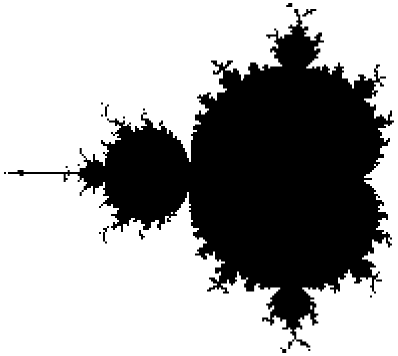
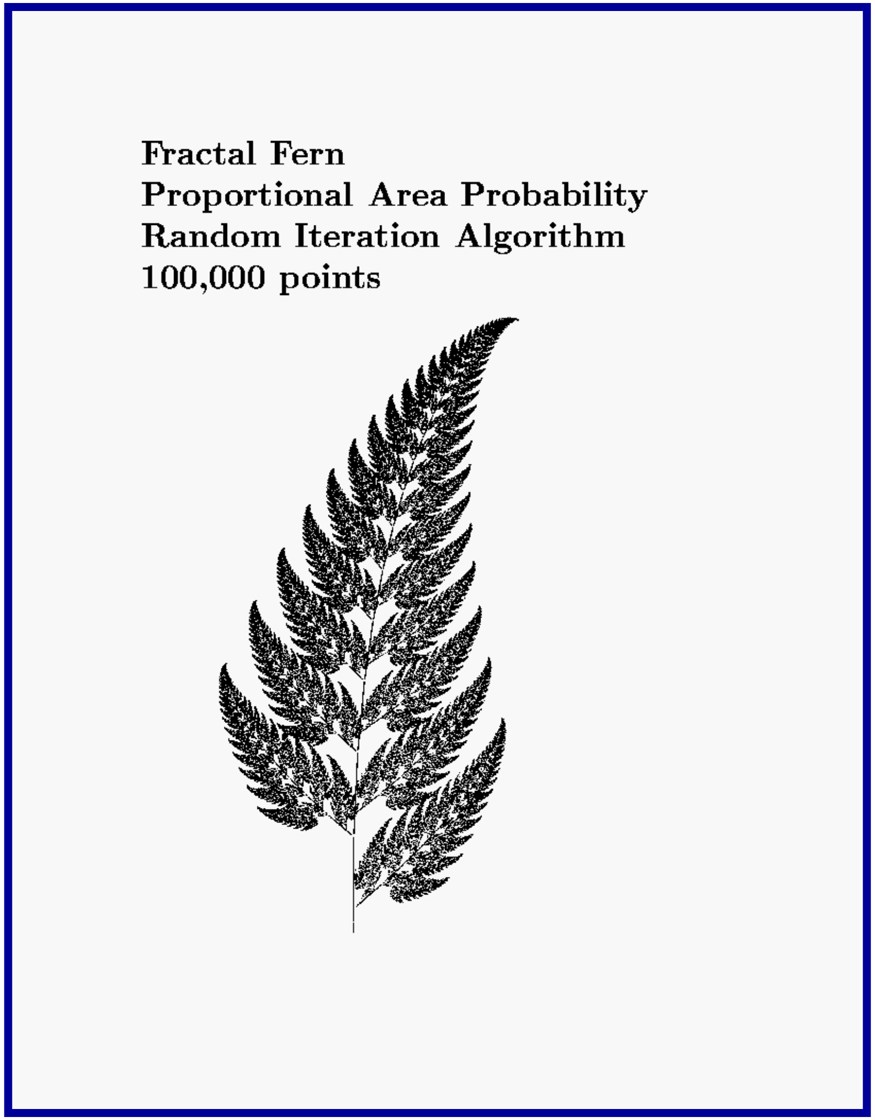
Fractal Image Coding (IFS)





Nimrod Peleg
Update: Mar. 2008

What is a fractal ?

- A fractal is a geometric figure, often characterized as being “self-similar”: irregular, fractured, fragmented, or loosely connected in appearance.
- Benoit Mandelbrot coined the term fractal to describe such figures, deriving the word from the Latin "fractus": broken, fragmented, or irregular.

Why Fractals ?

- Fractals seem to provide an excellent description of many natural shapes
- Euclidean geometry provides concise accurate descriptions of man-made objects.

A coastline:
No characteristic sizes,
Hence a fractal

Euclidean Vs. Fractals

- Euclidean shapes have one, or several, characteristic sizes (the radius of a sphere, the side of a cube)
- fractals possess no characteristic sizes: the most important difference is that fractal shapes demonstrate self similarity of metric properties, up to being <u>independent of scale or scaling.</u>

Self Simliarity

- defined by the relation:

$$M(r \cdot x) = r^{f(D)} M(x)$$

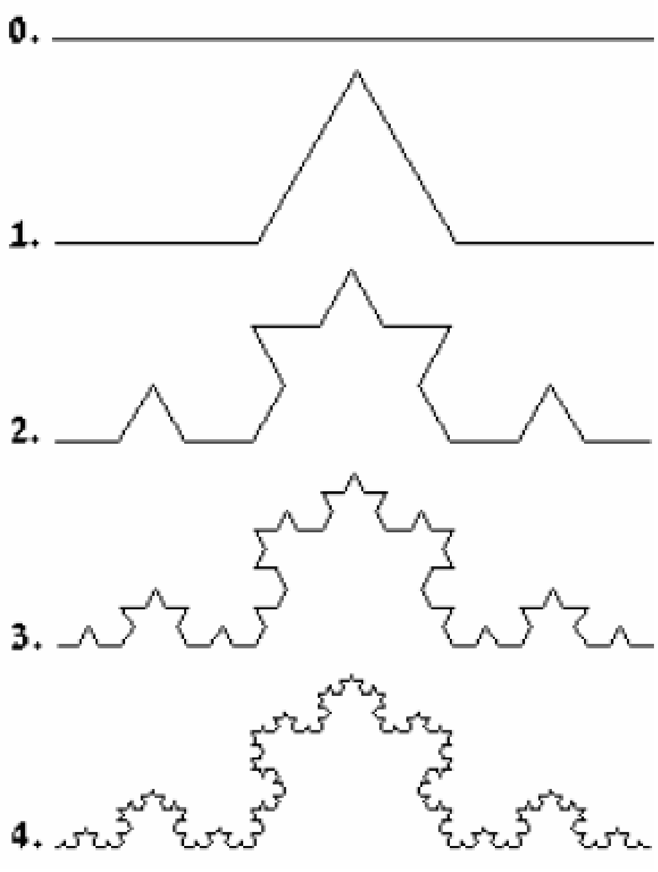
- M(x) represents any metric property of the fractal (e.g. area or length),
- *x* denotes the scale of measurement of the metric property,
- ***r*** is a scaling factor, such that $0 \leq r \leq 1$
- *f(D)* is a function of the fractal dimension D for the given metric property.

Where did it start…?
The Von-Koch curve

- The von Koch snowflake is a famous fractal curve that was first proposed in 1904 by the Swedish mathematician Helge von Koch.
- Starting with a line segment, the curve is produced by recursive steps
- The single line segment, shown in Step 0, is broken into four equal-length segments, as seen in Step 1.

The Von-Koch curve (2)

- The same "rule" is applied an infinite number of times, resulting in a figure with an infinite perimeter.



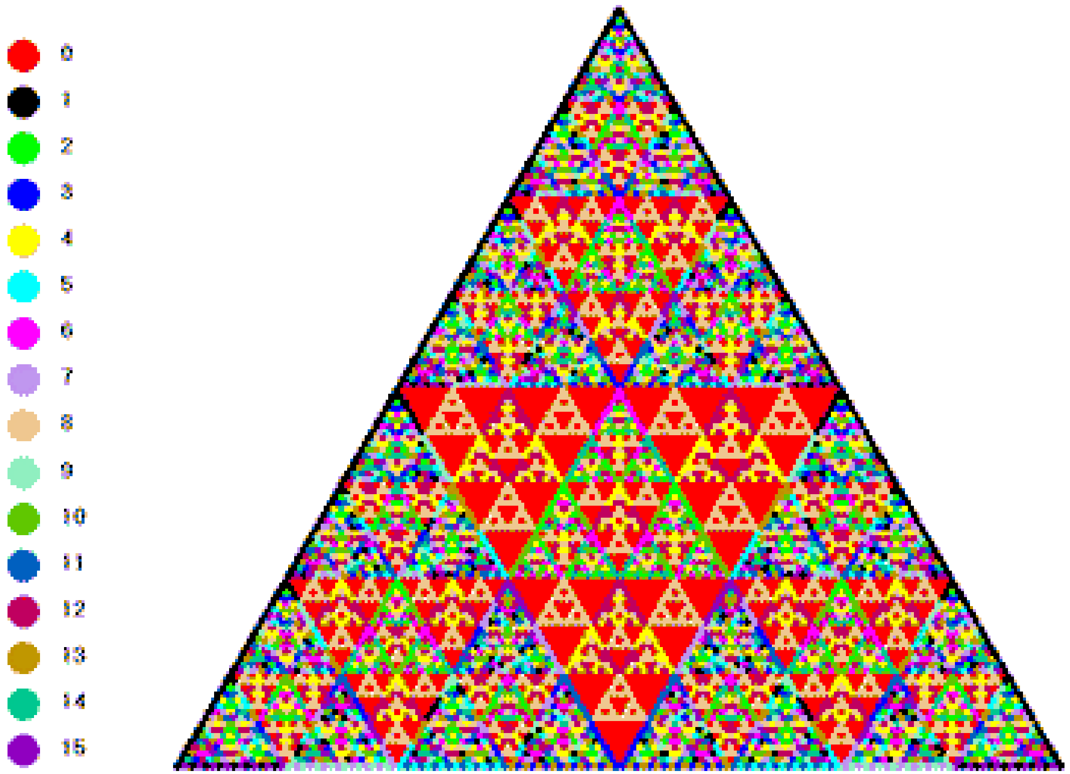
Analysis of *The* *Von-Koch curve (3)*

- Suppose the original line segment was of length L, after the first step each line segment is of length L/3.
- For the second step, each segment has a length L/32, and so on. After the first step, the total length of the curve is 4L/3.
- Following the second step, the total length is 42L/32.
- Finally, the length of the curve will be 4kL/3k after the k^{th} step. The length of the curve grows by a factor of 4/3 after each step !

