Basics of DCT, Quantization and Entropy Coding

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Discrete Cosine Transform (DCT)

• First used in 1974 (Ahmed, Natarajan and Rao).
• Very close to the Karunen-Loeve* (KLT) transform (or: Hotelling** Transform), that produces un-correlated coefficients.
• Coefficients can be quantized using visually-weighted quantization values.
• A fast algorithm is available and similar to FFT.

* named after Kari Karhunen and Michel Loève
** in honor of Harold Hotelling
DCT Based Compression

- HVS response is dependent on spatial frequency.
- We decompose the image into a set of waveforms, each with a particular spatial frequency.
- DCT is a real transform.
- DCT de-correlating performance is good.
- DCT is reversible (with IDCT).
- DCT is a separable transform.
- DCT has a fast implementation.
DCT (Discrete Cosine Transform)

- The DCT helps separate the image into parts (or spectral sub-bands) of differing importance, with respect to the image's visual quality.

\[
F_{(u)} = \left(\frac{2}{N}\right)^{1/2} \sum_{i=0}^{N-1} A_{(i)} \cos \left(\frac{\pi u}{2N} (2i + 1)\right) f_{(i)}
\]

\[
A_{(i)} = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{for } i=0 \\
1 & \text{Otherwise}
\end{cases}
\]
2D - DCT

• Image is processed one 8x8 block at a time
• A 2-D DCT transform is applied to the block:

\[ F(u, v) = \frac{C(u)C(v)}{4} \sum_{x=0}^{7} \sum_{y=0}^{7} f(x, y) \cos\left(\frac{(2x+1)u\pi}{16}\right) \cos\left(\frac{(2y+1)v\pi}{16}\right) \]

\[ C(n) = \begin{cases} 
  \frac{1}{\sqrt{2}} & n = 0 \\
  1 & n \neq 0 
\end{cases} \]

- Perform nearly as well as the ideal K-L transform
- Closely related to the DFT transform
- Computationally efficient
2D-DCT Basic Functions
Why DCT and not FFT?

- DCT is similar to the Fast Fourier Transform (FFT), but can approximate lines well with fewer coefficients.

A better “De-Correlator”:
Concentrates energy
In the lower “Frequencies”
Computing the 2D DCT

- Factoring reduces problem to a series of 1D DCTs:
  - apply 1D DCT (Vertically) to Columns
  - apply 1D DCT (Horizontally) to resultant Vertical DCT
- Or alternatively Horizontal to Vertical.

Most software implementations use fixed point arithmetic. Some fast implementations approximate coefficients so all multiplies are shifts and adds.

1-D DCT Correlation Removal
(original signal: yellow, DCT: green)
And in numbers...

Original Values

20  60  10  -90  80  120  20  115

DCT Coeff. Values (Not normalized)

670.0000  -308.3878  229.6567  231.0802
-120.2082  -509.6407  203.3661  69.0309

Compression ???
And the Answer ...
Quantization of DCT Coeff.

• Quantization reduces accuracy of the coefficients representation when converted to integer.
• This zero’s many high frequency coefficients.
• We measure the threshold for visibility of a given basis function (coefficient amplitude that is just detectable by human eye).
• We divide (Quantize) the coefficient by that value (+ appropriate rounding to integer).
What is Quantization?

- Mapping of a continuous-valued signal value $x(n)$ onto a limited set of discrete-valued signal $y(n)$: $y(n) = Q[x(n)]$

  such as $y(n)$ is a “good” approximation of $x(n)$

- $y(n)$ is represented by a limited number of bits
- We define **Decision levels** and **Representation levels**
Inverse Quantization

Un-Quantized DCT Coeff.: -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
Quantization Intervals:
Quantized DCT Coeff.: -2 -1 0 1 2
De-Quantized DCT Coeff.: -8 -4 0 4 8
Inverse Quantization  (Cont’d)

Note that:

- **Quantization factor** is 4 (for the given example)
  - A half of the Q-factor (i.e. 2 for the above) is added to coefficient magnitude before it is divided and truncated
  - done in JPEG

- **Linear quantization**: for a given quantization value, steps are uniform in size

- **8-bit sample transforms to 11-bit quantized DCT coefficients**, for Q value of 1
  - for an 8-bit sample, quantization value of 16 produces 7-bit DCT coefficients)
## Luminance Q-Table

