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Δ_2^0 OPERATORS AND ALTERNATING SENTENCES IN ARITHMETIC

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§0. **Introduction.** Determining the truth value of self-referential sentences is an interesting and often tricky problem. The Gödel sentence, asserting its own unprovability in $\boldsymbol{P}$ (Peano arithmetic), is clearly true in N (the standard model of $\boldsymbol{P}$), and Löb showed that a sentence asserting its own provability in $\boldsymbol{P}$ is also true in N (see Smorynski [Sm, 4.1.1]). The problem is more difficult, and still unsolved, for sentences of the kind constructed by Kreisel [Kl], which assert their own falsity in some model N^* of $\boldsymbol{P}$ whose complete diagram is arithmetically defined. Such a sentence χ has the property that $N \models \chi$ iff $N^* \not\models \chi$ (note that $\neg\chi$ has the same property).

We show in §1 that the truth value in N of such a sentence χ, after a certain normalization that breaks the symmetry between it and its negation, is determined by the parity of a natural number, called the rank of N, for the particular construction of N^* used. The rank is the number of times the construction can be iterated starting from N and is finite for all the usual constructions. We also show that modifications of, e.g., Henkin's construction (in his completeness proof of predicate calculus) allow arbitrary finite values for the rank of N. Thus, on the one hand the truth value of χ in N, for a given "nice" construction of N^*, is independent of the particular (normalized) choice of χ, and we shall see that χ is unique up to (provable) equivalence in $\boldsymbol{P}$. On the other hand, the truth value in question is sensitive to minor changes in the definition of N^* and its determination seems to be largely a combinatorial problem.

Kreisel [K2, footnote 43, pp. 382–383] observed the constructions of models of a certain type cannot be iterated indefinitely (hence the ranks referred to above are finite numbers). He then used this observation to give a model-theoretic proof of Gödel's second incompleteness theorem (see [Sm, 6.2.4]). In §2 we generalize the observation to all operators satisfying a recursion theoretic condition, namely the bounded Δ_2^0 operators. These operators are well behaved in other ways too. Each of them induces tree structures on models M of $\boldsymbol{P}$, their theories $\mathrm{Th}(M)$ and their universal theories $\mathrm{Th}_\forall(M)$. These are trees of finite height and the standard model N behaves especially nicely in that N, $\mathrm{Th}(N)$ and $\mathrm{Th}_\forall(N)$ are maximal nodes in the respective trees.

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In §3 we show that the good behaviour of bounded Δ_2^0 operators is not shared by definable operators in general and describe some peculiar Δ_2^0 and Δ_3^0 operators. These examples, together with the fact that all the usual methods of defining (complete diagrams of) models of $\boldsymbol{P}$ within $\boldsymbol{P}$ give rise to bounded Δ_2^0 operators, give some weight to the belief that the class of bounded Δ_2^0 operators is a natural class worthy of investigation.

§1. Truth values of alternating sentences. We denote by $\mathrm{Mod}(\boldsymbol{P})$ the class of all models of Peano's (first-order) arithmetic $\boldsymbol{P}$. Members of $\mathrm{Mod}(\boldsymbol{P})$ are denoted by M, M_1, M_2, etc. The standard model is denoted by N $(= \langle N, 0, ', +, \cdot \rangle)$ and we often assume implicitly that $M \supseteq N$. Partial mappings from $\mathrm{Mod}(\boldsymbol{P})$ into $\mathrm{Mod}(\boldsymbol{P})$ will be referred to as *operators*.

1.1. Definition. Let $\boldsymbol{O}$ be an operator. $\boldsymbol{O}^0(M) = M$ and $\boldsymbol{O}^{n+1}(M) = \boldsymbol{O}(\boldsymbol{O}^n(M))$ (if it is defined). The largest $n \in \omega$ such that $\boldsymbol{O}^n(M)$ is defined is denoted by $d_{\boldsymbol{O}}(M)$ (the *rank* or "*depth*" of M with respect to $\boldsymbol{O}$); $d_{\boldsymbol{O}}(M) = \infty$ if $\boldsymbol{O}^n(M)$ is defined for all n.

1.2. Definition. A sentence φ of $L(\boldsymbol{P})$ (the first-order language of arithmetics) will be called an alternating sentence for the operator $\boldsymbol{O}$ when (i) for every $M \in \mathrm{dom}(\boldsymbol{O})$, $M \models \varphi \Leftrightarrow \boldsymbol{O}(M) \not\models \varphi$, and (ii) if $M \notin \mathrm{dom}(\boldsymbol{O})$ then $M \not\models \varphi$.

