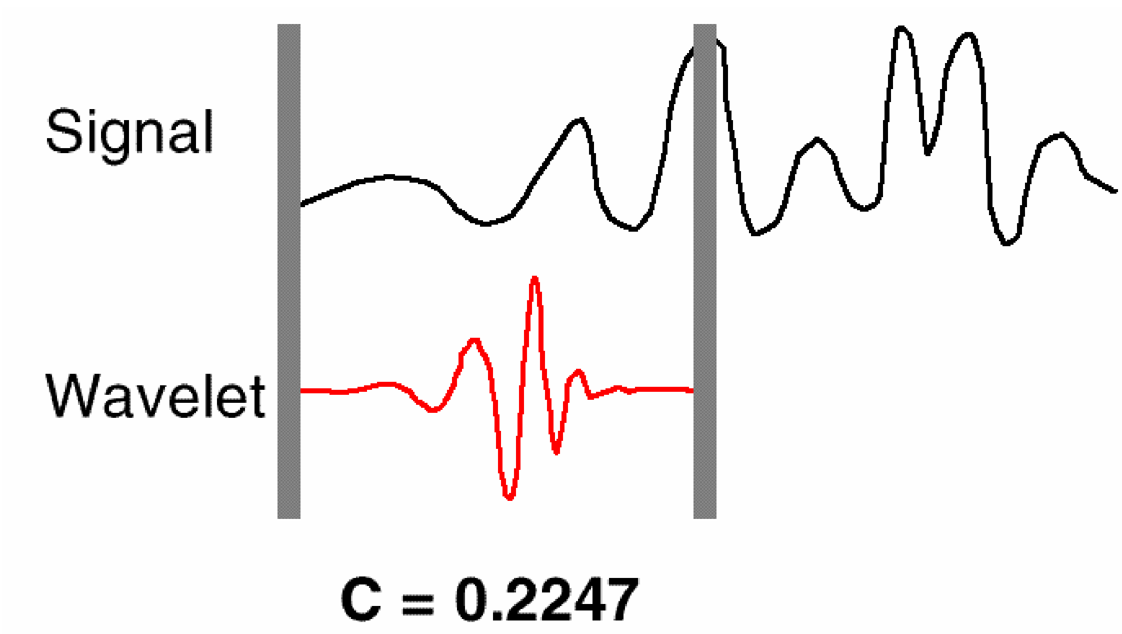
CWT: The Process (Cont'd)

- Step 3: Shift the wavelet to the right and repeat steps 1-2 until you've covered the whole signal



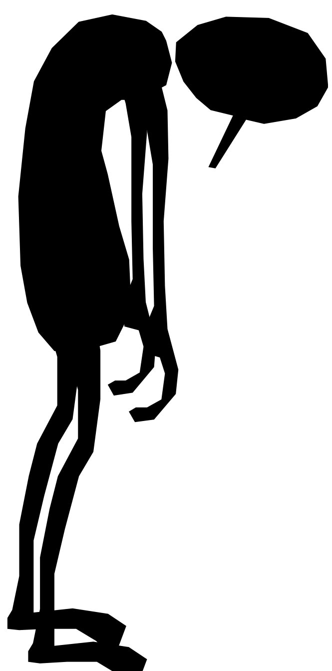
CWT: The Process (Cont'd)

- Step 4: Scale (stretch) the wavelet and repeat steps 1-3



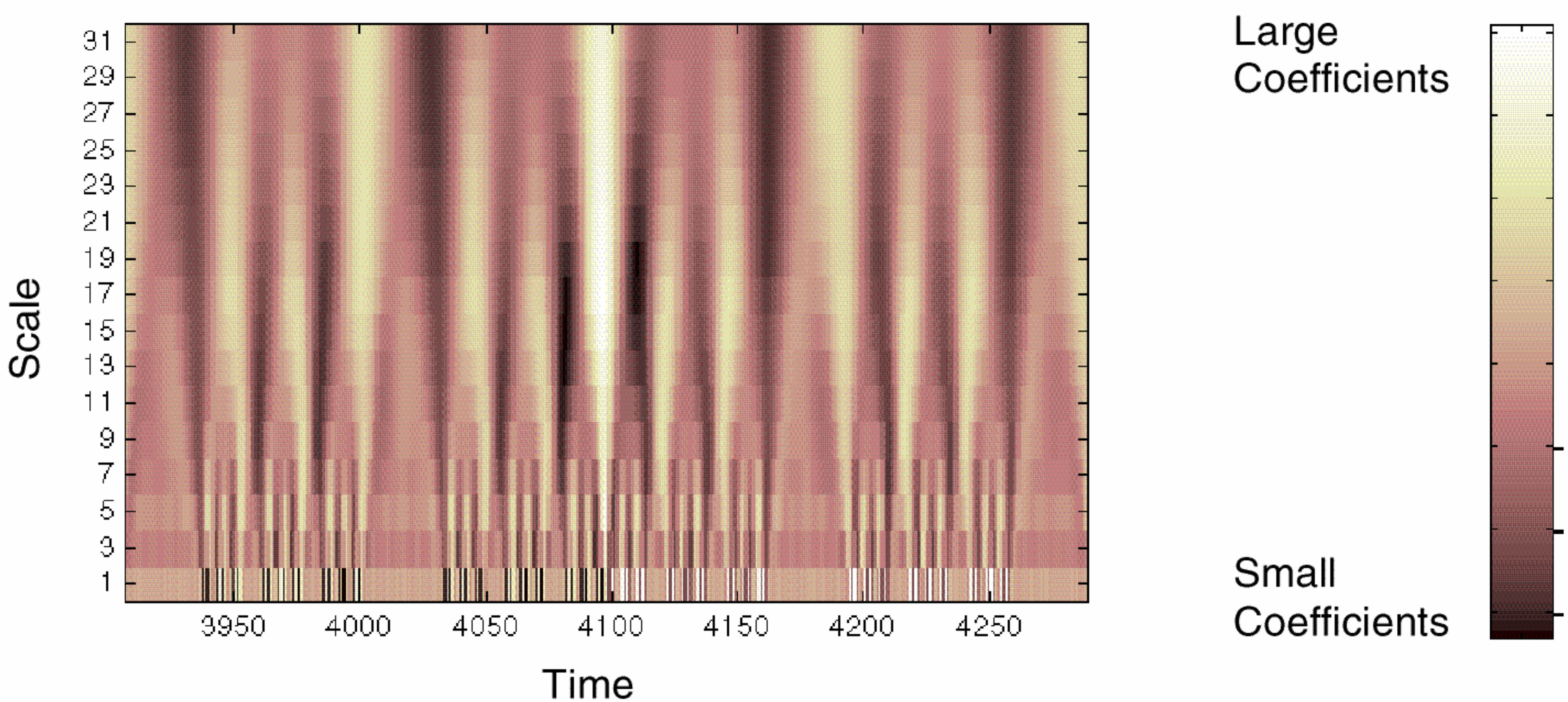
CWT: The Process (Cont'd)

- Step 5: Repeat steps 1-4 for all scales...