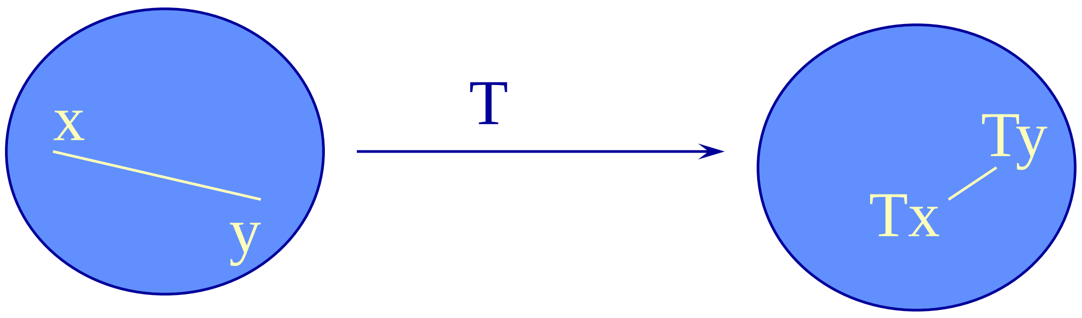
And after infinite number of iterations…

- When repeated an infinite number of times, the perimeter becomes infinite…
- a repetition of a very simple rule ca produce seemingly complex shapes with some highly unusual properties. Unlike Euclidean shapes, this curve has detail on all length scales - the closer one looks, the more detail one finds.
- Of greater importance is the self-similarity quality, which the curve possesses. Each small portion, when magnified, can exactly reproduce a larger portion. The curve is said to be invariant under changes of scale.

Generating a Fractal…



And now…Contractive Transform



- There ***exists*** a ***unique*** Fixed-Point x^f :

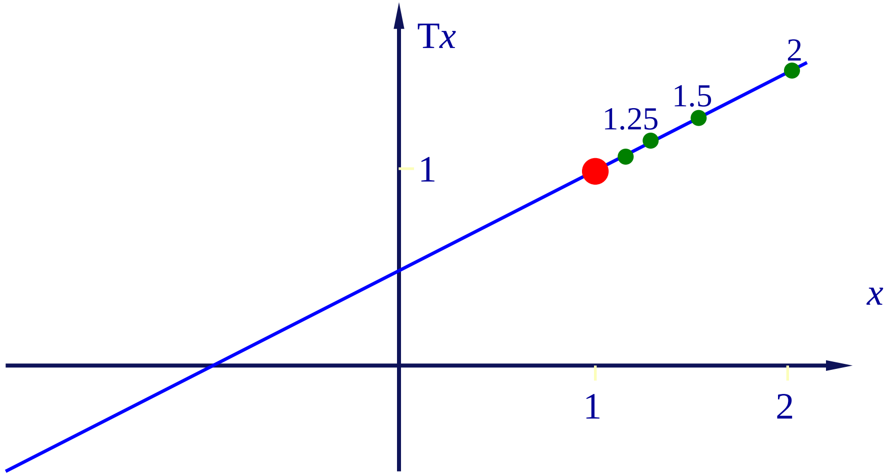
$T\{x^f\} = x^f$

E.g. : $\mathrm{T}\{\mathrm{x}\} = \frac{\mathrm{x}}{2}$ ⇨ **X=0** is the fixed point of the transformation T{x}

And one more

1D Affine contraction and iterative approximation of the fixed point

$$Tx = 0.5x + 0.5$$



Fixed point: $x_* = 1$

Contractivity: $\alpha = 0.5$

Let

$(\mathbf{M}, d)$ be a complete metric space

$T:\mathbf{M} \to \mathbf{M}$ be a **contractive mapping**

Then

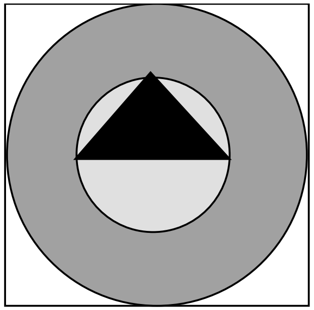
T has one and only one **fixed point**

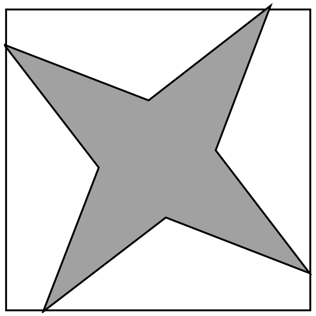
$$x_* = Tx_*$$

The fixed point can be found by repeated application of T on an arbitrary vector x_0, having:

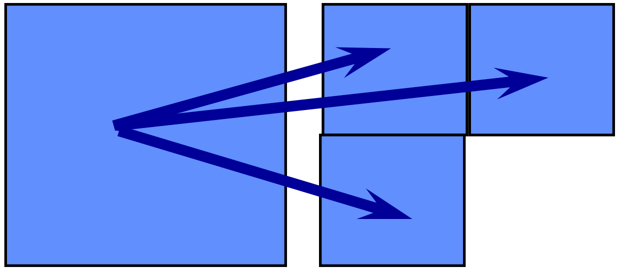
$$\lim_{n\to\infty} T^n x_0 = x_*$$

How to generate a Fractal ?



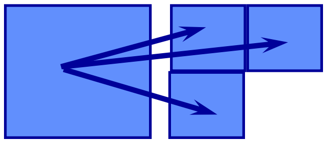


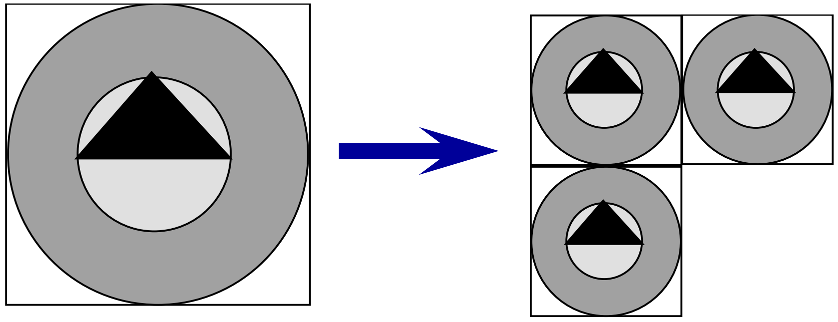
Basic "start" point

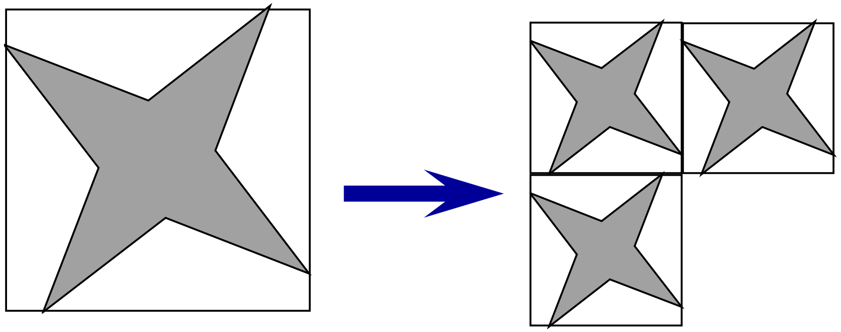


Contracting transformation by
Iterated Function System (IFS)

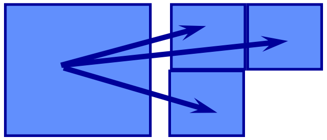
Fractal Generation (Cont'd)