Remark. (ii) is a normalization condition; as a result the negation of an alternating sentence for $\boldsymbol{O}$ is not an alternating sentence for $\boldsymbol{O}$ unless $\boldsymbol{O}$ is total, i.e., $\mathrm{dom}(\boldsymbol{O}) = \mathrm{Mod}(\boldsymbol{P})$.

1.3. Lemma. *If φ is an alternating sentence for $\boldsymbol{O}$ and $d_{\boldsymbol{O}}(M) < \infty$ then $M \models \varphi$ iff $d_{\boldsymbol{O}}(M)$ is an odd number.*

Proof. Immediate from definitions. □

We now define a type of operator that always has an alternating sentence. Briefly, it is required that the complete diagram of $\boldsymbol{O}(M)$ is defined in M by a fixed formula (independent of M).

1.4. Definition. A completely arithmetically definable operator (cado) is an operator $\boldsymbol{O}$ for which there exist formulas $\mathrm{Nat}(x)$ and $\mathrm{Sat}(y, z)$ and a sentence δ such that $\mathrm{dom}(\boldsymbol{O}) = \{M \in \mathrm{Mod}(\boldsymbol{P}) \mid M \models \delta\}$ and if $M \in \mathrm{dom}(\boldsymbol{O})$ then

(i) The underlying set of $\boldsymbol{O}(M)$ is $\mathrm{Nat}^{(M)} = \{m \in M \mid M \models \mathrm{Nat}[m]\}$.

(ii) If $\varphi(x_1, \ldots, x_n)$ is a formula of $L(\boldsymbol{P})$ and $a_1, \ldots, a_n, \in \mathrm{Nat}^{(M)}$ then $\boldsymbol{O}(M) \models \varphi\ [a_1, \ldots, a_n]$ iff $M \models \mathrm{Sat}\ [\ulcorner\varphi\urcorner, a]$ where $\ulcorner\varphi\urcorner$ is the Gödel number of φ and a is the sequence number $\langle a_1, \ldots, a_n\rangle$ evaluated in M.

Remarks. (1) The triple of formulas (δ, Nat, Sat) clearly determines $\boldsymbol{O}$ completely.

(2) It is *not* required that every $M \in \mathrm{dom}(\boldsymbol{O})$ satisfies the sentence expressing that Sat is a satisfaction relation for a model with underlying set Nat (e.g. that a conjunction is satisfied iff both conjuncts are, etc.). Adding this requirement would not affect the results of this paper. One could, however, require this and moreover that $M \models \forall x(x \text{ is an axiom of } \boldsymbol{P} \rightarrow \mathrm{Sat}(x, \langle\ \rangle))$. Let us call an operator $\boldsymbol{O}$ satisfying this a *strict cado*. If $\boldsymbol{O}$ is a strict cado then $M \models \mathrm{Con}(\boldsymbol{P})$ for every $M \in \mathrm{dom}(\boldsymbol{O})$, where $\mathrm{Con}(\boldsymbol{P})$ expresses the consistency of $\boldsymbol{P}$. All examples of cados mentioned in this paper, except in §3, are examples of strict cados.

1.5. Lemma. *If $\boldsymbol{O}$ is a cado then there exists an alternating sentence for $\boldsymbol{O}$.*

PROOF. Let $\boldsymbol{O}$ be determined by the triple (δ, Nat, Sat) and let False(x) be the formula $\neg$Sat(x, $\langle\ \rangle$). By diagonalization find a sentence φ such that $\boldsymbol{P} \vdash [\varphi \leftrightarrow \delta \wedge \text{False}(\ulcorner\varphi\urcorner)]$. Clearly, any such φ is an alternating sentence for $\boldsymbol{O}$. $\square$

1.6. LEMMA. *If $\boldsymbol{O}$ is a cado and $M \in \text{dom}(\boldsymbol{O})$ then $\boldsymbol{O}(M)$ is isomorphic to a proper end extension of M.*

PROOF. We can write down a formula $\chi(x, y)$ which intuitively says that y is the xth element of the model determined by Nat and Sat. Then in every model $M \in \text{dom}(\boldsymbol{O})$, χ defines the graph of an embedding of M in $\boldsymbol{O}(M)$ as an initial segment. By the previous lemma, $\boldsymbol{O}(M) \not\cong M$, hence $\boldsymbol{O}(M)$ is isomorphic to a *proper* end extension of M.