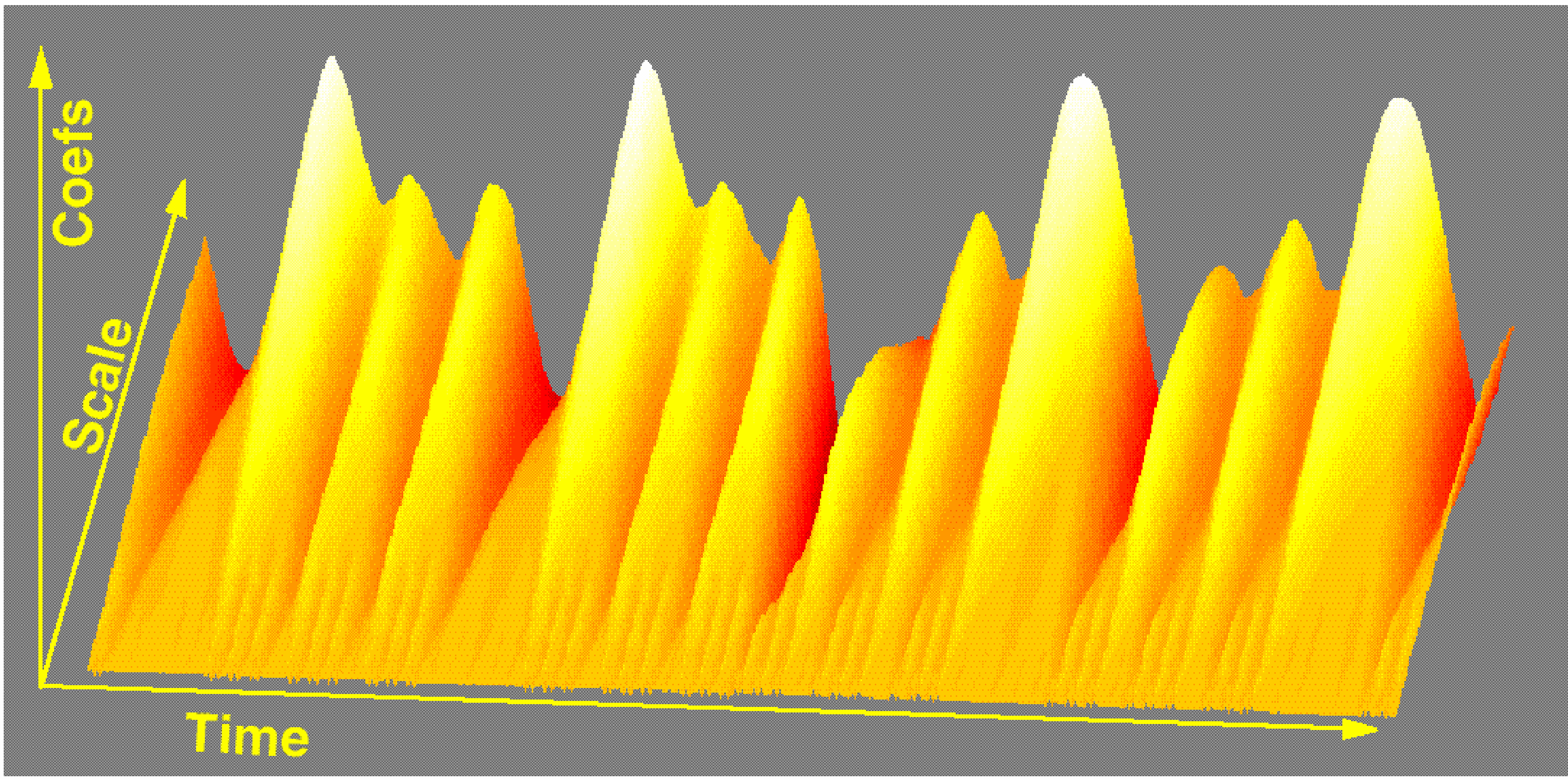


And when you are done...

- You'll get the coefficients produced at different scales by different sections of the signal:

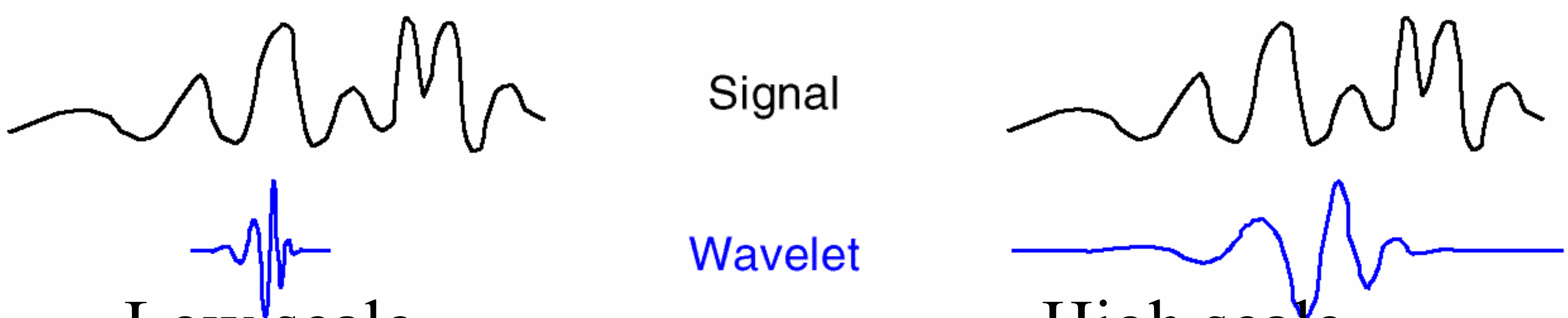


A “side” look at the surface:



Scale and Frequency

- In the former example, the "scale" run from 1 to 31, when higher scale correspond to the most "stretched" wavelet.
- The more stretched the wavelet - the longer the portion of the signal with which it is being compared, and thus, the coarser the signal features being measured by the wavelet coefficient.



Low scale High scale

Scale and Frequency (Cont'd)

- Low scale *a* : Compressed wavelet :Fine details (rapidly changing) : High frequency

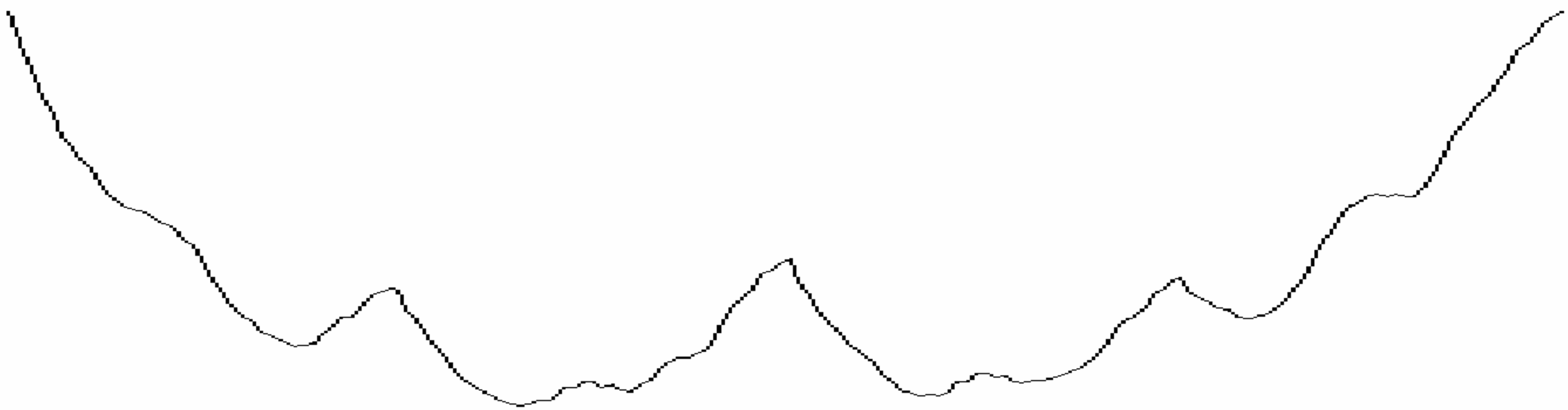
- High scale *a* : Stretched wavelet: Coarse details (Slowly changing): Low frequency

Why Scale ?

- <u>Time-Scale</u> is a different way to view data… but it s more than that !