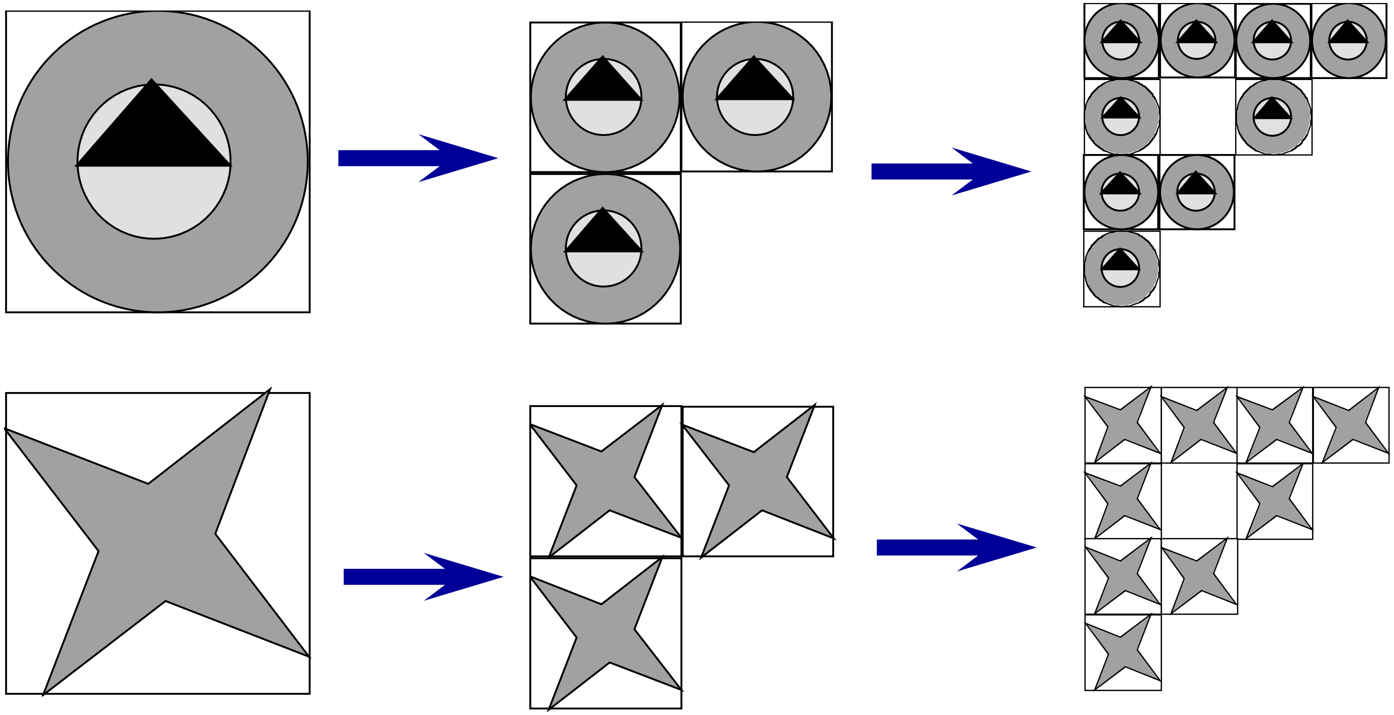




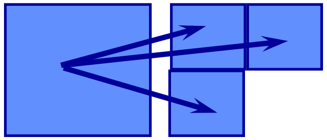


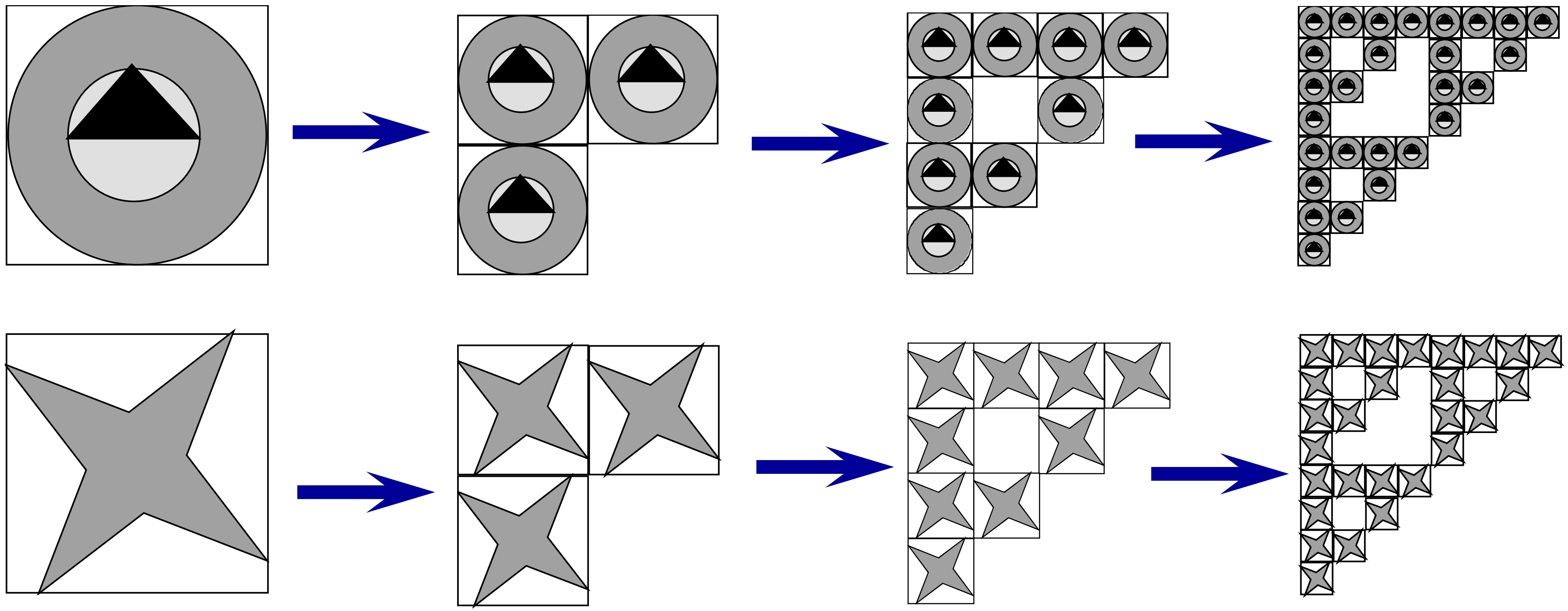
Fractal Generation (Cont'd)



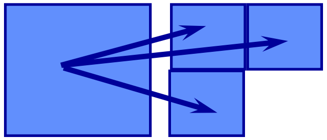


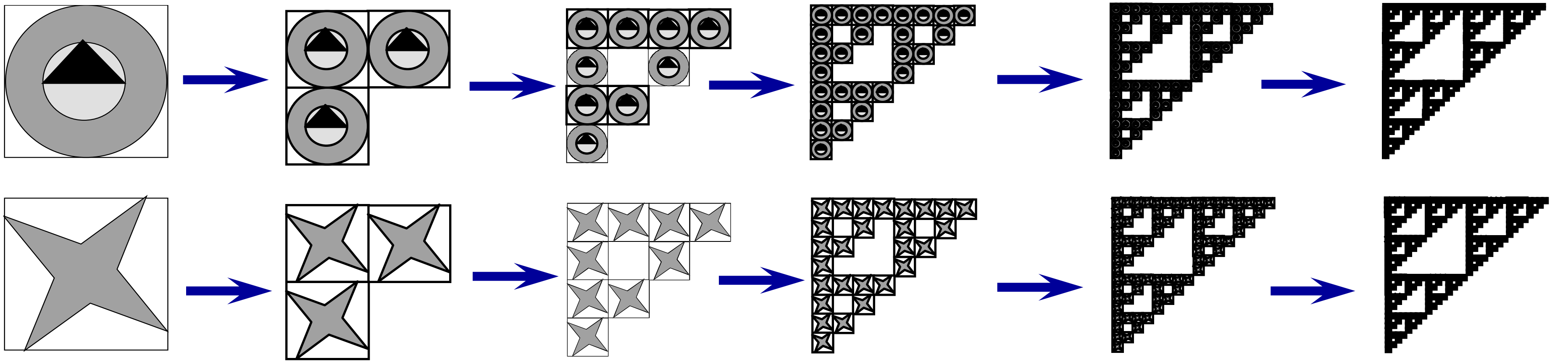
Fractal Generation (Cont'd)





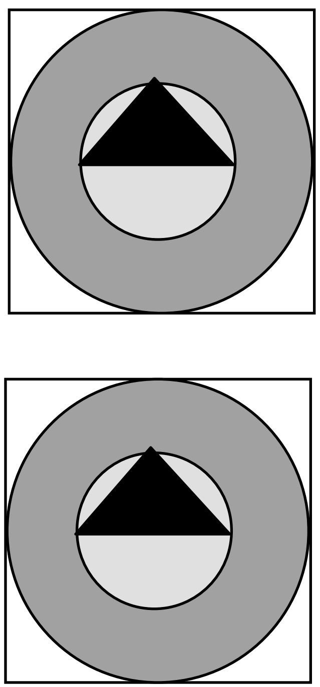
Fractal Generation (Cont'd)