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
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## Chrominance Q-Table

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<th>24</th>
<th>47</th>
<th>99</th>
<th>99</th>
<th>99</th>
<th>99</th>
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<td>99</td>
<td>99</td>
<td>99</td>
<td>99</td>
<td>99</td>
</tr>
</tbody>
</table>
Quality Factor

- For other quality factors, the elements of these quantization matrices are multiplied by the compression factor $\alpha$, defined as:

$$\alpha = \begin{cases} \frac{50}{q_{\_ JPEG}} & 1 \leq q_{\_ JPEG} \leq 50 \\ 2 - \frac{2q_{\_ JPEG}}{100} & 51 \leq q_{\_ JPEG} \leq 99 \end{cases}$$

- The minimum legal $\alpha \cdot Q(u,v)$ is 1
- For $\alpha=100$, all $Q(u,v)$ equal 1
  - This cannot be a lossless compression !!!
Notes on Q-Tables

• **Visibility** measured for:
  – 8x8 DCT basis functions
  – Luminance resolution: 720x576
  – Chrominance resolution: 360x576
  – Viewing distance: 6 times screen width

• **Noticeable artifacts** on high-quality displays, since little is known about visibility thresholds when more than one coefficient is zero (simultaneously)
More About JPEG DCT Process

• Zero-shift on input samples to convert range from 0:255 to -128:127, reduces internal precision requirements in DCT calculations.

• DCT is always done on 8x8 blocks:
  – Spatial frequencies in the image and of the cosine basis are not precisely equivalent.
  – “Blocking artifacts” appear if not enough AC coeff. are taken.
DCT Computational Complexity

• Straightforward calculation for 8x8 block: each coeff. needs 64 multiplies and 63 additions. Quantization requires one more addition and a division.

• Fast algorithm requires 1 multiply and 9 additions.

• Modern pipelined processors architectures makes the calculation very economic and fast.
Entropy Coding: VLC

How are you? I just wanted to say hi. If you can see this, I don't think you are blind.

בתרמודינמיקה: רמת אינפרא במעורכת התרמודינמיקה נמנים עם מידה לעבודה

בתרמודינמיקה: נמדע לckett מטרני האנרגיה שיאינה.iterator לעבודה

Second Law of Thermodynamics

Entropy (simplicity) increases in closed system
Entropy Encoding

• Amount of information in a symbol: \( F = \log_2 \frac{1}{P_i} \)

• **Entropy** of an Image:
  “Average amount of information”

• A source with \( M \) possible symbols, With a uniform distribution:
  \[
  H_1 = -\sum_{i=1}^{M} P_i \log_2 P_i \quad \text{[bit]}
  \]

• Shannon Theorem:
  \( H = \text{minimum number of bits to code the source} \)
The Shannon Theorem

• Shannon (1948) showed that for the coding of a discrete (independent) source, a uniquely decodable binary code exists, with average codeword length: \[ H(Y(n)) \leq S_{ave} \]

• Shannon also showed that for this source, a binary prefix code (such as Huffman) exists, and holds: \[ S_{ave} \leq H(Y(n)) + 1 \]
The Shannon Theorem (Cont’d)

• If N independent symbols are taken together we get: \( S_{\text{ave}} \leq H(Y(n)) + \frac{1}{N} \)

• For very large N, \( S_{\text{ave}} \) is very close to the entropy, but the practical problems become severe:
  – coding delay
  – complexity
  – size of VLC tables
k-th order entropy \( \frac{1}{2} \)

- For an ‘alphabet’ \( A \) (set of symbols):

\[
H_k = -\frac{1}{k} \sum_{x_1 \ldots x_k \in A^k}^M P(x_1, \ldots x_k) \log_2 P(x_1, \ldots x_k) \quad \text{[bit]}
\]

- The important property of the entropy is:

\[
\log 2^{\left| A \right|} \geq H_k(X) \geq H_{k+1}(X) \geq 0
\]

- The expected value (per-letter in ‘A’ !) is always greater than or equal to entropy:

\[
\frac{E\left\{ L(X_1, \ldots X_n) \right\}}{n} \geq H_n(X) \quad \text{Shannon, Theorem 1}
\]
k-th order entropy

• Shannon, Theorem 2
There exists a coding algorithm such that:

\[
\frac{E\{L(X_1, \ldots, X_n)\}}{n} < H_n(X) + \frac{1}{n}
\]

• Important to remember:
for a “memoryless” source (symbols are independent): entropy of any order is equal to the first order entropy!
Influence of block length

- Does theorem 2 mean that for memory-less source, we can reach entropy by 1-symbol blocks?