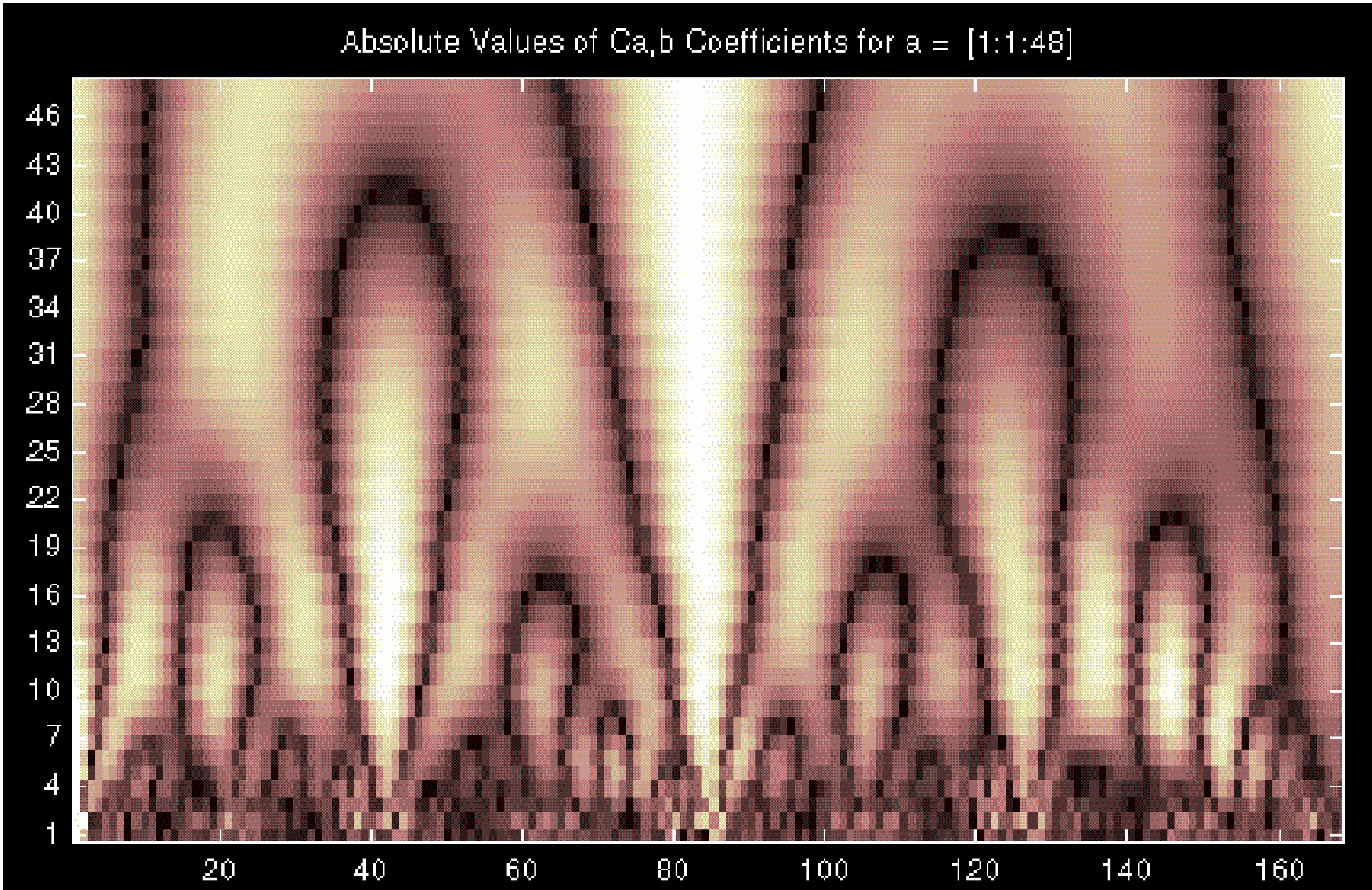
Time-Scale is a very natural way to view data deriving from a great number of natural phenomena !

Self Similarity



A simulated lunar landscape. ragged surface.

CWT of the “Lunar landscape”

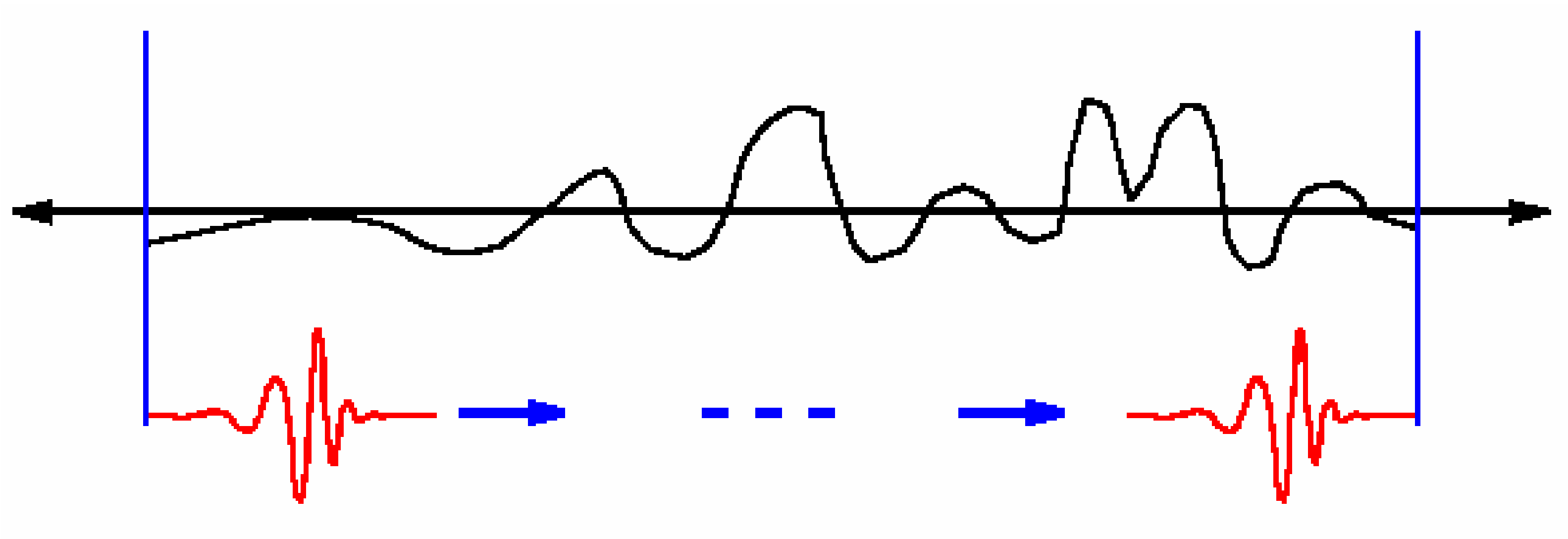


A "Continuous" transform?

The CWT is continuous in 2 means:

- It can operate at every scale, up to some maximum scale you determine (trade off between detailed analysis and CPU time…).
- During analysis the wavelet is shifted smoothly over the analyzed function.

Shift Smoothly over the analyzed function



The DWT

- Calculating the wavelets coefficients at every possible scale is too much work
- It also generates a very large amount of data

Solution: choose only a subset of scales and positions, based on power of two (*dyadic* choice)

Discrete Wavelet Transform

Mallat* Filter Scheme

- Mallat was the first to implement this scheme, using a well known filter design called "two channel subband coder", yielding a *'Fast Wavelet Transform'*

* Mallat S., A Theory for Multiresolution Signal Decomposition: The Wavelet Representation, IEEE Pattern Anal. and Machine Intelligence ., Vol.11 No.7 pp.674-693

One Stage Filtering

Approximations and details:

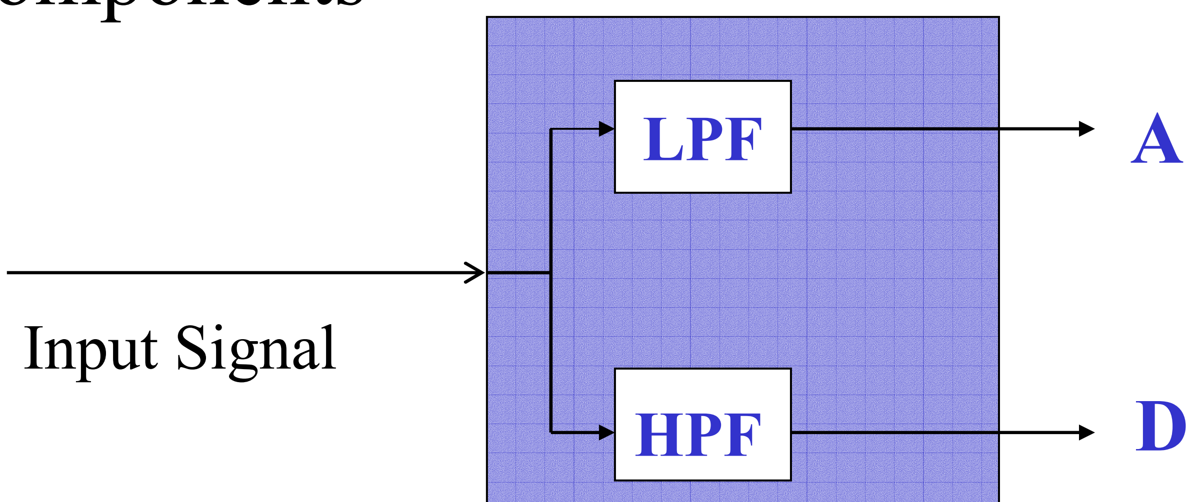
- The low-frequency content is the most important part in many applications, and gives the signal its identity.