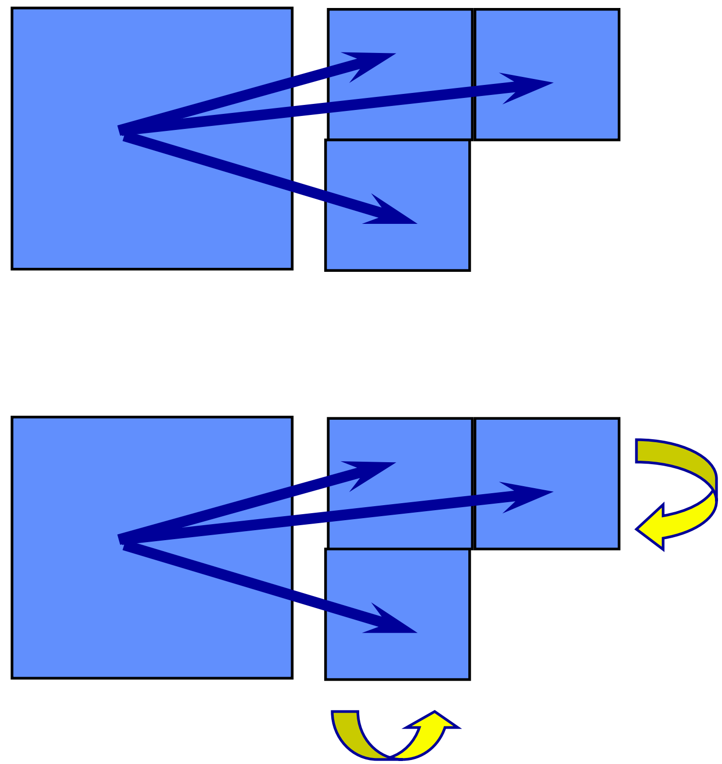


Sierpinski triangle

Another example

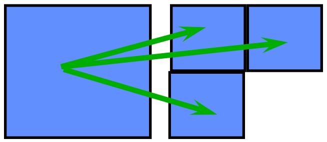


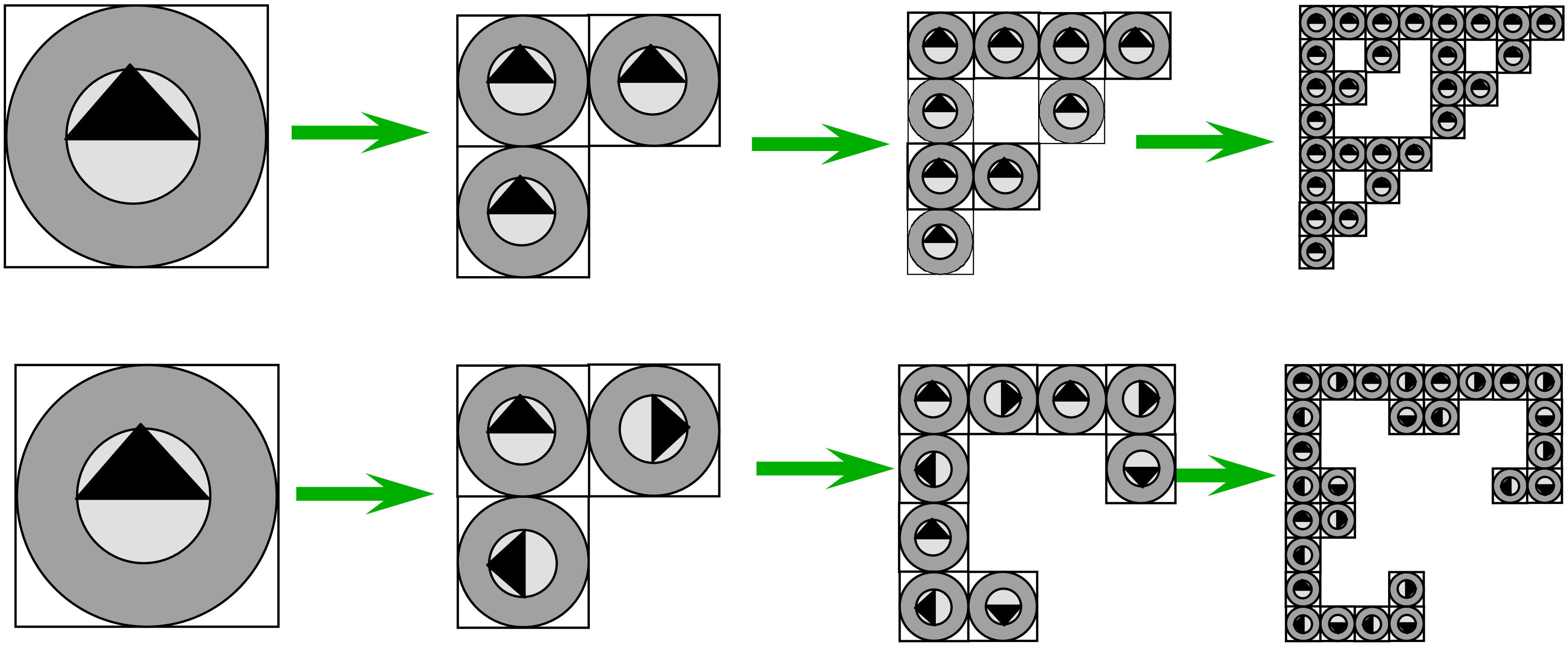
Basic “start” point



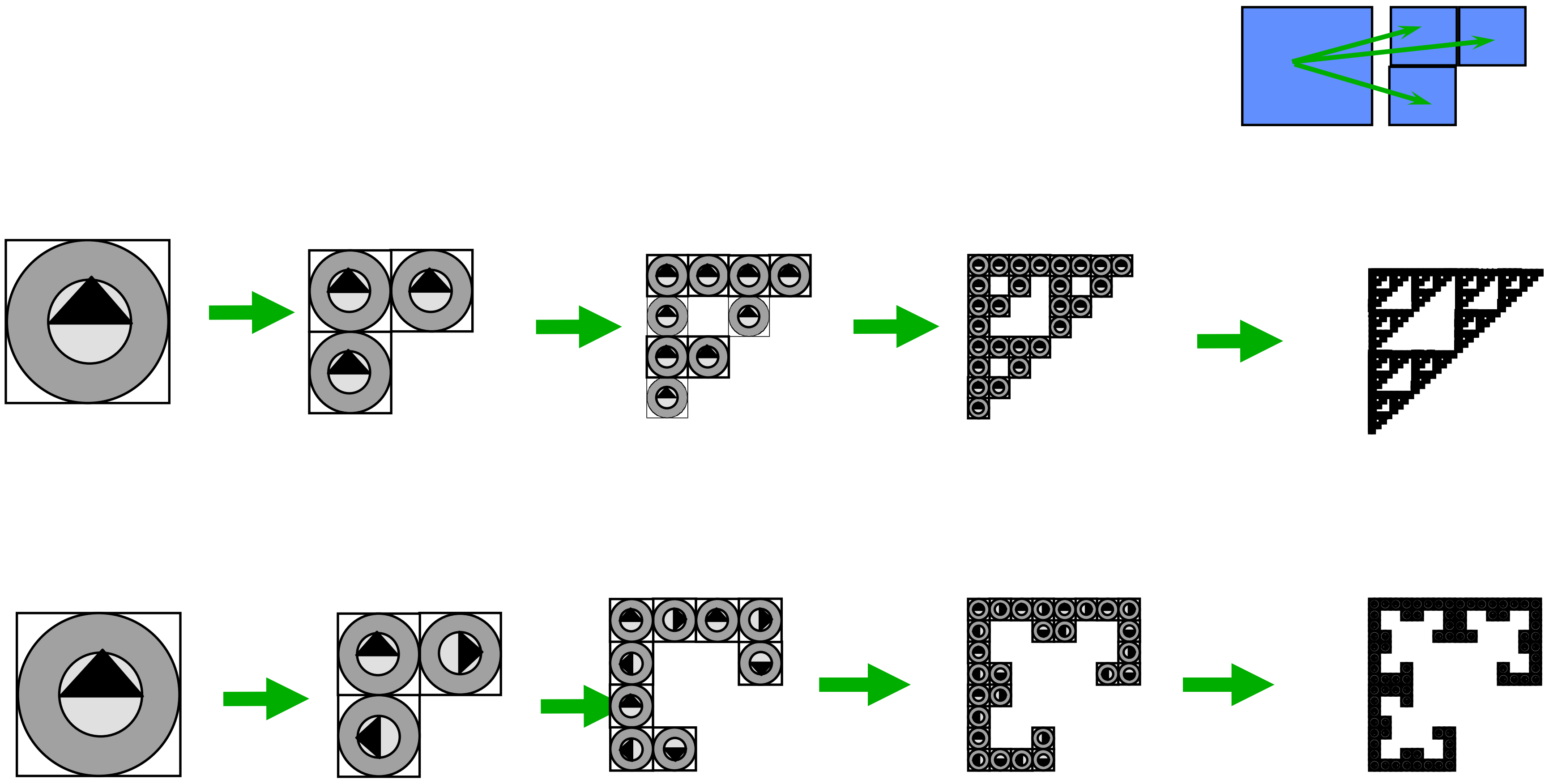
Contracting transformation by
Iterated Function System (IFS)

2nd Fractal Generation (Cont'd)

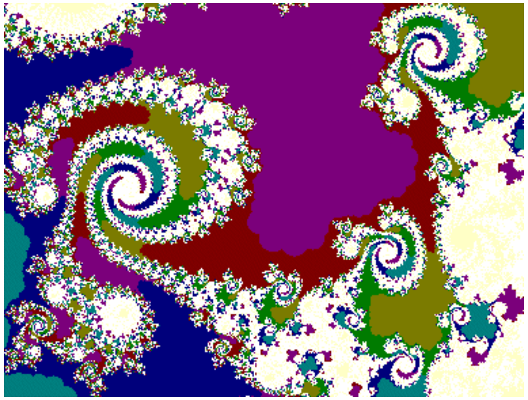


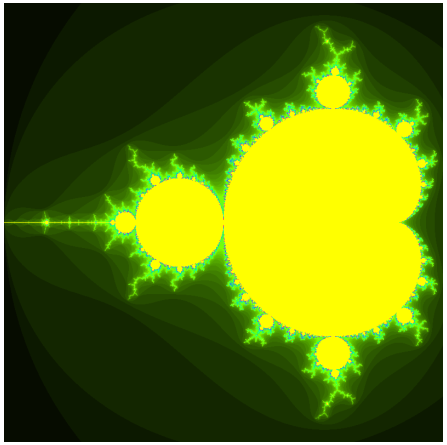


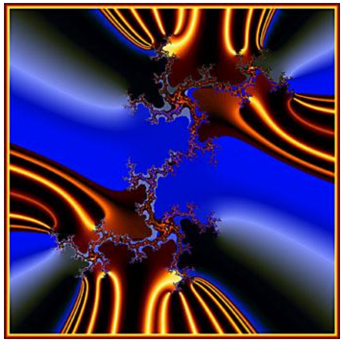
2nd Fractal Generation (Cont'd)