Consider a binary source with entropy: $H_1=0.4$
According to the above, the lower bound is $H_1 + \frac{1}{n}$, with $n=1$ we get: $0.5 + 1 = 1.5$ bpp

But …

we can easily do it with 1 bpp ….
Huffman Coding

– The symbol with the highest probability is assigned the shortest code and vice versa.
– Codewords length is not fixed (VLC)
– A codeword cannot be a prefix for another codeword (Self Synchronized Code)
### Huffman Coding Example

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Pi</th>
<th>Huffman</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>0.4</td>
<td>1</td>
<td>000</td>
</tr>
<tr>
<td>Q2</td>
<td>0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q3</td>
<td>0.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q4</td>
<td>0.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q5</td>
<td>0.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q6</td>
<td>0.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q7</td>
<td>0.03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q8</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Diagram of Huffman Coding:

- **Symbol**
- **Pi** (Probability)
- **Huffman** (Code)
- **Binary** (Code)
Huffman Coding Example (Cont’d)

- Source Entropy ($H_0$):
  \[ H = - \sum_{i=1}^{M} P_i \log_2 P_i = 2.45 \]

- Huffman Rate:
  \[ R = \sum P_i L_i = 2.52 \]

- No Coding: 3 bps
- Compression Ratio: 2.52/3 = 0.84

• Problems:
  - Sensitive to BER
  - Statistics Dependent
  - Not Unique
Other Coding Techniques

• **Run-Length**
  Coding of strings of the same symbol
  (Used in JPEG)

• **Arithmetic (IBM)**
  Probability coding  (A JPEG Option)

• **Ziv-Lempel (LZW)**
  Used in many public/commercial applications
  such as ZIP etc..  *(LZ Demo - Java Applet)*
So...how JPEG do it?

- The theoretical entropies of each quantized DCT sub-band are known.
- In practice this would be a poor way to code the data because:
  - 64 separate entropy codes would be required, each requiring many extra states to represent run-length coding of zeros.
  - The statistics for each code are likely to vary significantly from image to image.
  - To transmit the code table for each sub-band as header information would involve a large coding overhead.
  - Coding the sub-bands separately does not take account of the correlations which exist between the positions of the non-zero coefficients in one sub-band with those of nearby sub-bands.
Dc Coefficient: DPCM

- The first coefficient (0) of each block is the DC coef, which represents the mean value of the pixels in the block.
- The DC coefficients have significant local correlations, so differential coding is used: the *difference* between the current DC coefficient and that of the previous block.
  - The blocks are scanned from left to right, row by row. The first block in each row is coded with respect to zero.

Next slide:
- The histogram of entropies of the DC coefficients differences is compared with that of the raw DC coefs.
- We note the histogram peak around zero and see that the entropy is reduced from 6.42 bits to 6.07 bits.
Entropy Histograms of DC Coefficients

With Differential Coding

Without differential coding, entropy = 6.42 bits

With differential coding, entropy = 6.07 bits
Coding the DC Coefficients

• The size of the differences can in theory be up to 
  \( \pm(255 \times 8) = \pm2040 \) if the input pixels occupy the range -128 to +127 (The DCT has a gain of 8 at very low frequencies)
• Hence the Huffman code table would have to be quite large.
• JPEG adopts a much smaller code by using a form of floating-point representation, where \( \text{Size} \) is the base-2 exponent and \( \text{Additional Bits} \) are used to code the polarity and precise amplitude
• There are only 12 Sizes to be Huffman coded, so specifying the code table can be very simple and require relatively few bits in the header: 16 header bytes + 12 codes = 28 bytes for the DC table.
The DC Codes