This part is called "*Approximations*"

- The high-frequency gives the 'flavor', and is called "*Details*"
- e.g. Human voice

Approximations and Details:

- Approximations: High-scale, low-frequency components of the signal
- Details: low-scale, high-frequency components

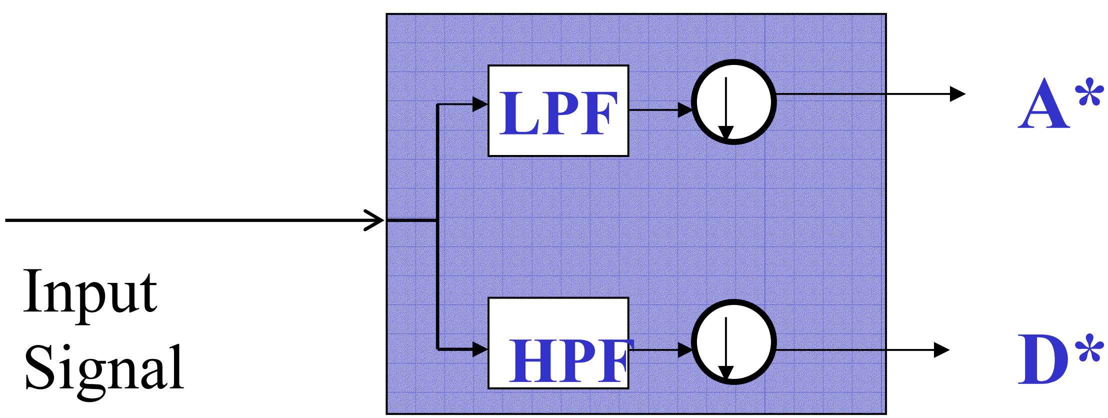


Decimation

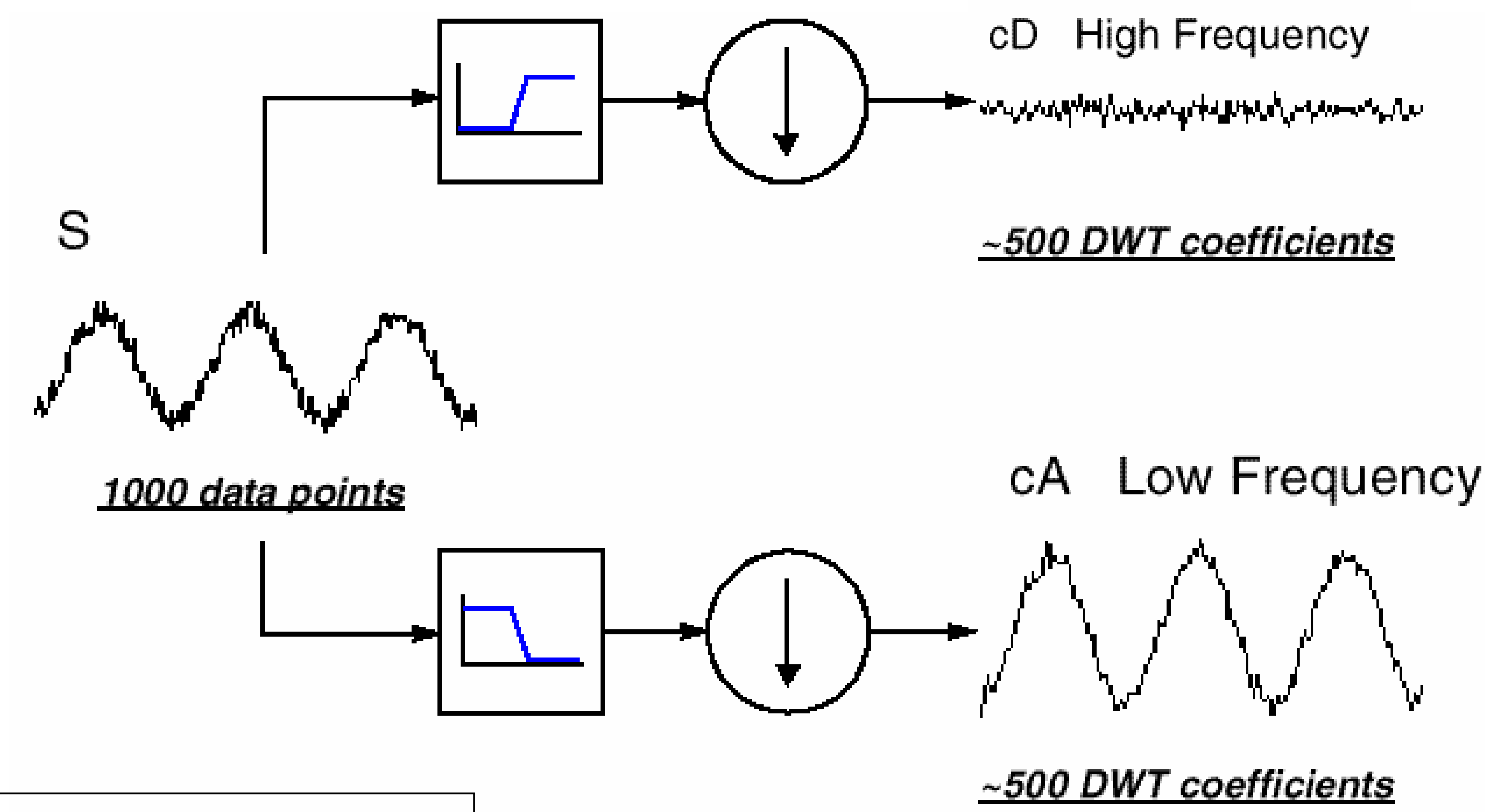
- The former process produces twice the data it began with: N input samples produce N approximations coefficients and N detail coefficients.
- To correct this, we *Downsample (*or: *Decimate)* the filter output by two, by simply throwing away every second coefficient.

Decimation (cont'd)

So, a complete one stage block looks like:



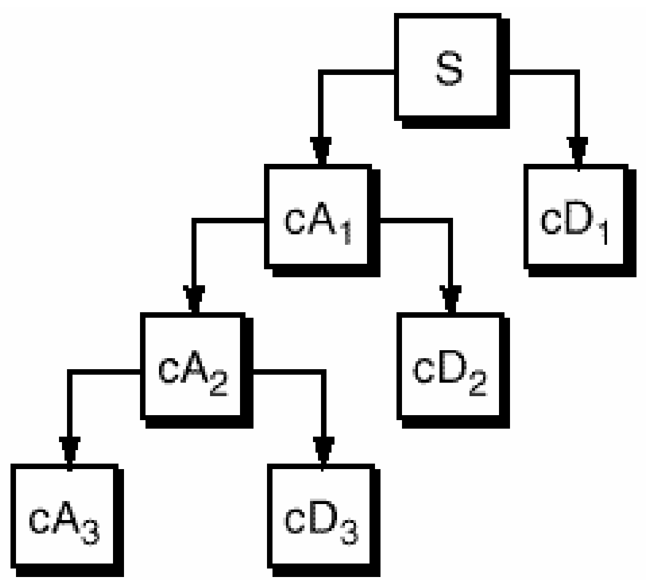
Example*:



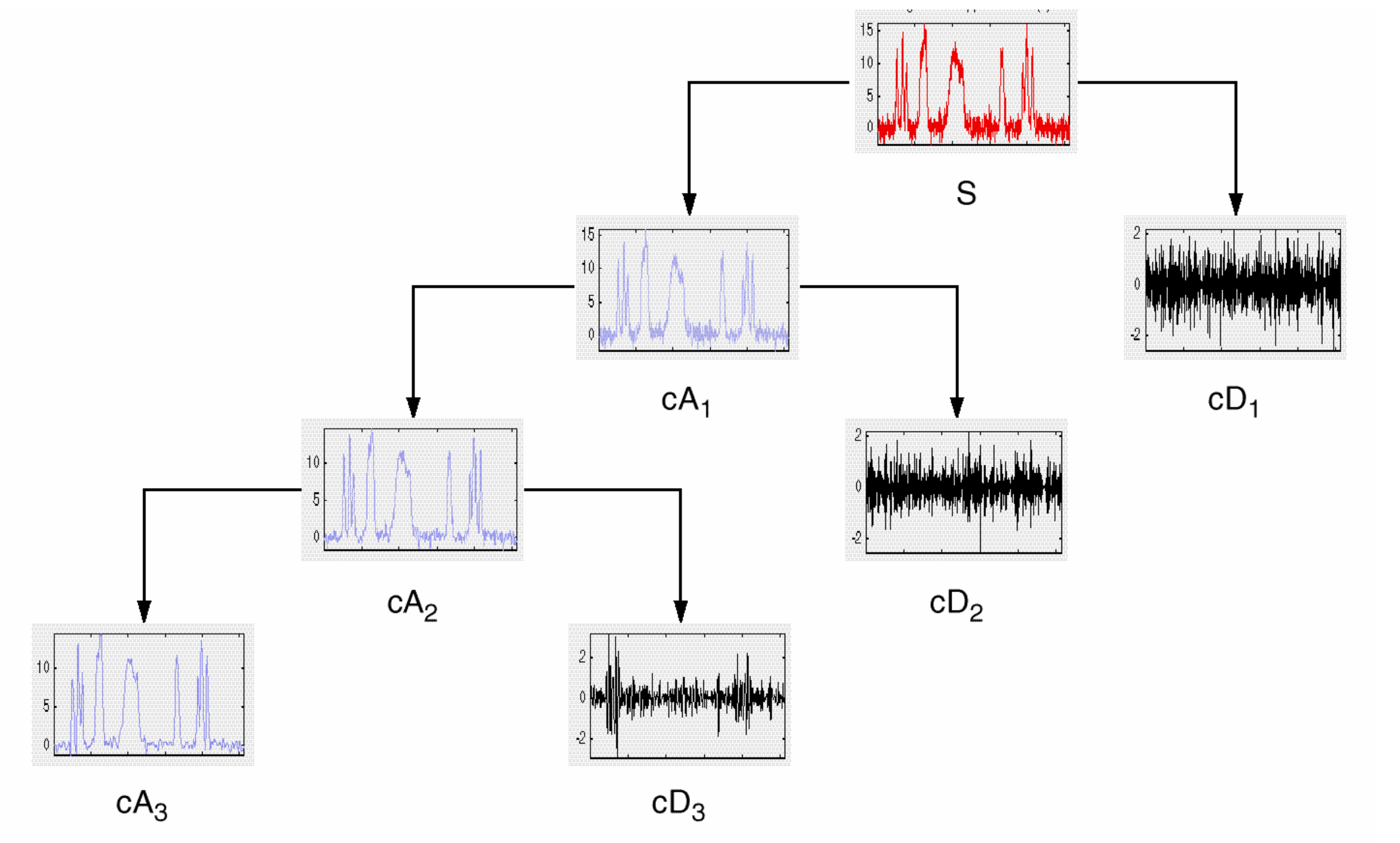
* Wavelet used: db2

Multi-level Decomposition

- Iterating the decomposition process, breaks the input signal into many lower-resolution components: *Wavelet decomposition tree*:



Signal's Wavelet decomposition tree

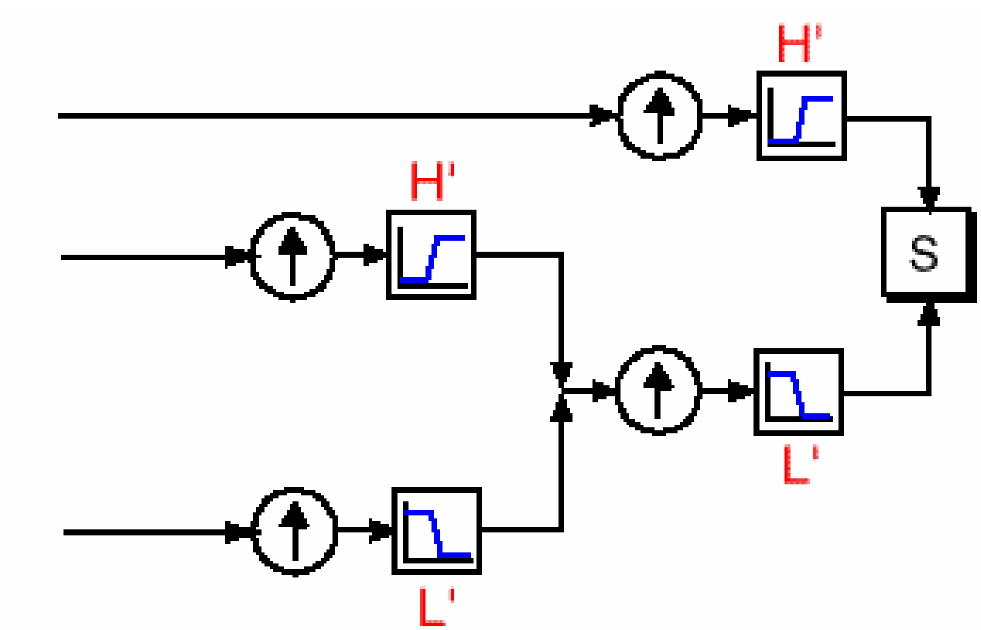


Number of Levels

- Theoretically, The process can be continued indefinitely, until one sample is left.
- In practice, the number of levels is based on the nature of the signal, or a relevant criterion (e.g. entropy).

Wavelet reconstruction

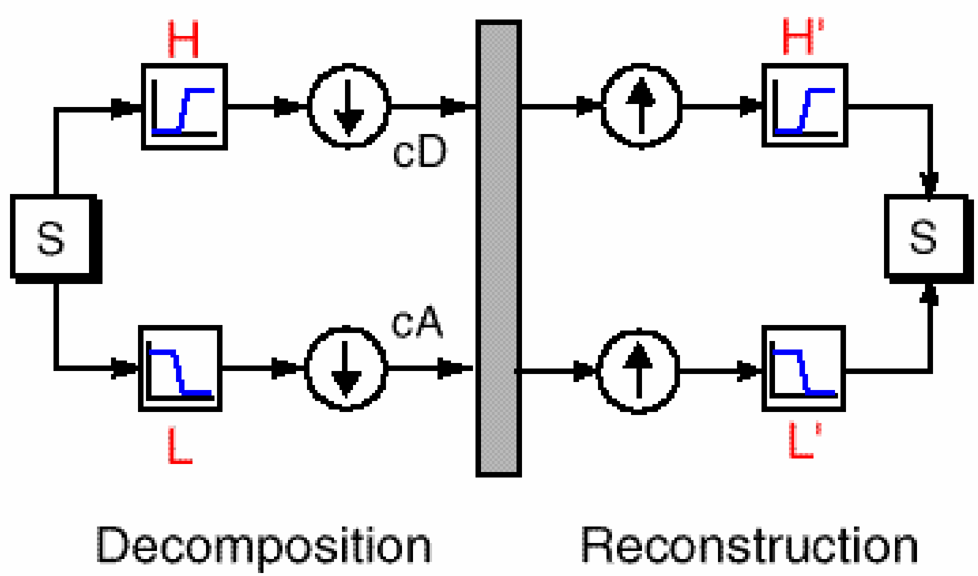
- Reconstruction (or synthesis) is the process in which we assemble all components back



Upsampling (or interpolation) is done by zero padding between every two coefficients

Filter Design

The decomposition and reconstruction filters design is based on a very well known technique called *"Quadrature Mirror Filters"*

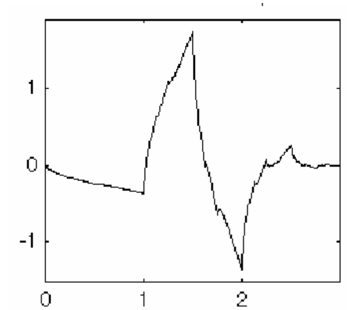


Relationship of Filters to Wavelet Shape

- Choosing the correct filter is most important.
- The choice of the filter determines the shape of the wavelet we use to perform the analysis.
- Usually, we first design the QMF, and then use them to create the waveform.

Example

- A low-pass reconstruction filter (L') for the db2 wavelet:



The filter coefficients (obtained by Matlab *dbaux* command:

0.3415 0.5915 0.1585 -0.0915

reversing the order of this vector and multiply every second coefficient by -1 we get the high-pass filter H':

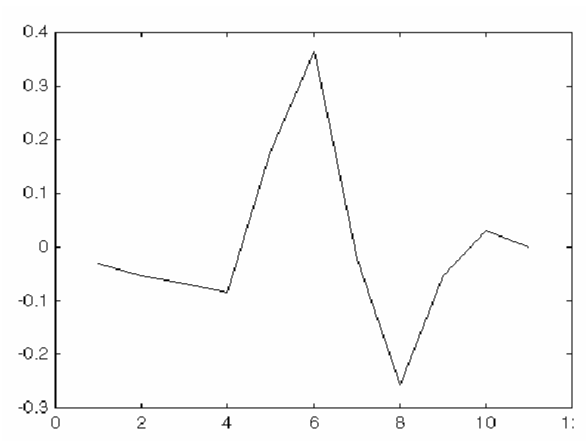
-0.0915 -0.1585 0.5915 -0.3415

Example (Cont'd)

- Now we up-sample the H' coefficient vector:

-0.0915 0 -0.1585 0 0.5915 0 -0.3415 0

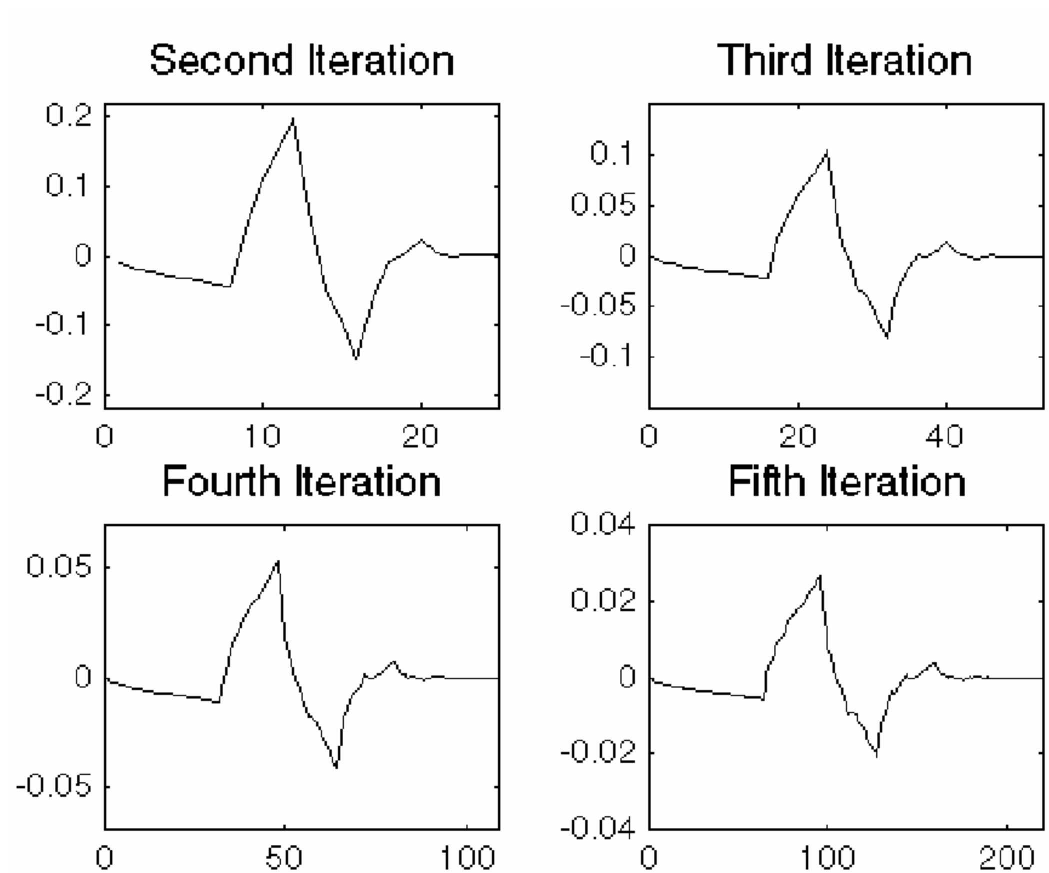
- and Convolving the up-sampled vector with the original low-pass filter we get:



Example (Cont'd)

- Now iterate this process several more times, repeatedly up-sampling and convolving the resultant vector

with the original low-pass filter, a *pattern* begins to emerge:



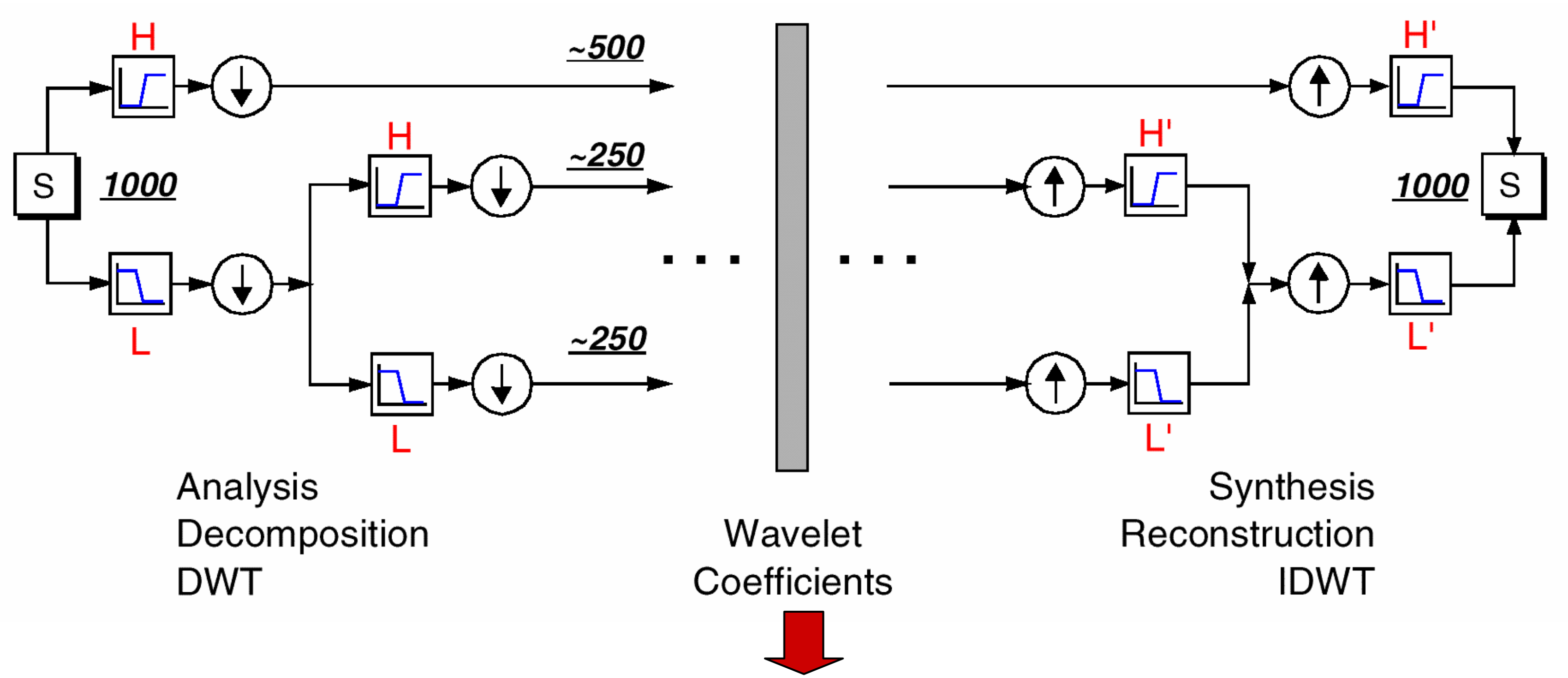
Example: Conclusion

- The curve begins to look more like the *db2* wavelet: the wavelet shape is determined entirely by the coeff. Of the reconstruction filter
- You can't choose an arbitrary wavelet waveform if you want to be able to reconstruct the original signal accurately !

You should choose a shape determined by quadrature mirror decomposition filters

Multistep Decomposition and Reconstruction

- A multistep analysis-synthesis process:



Process: compression, feature extraction etc.

Wavelet Packet Analysis

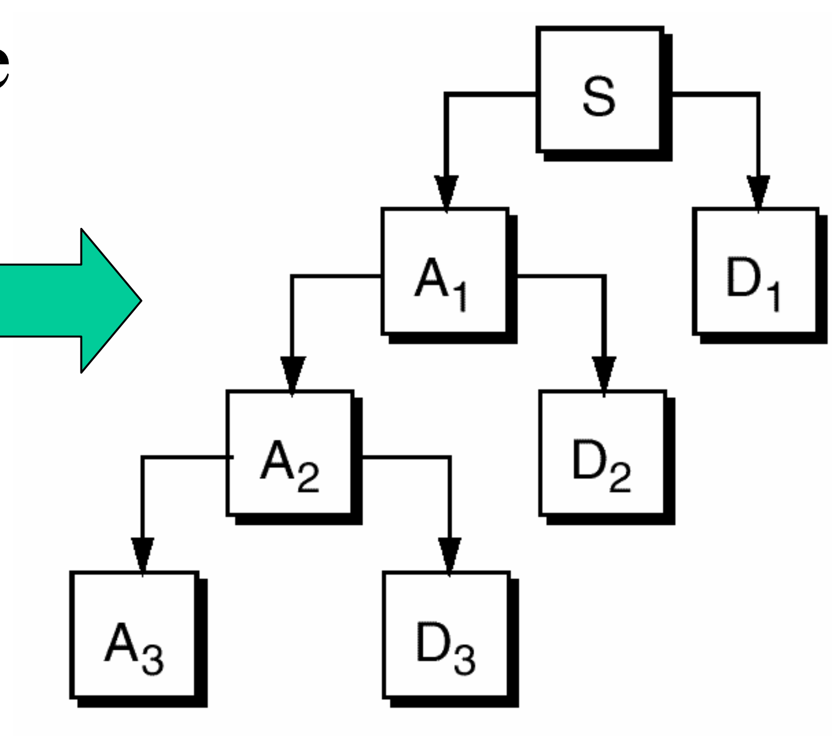
- A method of generalization of the wavelet decomposition that offers richer range of possibilities for signal analysis
- in wavelet analysis we