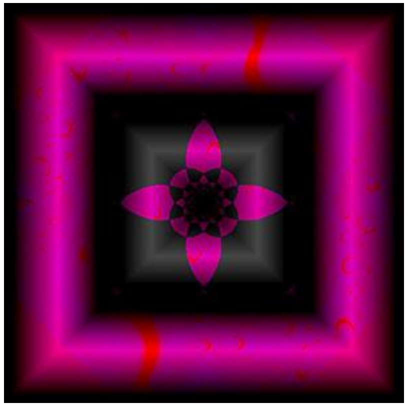


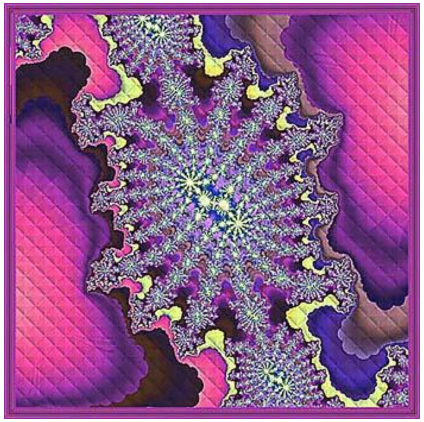
Some Nice Fractals



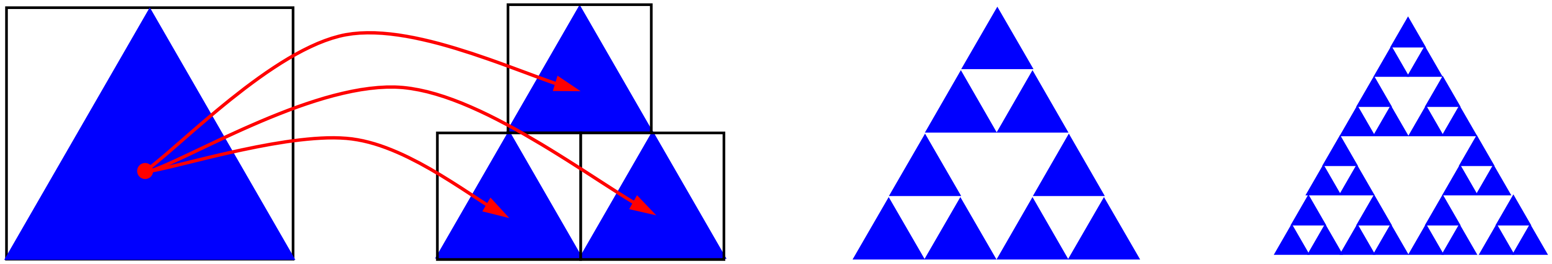




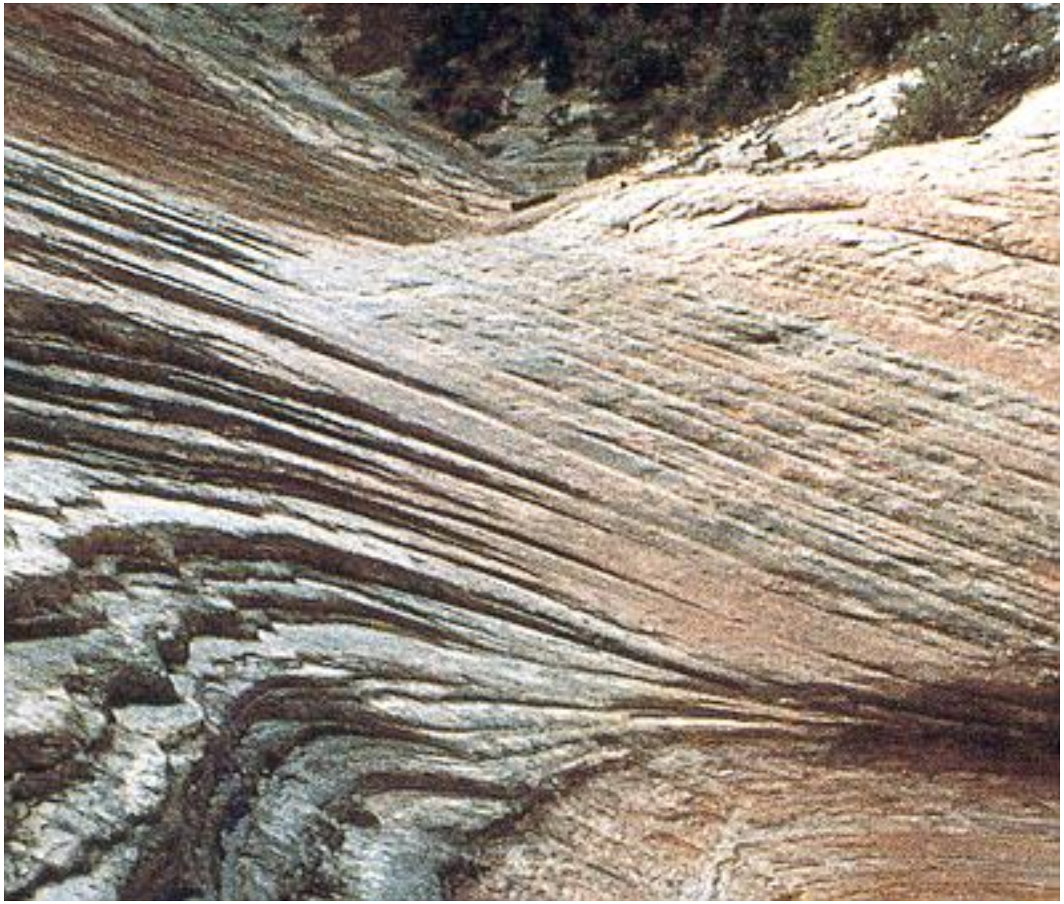




And the Sierpinski fractal



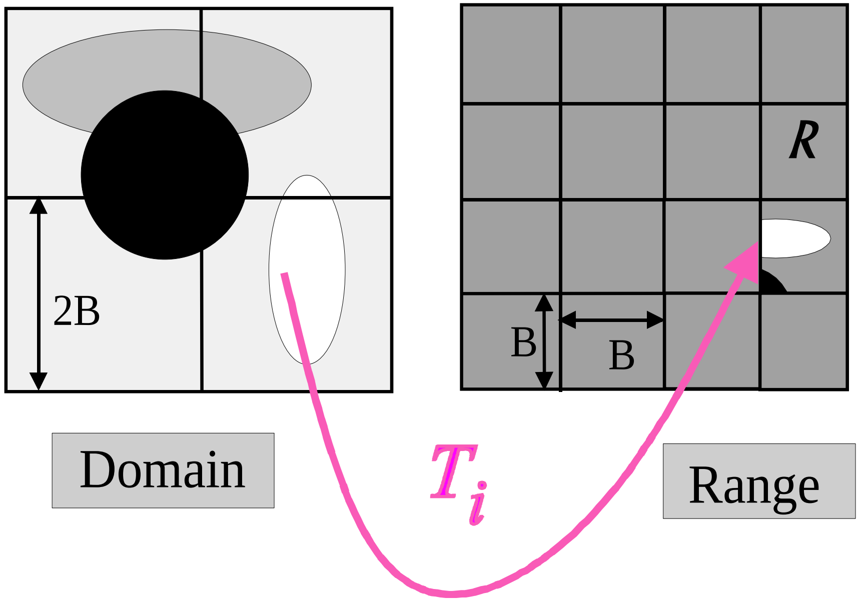
A Natural “Fractal”



Fractal Image Coding [Jacquin 1989]

- Find a Contractive T such that

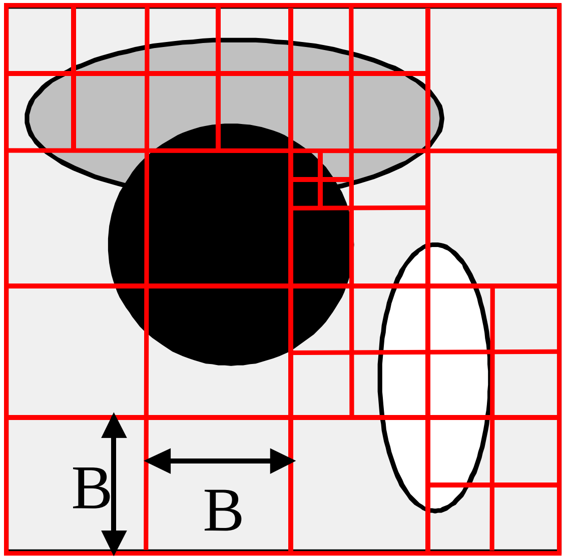
$$\chi^{f} \approx \chi$$



Quad-Tree Approach

- range blocks of different sizes

[*Fisher* 92]



Splitting criteria is a key issue, of course...

Comparison of Splitting Criteria

compression ratio ≃ 8:1

Threshold- based criterion

PSNR ≃ 35.9 [dB]

Rate-Distortion - based criterion

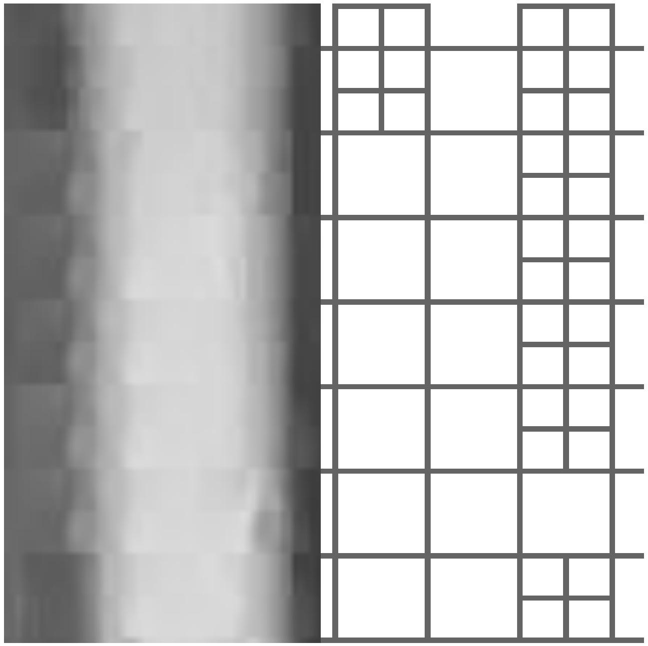
PSNR ≃ 36.9 [dB]

Range block sizes = 16x16, 8x8, 4x4, 2x2

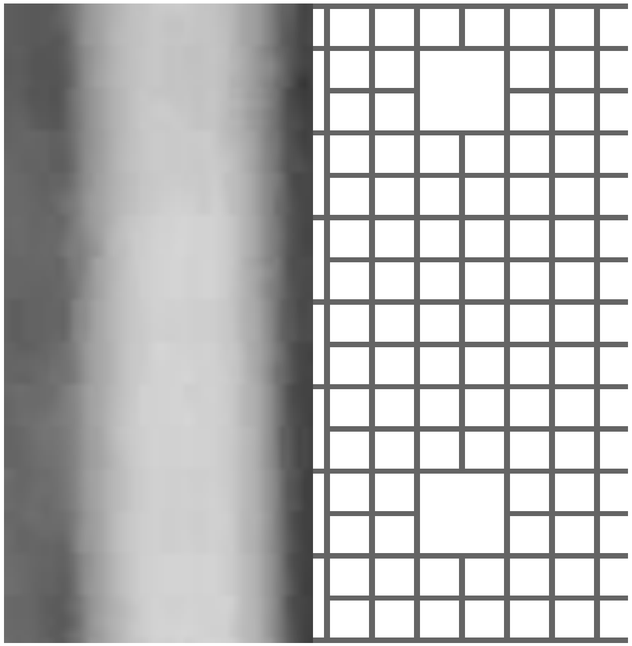
Comparison of Splitting Criteria (cont'd)

compression ratio ≃ 1:8

Threshold- based criterion



Rate-Distortion - based criterion



PSNR ≃ 35.9 [dB]