<table>
<thead>
<tr>
<th>DC Coef Difference</th>
<th>Size</th>
<th>Typical Huffman codes for Size</th>
<th>Additional Bits (in binary)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>00</td>
<td>-</td>
</tr>
<tr>
<td>-1,1</td>
<td>1</td>
<td>010</td>
<td>0,1</td>
</tr>
<tr>
<td>-3,-2,2,3</td>
<td>2</td>
<td>011</td>
<td>00,01,10,11</td>
</tr>
<tr>
<td>-7,...,-4,4,...,7</td>
<td>3</td>
<td>100</td>
<td>000,...,011,100,...111</td>
</tr>
<tr>
<td>-15,...-8,8,...,15</td>
<td>4</td>
<td>101</td>
<td>0000,...,0111,1000,...,111</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1023,...-512,512,...1023</td>
<td>10</td>
<td>1111 1110</td>
<td>00 0000 0000,...,11 1111 1111</td>
</tr>
<tr>
<td>-2047,...-1024,1024,...2047</td>
<td>11</td>
<td>1 1111 1110</td>
<td>000 0000 0000,...,111 1111 1111</td>
</tr>
</tbody>
</table>

- Only **Size** needs to be Huffman coded in the above scheme: within a given Size, all the input values have sufficiently similar probabilities

- Each coded Size is followed by the appropriate number of **Additional Bits** to define the **sign and magnitude** of the coefficient difference exactly
The Run-Size Technique

• The remaining 63 quantized AC-coefficients usually contain many zeros and so are coded with a combined run-amplitude Huffman code.

• The codeword represents the run-length of zeros before a non-zero coefficient and the Size of that coefficient.

• This is followed by the Additional Bits which define the coefficient amplitude and sign precisely.

• This 2-dimensional Huffman code (Run, Size) is efficient because there is a strong correlation between the Size of a coefficient and the expected Run of zeros which precedes it: small coefficients usually follow long runs; larger coefficients tend to follow shorter runs.
The AC Run-Size Table

• To keep the code table size \( n \) below 256, only the following Run and Size values are coded: \( \text{Run}=1 \rightarrow 15 ; \text{Size}=1 \rightarrow 10 \). These require 150 codes.

• Two extra codes, corresponding to \((\text{Run,Size}) = (0,0)\) and \((15,0)\) are used for EOB (End-of-block) and ZRL (Zero run length).

• EOB is transmitted after the last non-zero coef in a 64-vector. It is omitted in the rare case that the final element is non-zero.

• ZRL is transmitted whenever \( \text{Run} > 15 \): a run of 16 zeros which can be part of a longer run of any length.
  • Hence a run of 20 zeros followed by -5 would be coded as: \((\text{ZRL}) \ (4,3) \ 010\)
<table>
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<tr>
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<tr>
<td>(0,1)</td>
<td>01</td>
<td>00</td>
<td>(0,6)</td>
<td>06</td>
<td>1111000</td>
</tr>
<tr>
<td>(0,2)</td>
<td>02</td>
<td>01</td>
<td>(1,3)</td>
<td>13</td>
<td>1111001</td>
</tr>
<tr>
<td>(0,3)</td>
<td>03</td>
<td>100</td>
<td>(5,1)</td>
<td>51</td>
<td>1111010</td>
</tr>
<tr>
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<td>00</td>
<td>1010</td>
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<td>61</td>
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<td>(0,7)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>31</td>
<td>111010</td>
<td>(ZRL)</td>
<td>F0</td>
<td>1111111001</td>
</tr>
<tr>
<td>(4,1)</td>
<td>41</td>
<td>111011</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Run-Level Coding

- After zizag scan:
  
  $$[-33, 21, -3, -2, -3, -4, -1, 0, 3, 2, 1, 1, 0, 0, 0, -2, -1, -1, 0, 0, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$

- After run-level coding:

  $$(0, -33) \ (0, 21) \ (0, -3) \ (0, -2) \ (0, 3) \ (0, -4) \ (0, -3) \ (1, 2) \ (0, 1) \ (1, 1) \ (1, -2) \ (0, -1) \ (0, -1) \ (3, -2) \ (11, 1)$$
A Coding Example

Assume the values of a quantized DCT matrix are given (Zig-Zag order):

42  16  -21  10  -15  0  0  0
3   -2   0   2   -3   0  0  0
0   0   2   -1   0   0  0  0
0   0   0   0   0   0  0  0
0   0   0   0   0   0  0  0
0   0   0   0   0   0  0  0
0   0   0   0   0   0  0  0
0   0   0   0   0   0  0  0
0   0   0   0   0   0  0  0
0   0   0   0   0   0  0  0
0   0   0   0   0   0  0  0
0   0   0   0   0   0  0  0
A Coding Example (Cont’d)