split the signal again

and again into

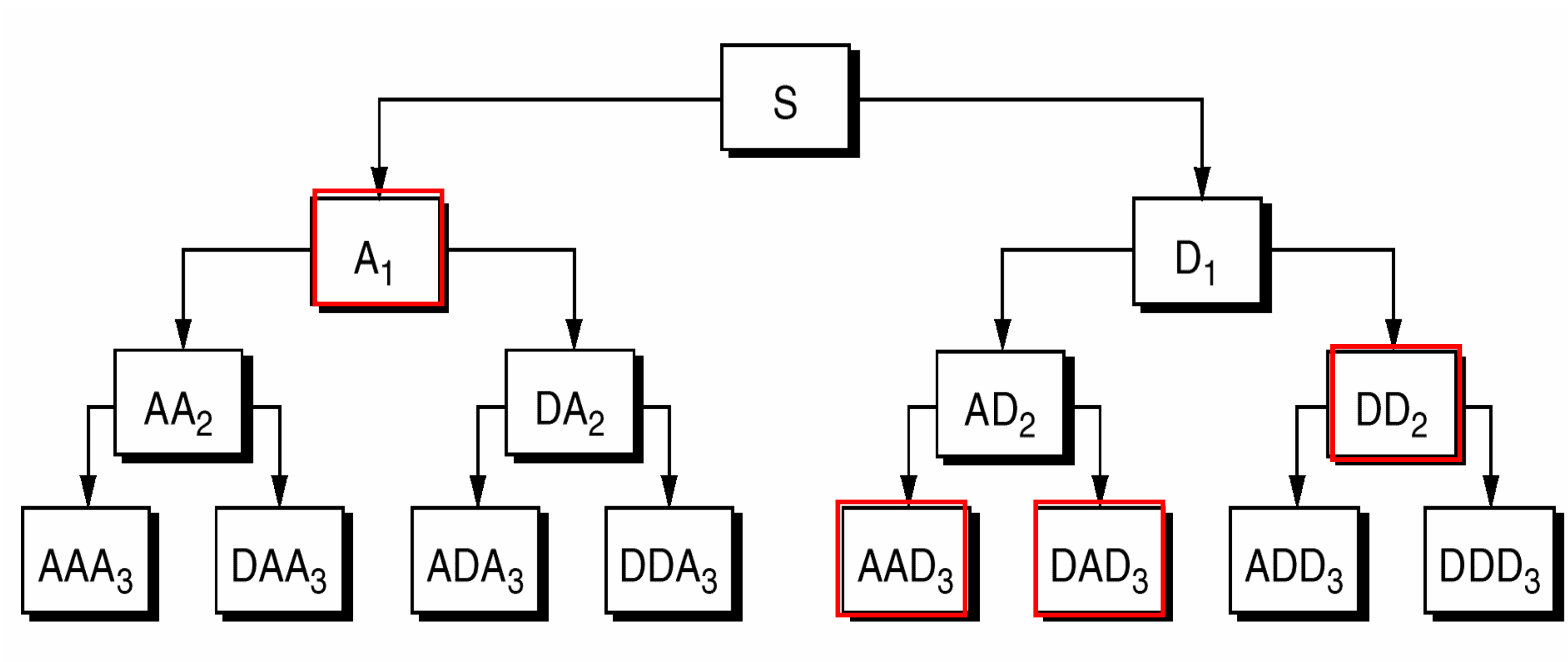
Approximations

and Details



Wavelet Packet Analysis (cont'd)

- In the wavelet packet analysis, both Approximations and Details can be split, so that there are 2^n different ways to encode a signal



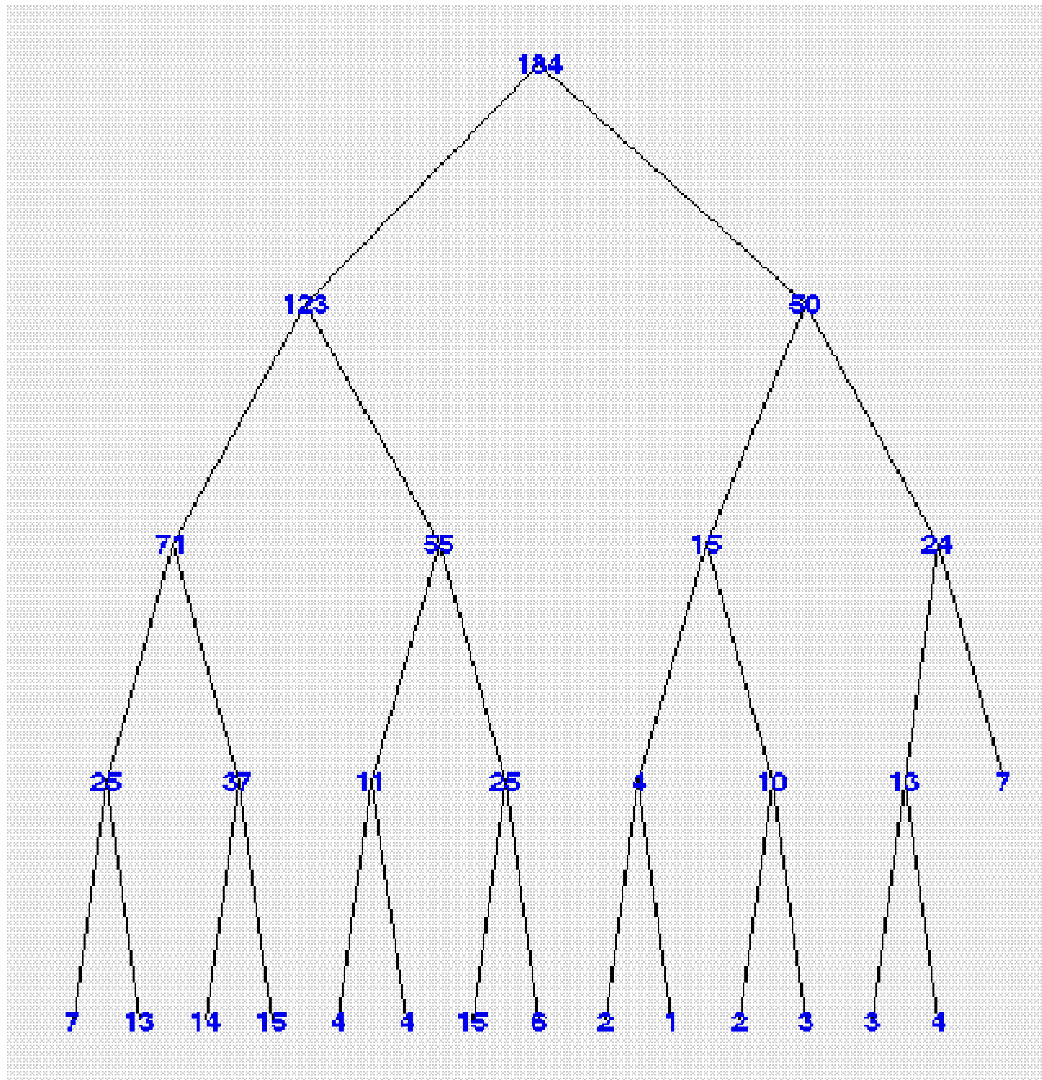
Wavelet Packet Analysis (cont'd)

- E.g. , Signal S can be represented as:

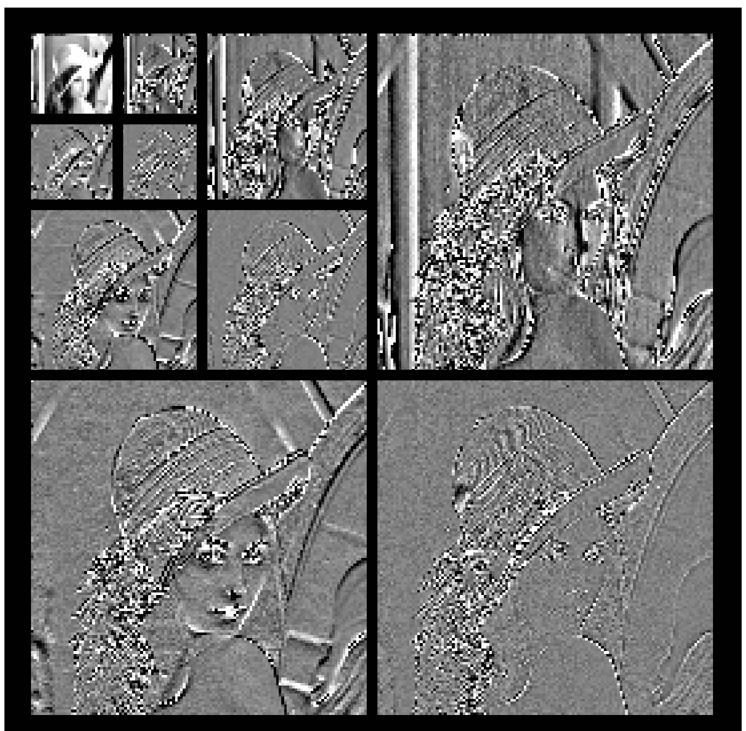
A1+AAD3+DAD3+DD2 , which is not possible in regular wavelet analysis.