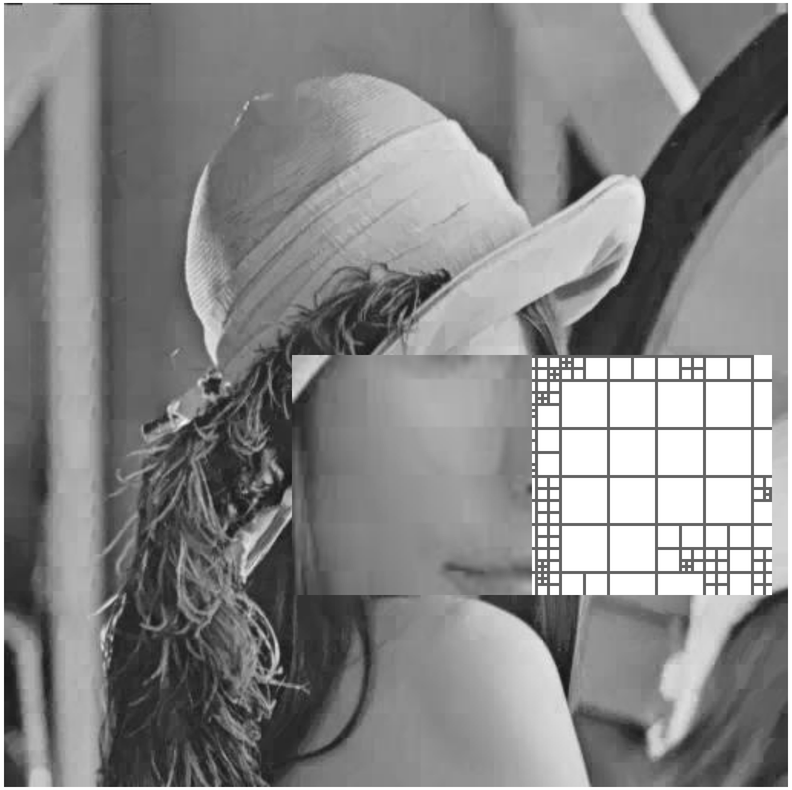
PSNR ≃ 36.9 [dB]

Range block sizes = 16x16, 8x8, 4x4, 2x2

Comparison of Splitting Criteria (cont'd)

compression ratio ≃ 1:8

Threshold- based criterion



PSNR ≃ 35.9 [dB]

Rate-Distortion - based criterion

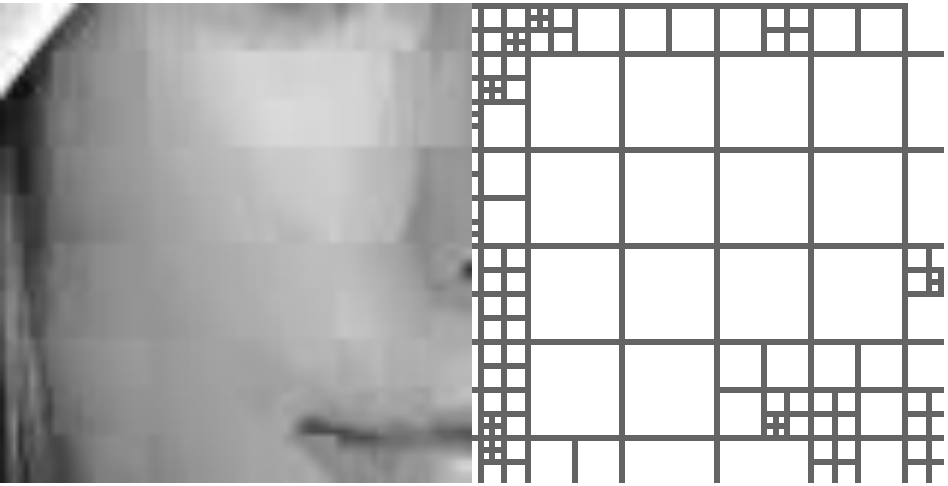
PSNR ≃ 36.9 [dB]

Range block sizes = 16x16, 8x8, 4x4, 2x2

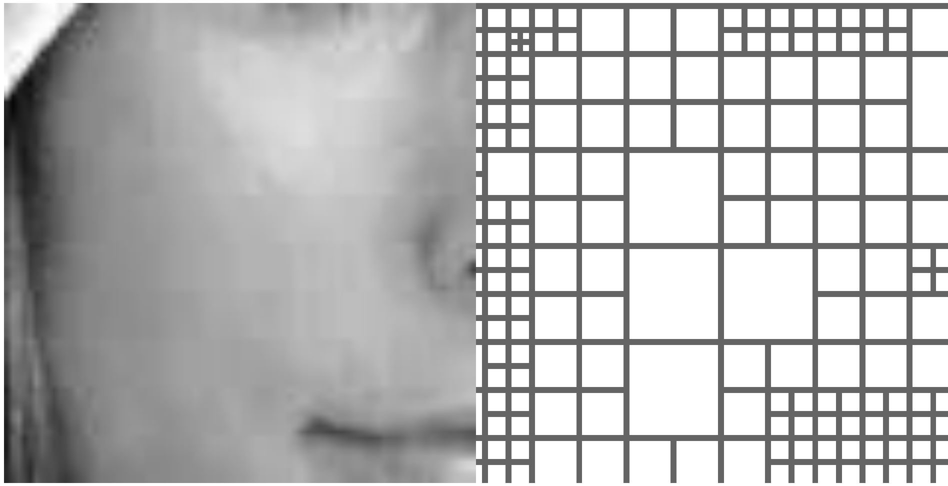
Comparison of Splitting Criteria (cont'd)

compression ratio ≃ 1:8

Threshold- based criterion



Rate-Distortion - based criterion



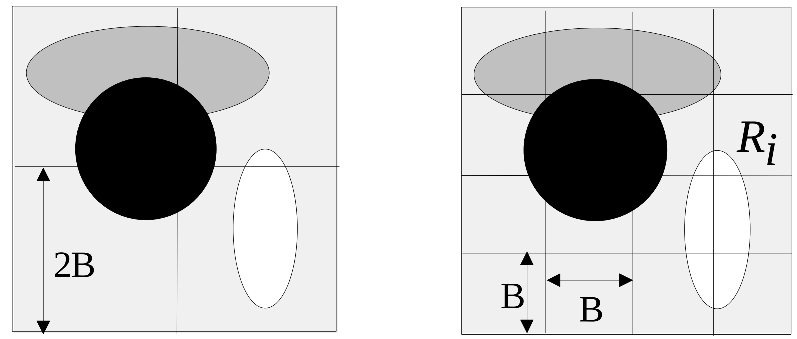
PSNR ≃ 35.9 [dB]

PSNR ≃ 36.9 [dB]

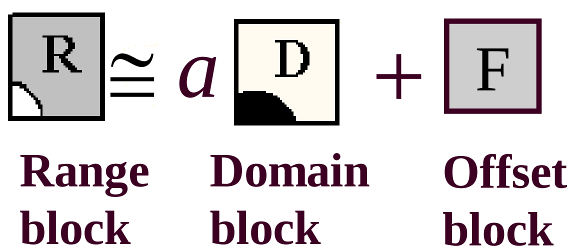
Range block sizes = 16x16, 8x8, 4x4, 2x2

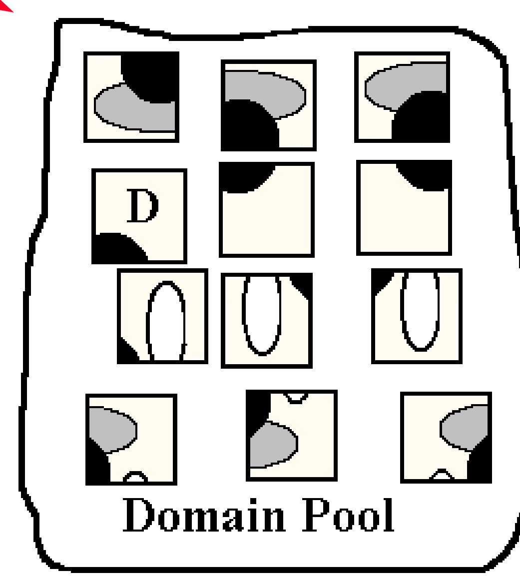
Conventional Image Fractal Coder

- Extract *domain pool*



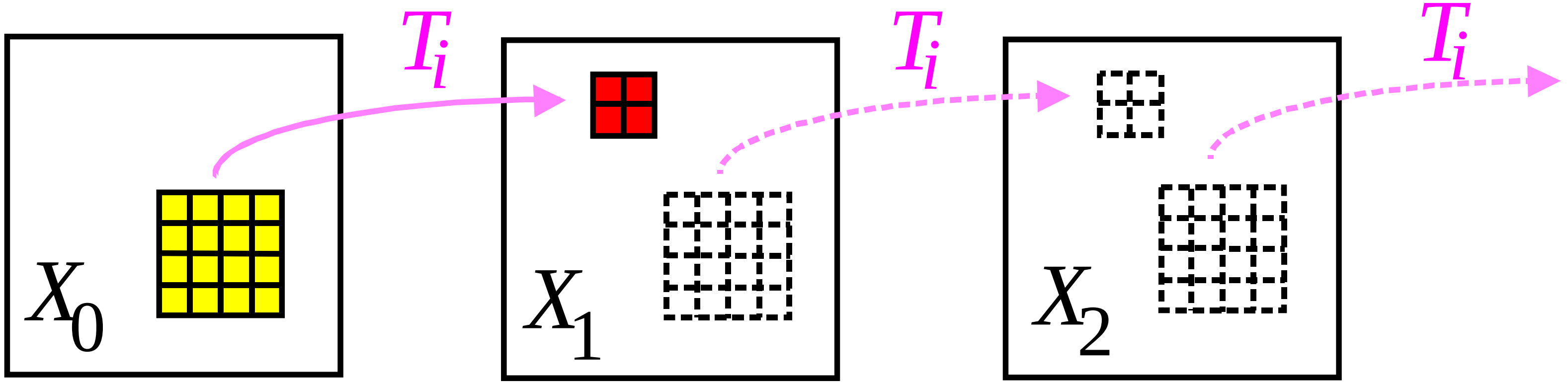
- For each R_i :