• If the DC value of the previous block was 44, then the difference is -2. If Huffman codeword for size=2 is 011 then the codeword for the DC is: 01101

• Using Annex K in the JPEG for the luminance AC coeff., we receive the following codewords:
A Coding Example (Cont’d)

<table>
<thead>
<tr>
<th>Value</th>
<th>Run/Size</th>
<th>Huffman</th>
<th>Ampl.</th>
<th>Total bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0/5</td>
<td>11010</td>
<td>10000</td>
<td>10</td>
</tr>
<tr>
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<td>0/5</td>
<td>11010</td>
<td>01010</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>0/4</td>
<td>1011</td>
<td>1010</td>
<td>8</td>
</tr>
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<td>00</td>
<td>4</td>
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<td>5/2</td>
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<td>13</td>
</tr>
<tr>
<td>-1</td>
<td>0/1</td>
<td>00</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>EOB</td>
<td>0/0</td>
<td>1010</td>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>
A Coding Example (Cont’d)

• Compression results:
  – DC coeff. : 5 bits
  – AC coeff. : 82 bits
  – Average bitrate: $87/64 = 1.36 \text{ bit/pixel}$
  – Starting from 8 bpp resolution, the compression ratio is : $8/1.36 = 5.88$
A Coding Example (Cont’d)

- **Run-length coding contribution:**

  Direct Huffman coding of DC differentials [-2047, 2047] and AC differentials [-1023, 1023] requires code tables with 4,095 and 2,047 entries respectively.

  By Huffman coding only the \((run, size)\) pairs, tables are reduced to 12 and 161 entries respectively.
(Run,Size) and Quantization

Histogram of the (Run,Size) codewords for the DCT of Lenna, quantised using $Q_{\text{lum}}$.

Histogram of the (Run,Size) codewords for the DCT of Lenna, quantised using $2Q_{\text{lum}}$.

The bin number represents the decoded byte value.
(Run, Size) and Quantization  Cont’d

• Note the strong similarity between the histograms, despite the fact that figure 2 represents only 2/3 as many events.

• Only the EOB probability changes significantly
  – because its probability goes up as the number of events (non-zero coefficients) per block goes down.

• It turns out that the (Run, Size) histogram remains relatively constant over a wide range of images and across different regions of each image.
  – This is because of the strong correlation between the run lengths and expected coefficients sizes.

• The number of events per block varies considerably depending on the local activity in the image, but the probability distribution of those events (except for EOB) changes much less.
Comparing the mean bit rates to code Lenna for the two quantization matrices with the theoretical entropies:

- We see the high efficiency of the \((\text{Run,Size})\) code at two quite different compression factors. This tends to apply over a wide range of images and compression factors and is an impressive achievement.

- There is very little efficiency lost if a single code table is used for many images, which can avoid the need to transmit the 168 bytes of code definition in the header of each image. Using the recommended JPEG default luminance tables (Annex K.3.3) the above efficiencies drop to 97.35% and 95.74% respectively.

<table>
<thead>
<tr>
<th>Q matrix</th>
<th>Mean Entropy (b/pel)</th>
<th>JPEG Bit Rate (b/pel)</th>
<th>JPEG efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_{\text{lum}})</td>
<td>0.8595</td>
<td>0.8709</td>
<td>98.7%</td>
</tr>
<tr>
<td>2 (Q_{\text{lum}})</td>
<td>0.5551</td>
<td>0.5595</td>
<td>99.21%</td>
</tr>
</tbody>
</table>
Arithmetic Coding

• Used in both JPEG and JBIG
• Also known as QM-Coder (A “child” of well-known Q-Coder)
• A Binary coder (1 or 0 symbols only)
• descriptor is decomposed into a tree of binary decisions
• Rather than assigning intervals to the symbols it assigns intervals to the most and less probable symbols (MPS, LPS) ...
Arithmetic Coding  (Cont’d)

...Then orders the symbols such that the LPS sub-interval is above the MPS:

A  

A-Pe*A

LPS Subinterval = Pe*A

Pe: LPS Probability Estimate

MPS Subinterval = (1-Pe)*A
Arithmetic Coding (Example)