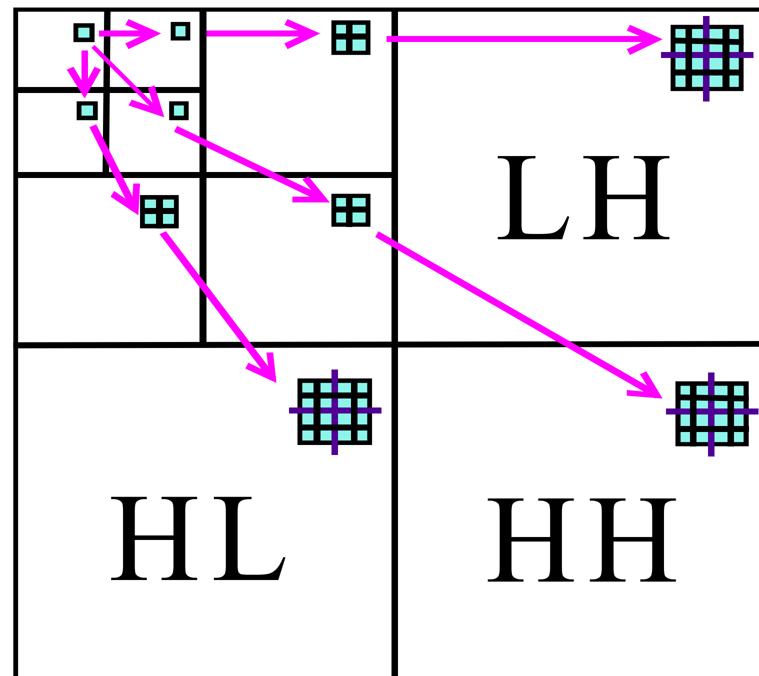
- The most suitable decomposition can be determined in various ways, for instance, The Matlab toolbox uses *entropy based criterion:* we look at each node of the tree and quantify the information we gain by performing each split.

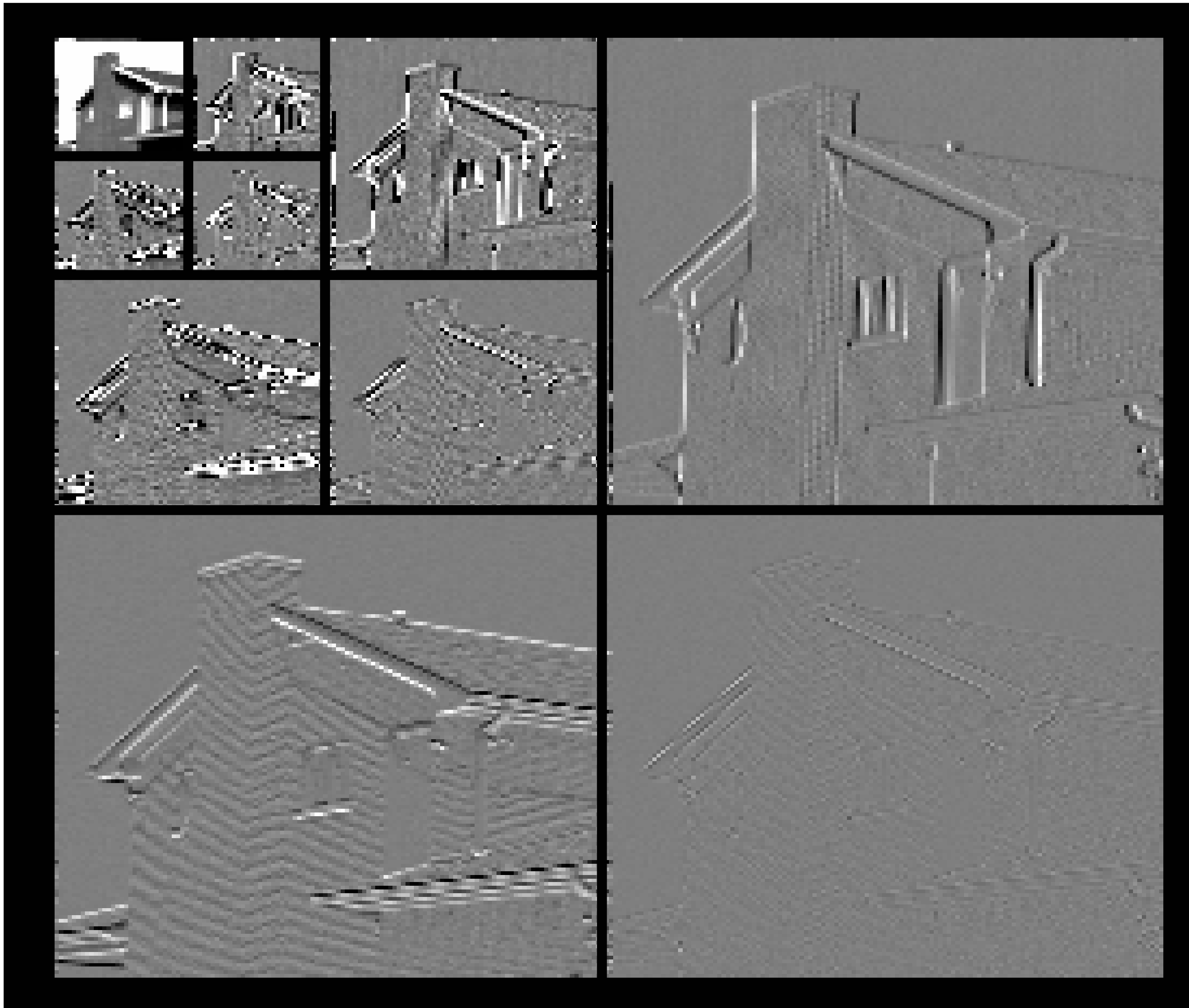
Example of Coding Tree



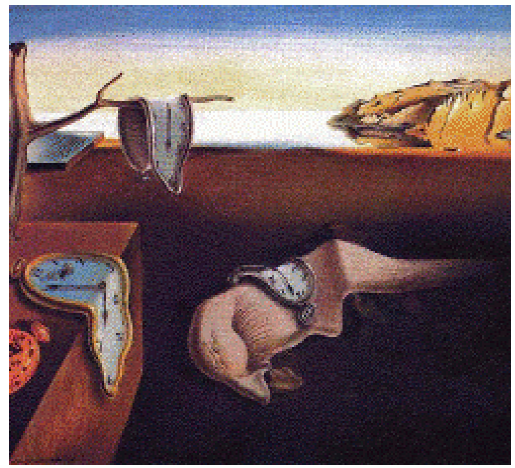
Sub-Band Example







Coding Example



Original @ 8bpp

DWT

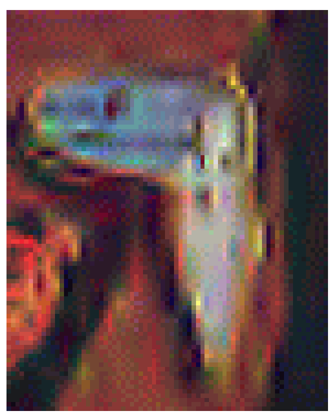
@0.5bpp

DCT

@0.5 bpp

Zoom on Details

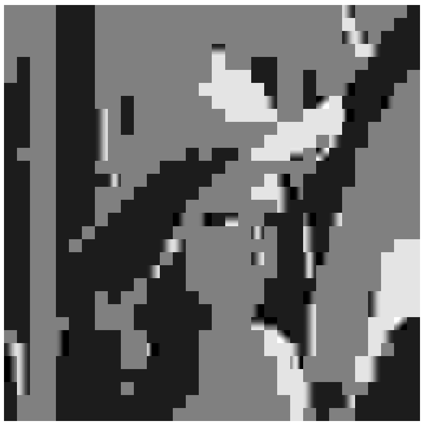
DWT



DCT

Another Example

(rana.usc.edu:8376/~kalocsai/wavelet.html)

	0.15bpp	0.18bpp	0.2bpp
DCT			
DWT