Finding the Fixed Point (Decoding)

- Start with any image X_0 .
- Apply T iteratively until the result converges



Example

T is found to represent "Lena"

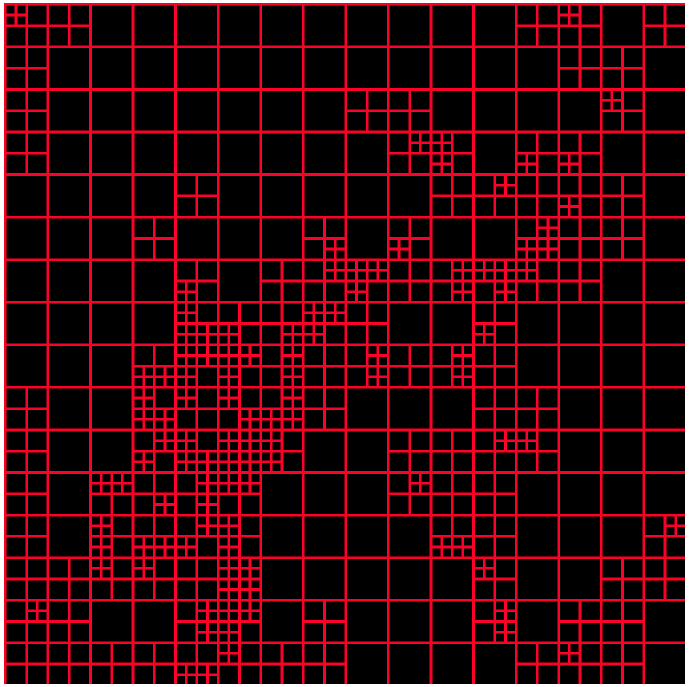
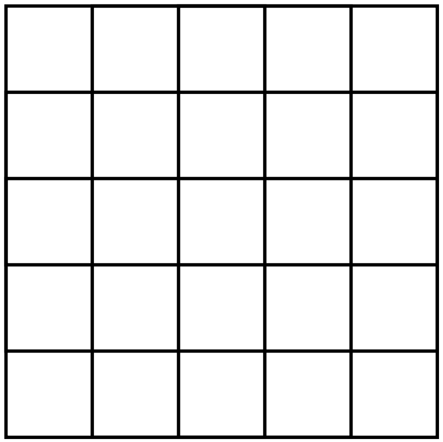
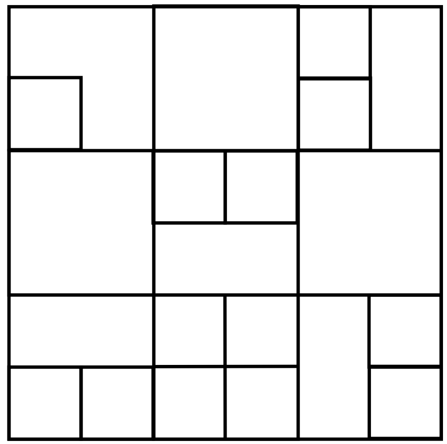