Assume quad-dictionary:

- **a** \((P_a=1/2 \rightarrow .100)\)
- **b** \((P_b=1/4 \rightarrow .010)\)
- **c** \((P_c=1/8 \rightarrow .001)\)
- **d** \((P_d=1/8 \rightarrow .001)\)

Codewords as joints of units interval:

\[
0 \quad 0.100 \quad 0.110 \quad 0.111 \quad 1
\]

- a b c d

- This is an accumulative probability function:
  \[ F(x) = \sum_{k=0}^{x} p(k) = p(0) + p(1) + \ldots + p(x) \]
- Each “code point” is the probability sum of previous probabilities
- The length of the interval on the right side of the point is equal to probability of the symbol
- Intervals in our case: \([0, 0.1)\), \([0.1, 0.11)\), ....
Arithmetic Coding (Example)

Successive Subdivision
of unit interval: $a \ a \ b \ ...$

New Code = Current Code + A*Pi

New_A = Current_A + Pi

A: interval length
Pi: probability
Arithmetic Coding Main Problems

• Needs ‘infinite precision’ for interval A
• Needs a multiplication for interval subdivision: $A \times P_e$
• Results about 10% better than Huffman but more complicated
### A Coding Example

- The accumulative distribution function of the random variable $X$ with optional values \{0,1,2,3,4\} are:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(x)$</th>
<th>$F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.2 + 0.1 = 0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.2 + 0.1 + 0.4 = 0.7</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.2 + 0.1 + 0.4 + 0.2 = 0.9</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
<td>0.2 + 0.1 + 0.4 + 0.2 + 0.1 = 1</td>
</tr>
</tbody>
</table>
A Basic coding and decoding algorithm

- We have \( n \) open segments \([a,b), \) where \( n \) is the size of the alphabet we use:
A Coding Example: 1/5

- We’ll represent the “message” “1432” using the accumulative distribution function we have:
A Coding Example: 2/5

- We’ll represent the “message” “1432” using the accumulative distribution function we have:
- First letter: 1
A Coding Example: 3/5

• We’ll represent the “message” “1432” using the accumulative distribution function we have:

• Second letter: 4
A Coding Example: 4/5

- We’ll represent the “message” “1432” using the accumulative distribution function we have:
- Third letter: 3.
A Coding Example: 5/5

• We’ll represent the “message” “1432” using the accumulative distribution function we have:
• Fourth letter: 2
Choosing the coding word

- The string “1432” should be represented by a binary word in the boundary $[0.2976, 0.2984)$
- For simplicity, let's choose the middle point: $0.298$
- $0.298_{10} = 0.0100110001001001101110\ldots_2$
- It can be shown that we need $\left\lceil -\log_2 p \right\rceil + 1$ bit to represent the exact message, where $p$ is the boundary size, in our case: $p = 0.0008$, so we need 12 bits
- Since the two zeros at the end mean nothing we can use a shorter string: $0.0100110001$
Decoding Procedure

We just need to “open” the boundary where 0.298 is in, the same way we encoded it, letter by letter.
Mathematical Formulation

• Through every step we are inside the boundary \([low, high)\)
• We begin in the area \([0, 1)\) so we initialize with \(low = 0, high = 1\)
• Every step, we get \(x\), and need to calculate the
• next boundary limits \(low', high'\)
• \(F(x)\) is the accumulative distribution function

\[
low' = low + (high - low) \cdot F(x - 1)
\]
\[
high' = low + (high - low) \cdot F(x)
\]
Pro’s

• **Adaptive** : optional, even “on-the-fly”, using symbols probability or other mathematical model
  – No need to send side information!
• The coding algorithm is **not depended on** the probability (or other) model
• Very efficient …
And Con’s …

- Computational **complexity**: need floating point arithmetic
- Sensitive to **BER**
- Decoding can’t be **parallelized**
- **Not progressive**: no output until the whole message is coded
- Message length or “**end-of-message**” sign are needed
Advanced Algorithms

• **Progressive**: we want to have the output start before the “end-of-message”
  – There are coding/decoding “predictive” algorithms

• **Fixed point arithmetic**
  – There are fixed point and table based methods
An excellent reference...

• By Ilan Sutskover,
can be found in the course web site