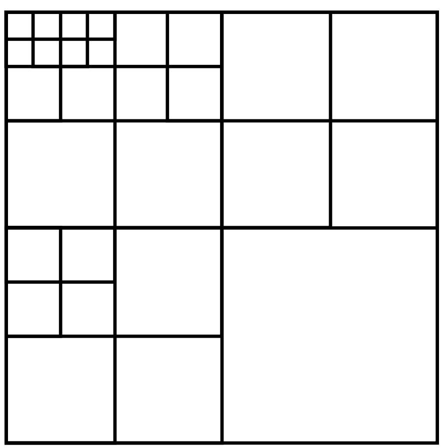
Image partition into range blocks



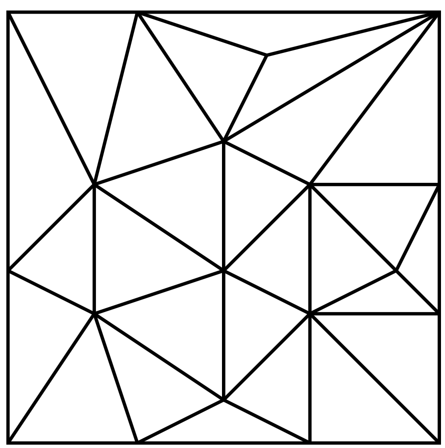
Fixed size blocks



Jacquin's 2-level partition



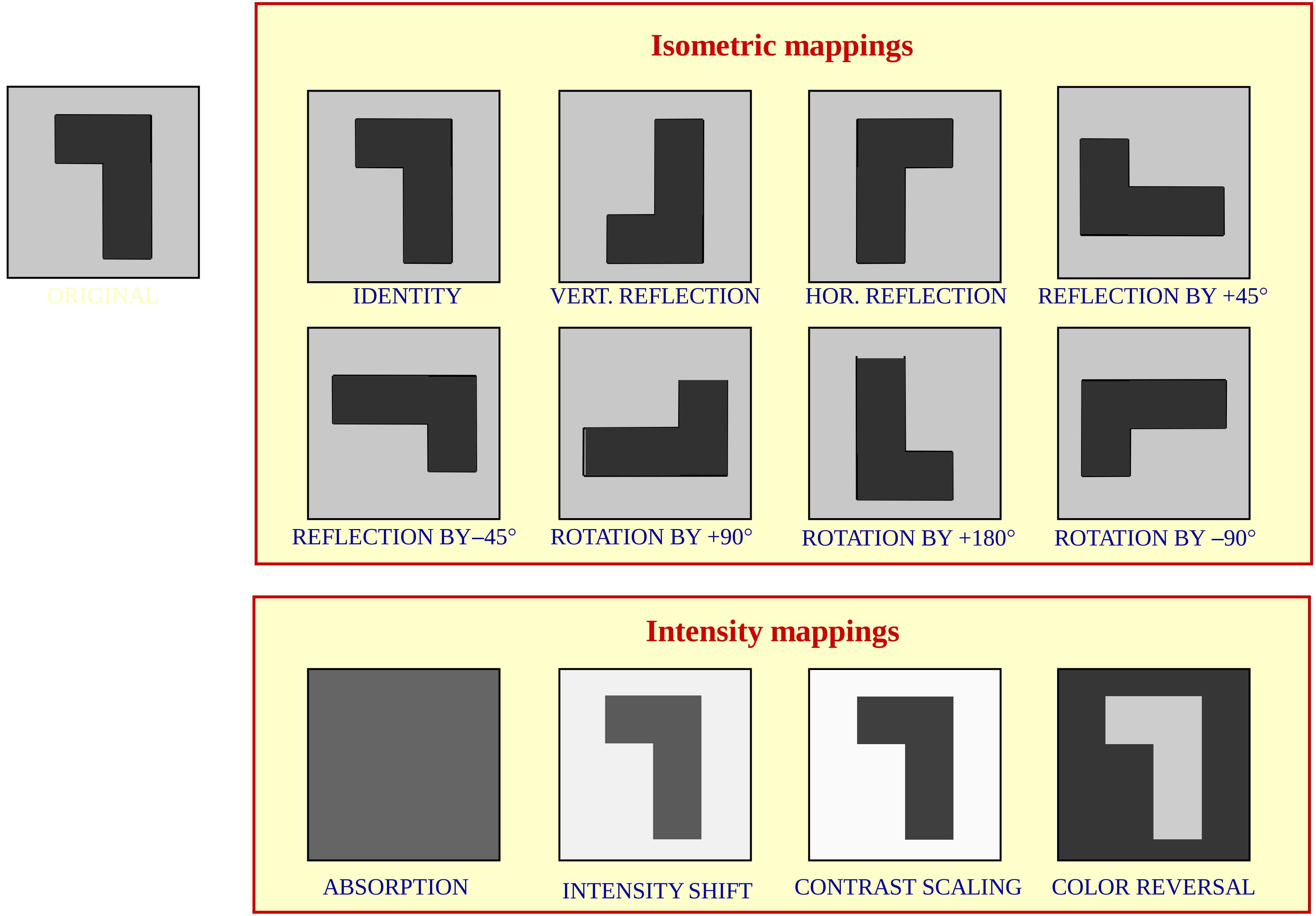
Quadtree



Delaunay triangulation

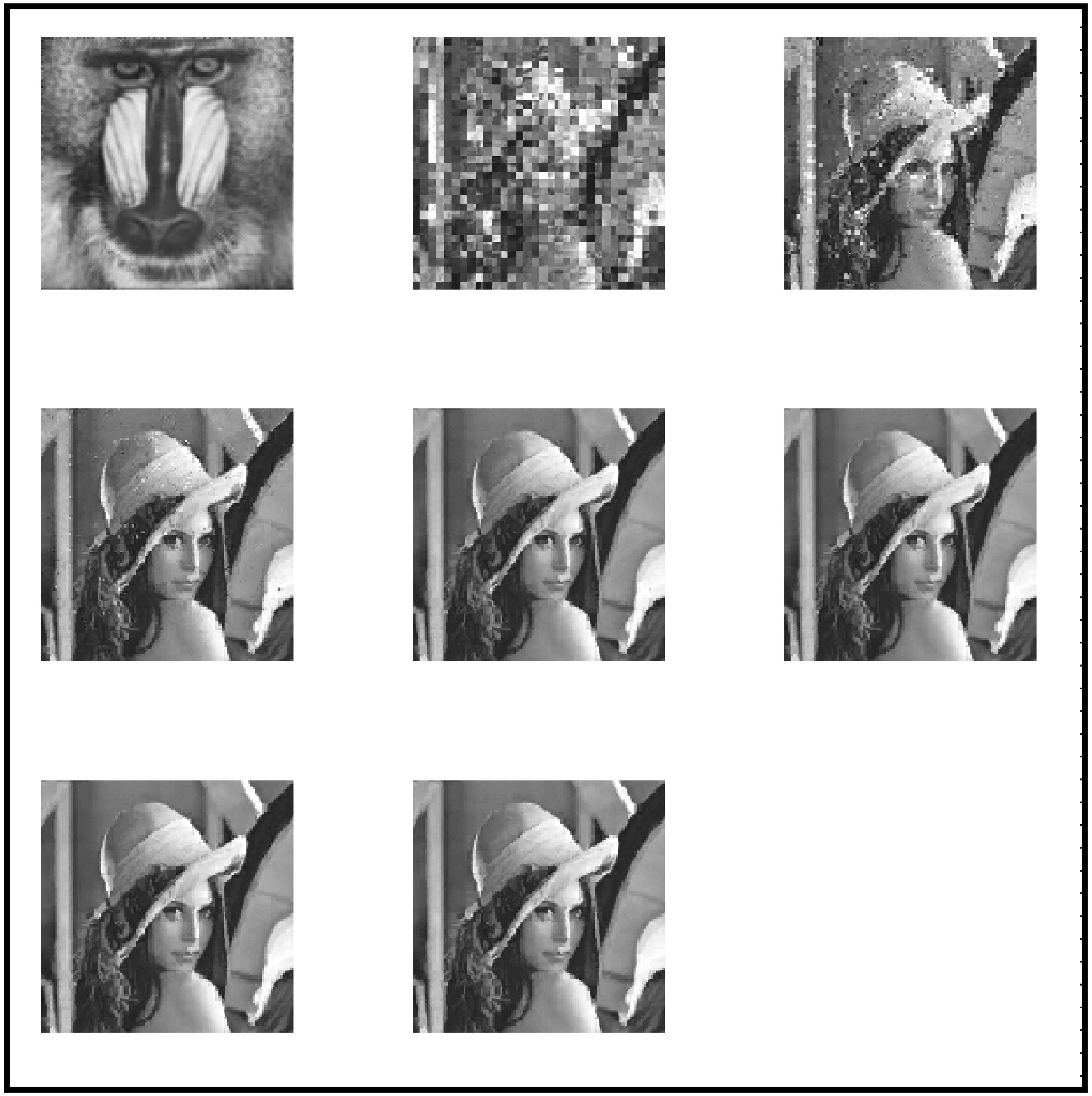
- Partition description is a part of the fractal code
- Simplest partition – non-overlapping fixed-size square *B*x*B* blocks
- Jacquin's 2-level partition – particular case of the Quadtree partition
- Other "exotic" partitions schemes, e.g. Delaunay triangulation

Possible mappings



Decoding: Starting from 'man'

Decoding: Starting from 'Baboon'



IFS Vs. JPEG

1: FIF Image , 26169B, Fractal Imager 1.1
(lena.fif)

2: JPEG Image, 26478B, Lview 3.1, (Q=64)
(lena.jpg)

3: JPEG Image, 17924B, Fractal Imager 1.1
(lenait.jpg)

4: JPEG Image, 17409B, Lview 3.1, (Q=37)
(lenaeq.jpg)

FIF Image , 26169B, Fractal Imager 1.1

JPEG Image, 26478B, Lview 3.1, (Q=64)

JPEG Image, 17924B, Fractal Imager 1.1

JPEG Image, 17409B, Lview 3.1, (Q=37)

FIF Image , 28:1 compression

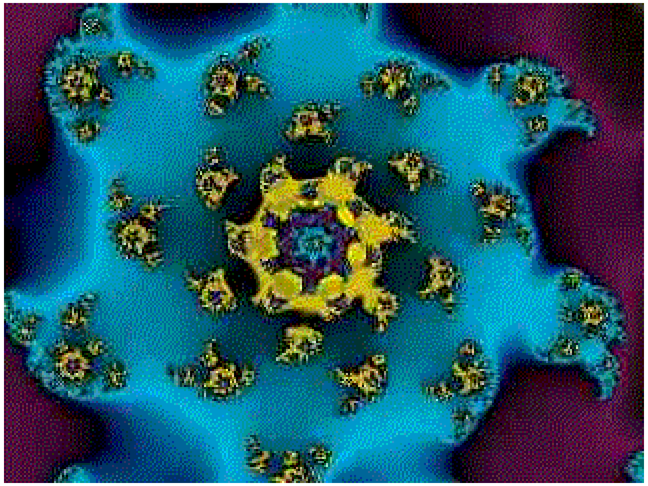
Pro's and Con's

- **Fast decoding**
- **Iterative** (progressive) decoding – the number of iterations dictates the image quality
- Compressed images are **visually more plausible** ("look better") compared to JPEG
- **Resolution independence** – can achieve very high "effective" compression ratios up to ~1000:1

- **Slow encoding**
- Decoding **dependent** on initialization
- Not in public domain (**patented**)

Thanks to...

Hagai Krupnik and Reuven Franco for some of the slides, taken from his graduate seminar for M.Sc . And Michael Bronstein.



References

- Barnsley, M. F., Jacquin, A. E., Application of recurrent iterated function systems to images, *Proc. SPIE Visual Comm. and Image Processing* (1988)
- Hutchinson, J., Fractals and self-similarity, *Indiana Univ. J. Math.*, Vol. 30, pp. 713-747 (1981).
- Jacquin, A. E., Fractal image coding: a review, *Proc. IEEE*, Vol. 81, No. 10, pp. 1451-1465 (1993)
- Jacquin, A. E., Image coding based on a fractal theory of iterated contractive image transformations, *IEEE Trans. Image Processing*, Vol. 1, No. 1, pp. 18-30 (1992)
- Mandelbrot, B., The fractal geometry of nature, *Freeman* (1977).
- Saupe, D., Hamazaoui, R., Hartenstein, H., Fractal image compression: an introductory overview, Fractal Models for Image Synthesis, Encoding, and Analysis, *SIGGRAPH '96 Course Notes* XX, New Orleans (1996)
- Turner, M. J., Blackledge, J. M., Andrews, P. R., Fractal geometry in digital imaging, *Academic Press* (1998).
- Wohlberg, B., de Jager, G., A review of fractal image coding literature, *IEEE Trans. Image Processing*, Vol. 8, No. 12, pp. 1716-1729 (1999)

Defining the Fractal Dimension

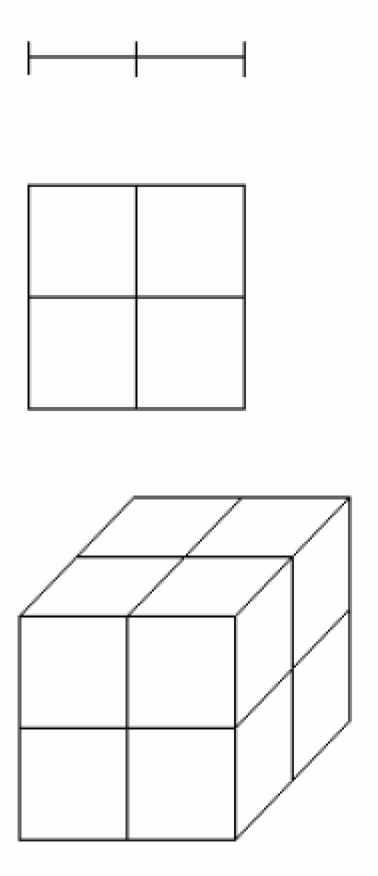
- The property of self-similarity or scaling, as in the Von Koch curve, is one of the central concepts of fractal geometry. It is closely connected with our intuitive notion of dimension.
- An object normally considered as one-dimensional, a line segment, for example, also posses a similar scaling property.
- It can be divided into N identical parts each of which is scaled down by the ratio $r = 1/\sqrt{N}$ from the whole.

The fractal dimension - cont'd

- A line can be divided into 2 lines each twice smaller than the original. Likewise, two dimensional objects, such as a square area in the plane, can be divided into N self-similar parts, each of which is scaled down by a factor of

$$r = 1 \; N.$$

Shapes of different Dimension



Scaling Down

- As demonstrated, a 2 dimensional object can be divided into N=4 equal parts, when each is a scaled down version of the original (by a factor of 2).
- A three-dimensional object like a solid cube may be divided into N little cubes, each is scaled down by a ratio: $r = \sqrt[3]{N}$
- A cube in 3D can be divided into N=8 scaled down versions of the original, again with a scaling factor of 2.

The general relation: N Vs. D

- These examples demonstrate the general relation between N and D, given by:

$$N \cdot r^{D} = 1$$

- E.g., one can easily verify that the relation holds for {N=4, r=0.5 D=2} which describe the 2D example by substituting the constants into the last equation:

$$4 \cdot \left(\frac{1}{2}\right)^{2} = 1$$

Self similarity & Fractal dimension

- With self-similarity, the generalization to fractal dimension is rather straightforward.

- A D-dimensional self-similar object can be divided into N smaller copies of itself each is scaled down by a factor of: $r = 1 / \sqrt[D]{N}$

The fractal dimension

- Thus, the fractal dimension of a self-similar object of N parts, scaled by a ratio r from the whole, is given by: $D = \frac{\log(N)}{\log(1/r)}$

- Unlike the more familiar notion of Euclidean dimension, the fractal dimension is not necessarily an integer.

Example: the Von Koch case

- Every segment is composed of four sub-segments, each is scaled down by a factor of 1/3 from its parent, as seen before.
- Substituting this into the last equation, we get: $$D = \frac{\log(N)}{\log(1/r)} = \frac{\log(4)}{\log(3)} \approx 1.26$$
- This non-integer dimension, greater than one but less than two, reflects the unusual properties of the curve.

References

- Wen-Shiung Chen, Shang-Yuan Yuan and Chin-Ming Hsieh, Two algorithms to estimates fractal dimension of gray-level images, Optical Engineering, 2003.

- Amar Ait-Kheddache, Classification of Textures Using Highter-Order Fractal Dimensions, Int'l conf. on Industrial and engineering applications of artificial intelligence and expert systems, 1988.

- Susan S. Chen, James M. Keller and Richard M. Crownover, On the calculation of fractal features from images, IEEE trans. on pattern analysis and machine intelligence, October 1993.

- Szu-Chu Liu and Shyang Chang, "Dimension Estimation of Discrete-Time Fractional Brownian motion with Applications to Image Texture Classification, IEEE trans. on image processing, August 1997.