1.7. EXAMPLE. The Henkin operator H. Consider Henkin's completeness proof for predicate calculus, specialized to show that if $\boldsymbol{P}$ is consistent then $\boldsymbol{P}$ has a model. There are two steps. First, one forms a conservative extension T_0 of $\boldsymbol{P}$, in a countable language L obtained for $L(\boldsymbol{P})$ by adding infinitely many new constants. T_0 is saturated in the sense that for each sentence $\exists x\varphi$ of L, T_0 has an axiom of the form $\exists x\, \varphi \rightarrow \varphi(c)$ for some constant c. The second step is to enumerate the sentences of L in a sequence $\varphi_0, \varphi_1, \ldots$ and let $T_{n+1} = T_n \cup \{\varphi\}$ if $T_n \cup \{\varphi\}$ is consistent, $T_{n+1} = T_n \cup \{\neg\, \varphi\}$ otherwise. Then $T_\omega = \bigcup_n T_n$ is the complete diagram of a model of $\boldsymbol{P}$ whose underlying set is the set of (equivalence classes of) closed terms of L. Formalizing this construction in $\boldsymbol{P}$ we get a (strict) cado determined by the triple (Con($\boldsymbol{P}$), Nat, Sat) where Nat and Sat are Δ_2^0 in $\boldsymbol{P}$.[2]

Kreisel [K2, ibid] observed that operators $\boldsymbol{O}$ corresponding to various standard proofs of the completeness theorem (e.g. Henkin's method, semantic tableaux, etc.) have the property that $d_O(M) < \infty$ for all M. The proof for $\boldsymbol{O} = H$, which is typical, appears in [Sm, 6.2.4]. The result applies also to the operator (call it K) corresponding to the construction in [K1], where a model of $\boldsymbol{P}$ is defined within any model of $\boldsymbol{P}$ + Con(BG) via a model of BG. A more general result is proved in §2.

Noting that the independent sentence constructed in [K1], normalized if necessary, is an alternating sentence for K, it follows from Lemma 1.3 that its truth value is determined by the parity of $d_K(N)$. The actual truth value is unknown and, in fact, we have been unable to determine $d_O(N)$, or its parity, for any natural choice of $\boldsymbol{O}$. The following example shows that in any case no uniform answer can be expected.

1.8. EXAMPLE. For $n \geq 1$, the operator H_n is defined exactly like H (see Example 1.7) except that instead of using a canonical enumeration $\varphi_0, \varphi_1, \varphi_2, \ldots$ of the sentences of L (e.g. by increasing Gödel numbers) one modifies this enumeration by putting the sentences $\text{Con}^1(\boldsymbol{P}), \text{Con}^2(\boldsymbol{P}), \ldots, \text{Con}^{n-1}(\boldsymbol{P}), \neg\, \text{Con}^n(\boldsymbol{P})$ at the start of the list. Here $\text{Con}^1(\boldsymbol{P}) = \text{Con}(\boldsymbol{P})$ and $\text{Con}^{k+1}(\boldsymbol{P}) = \text{Con}(\boldsymbol{P} + \text{Con}^k(\boldsymbol{P}))$. We leave it as an exercise for the reader to show that $d_{H_n}(N) = n$.

It seems to us that the problem of determining the truth value in N of an alternating sentence for a particular operator (in particular, of the sentence discussed

[2]This formalization of the completeness theorem goes back essentially to Hilbert-Bernays [Sm, 6.1.1]. Actually, they formalized Gödel's proof and got a Δ_2^0 definition of the diagram (not the complete diagram) of the constructed model. The formalization of Henkin's proof was done by Hasenjaeger (this JOURNAL, vol. 18 (1953), pp. 42–48), but compare also [K1] and Kleene's ***Introduction to metamathematics*** (1952), § 72.

in [K1]) is of a combinatorial rather than logical nature. The answer may be sensitive to very minor changes in the construction (although the changes made in the definition of H to get H_n are admittedly "artificial"). On the other hand, Lemma 1.3 shows that once $\boldsymbol{O}$ is determined, and $d_O(M) < \infty$, the truth value in M of an alternating sentence for $\boldsymbol{O}$ is uniquely determined and does not depend on the particular sentence chosen. This leads to

1.9. COROLLARY. *If $\boldsymbol{O}$ is a cado such that $d_O(M) < \infty$ for each model M of $\boldsymbol{P}$, then any two alternating sentences for $\boldsymbol{O}$ are equivalent in $\boldsymbol{P}$.*

§2. Bounded Δ_2^0 operators. We now describe a class of operators $\boldsymbol{O}$, including all operators mentioned in §1, such that $d_O(M) < \infty$ for all M. This is the class of bounded Δ_2^0 operators.

2.1. DEFINITION. For $n \geq 1$, *a* Π_n^0 (resp. Σ_n^0, Δ_n^0) cado is a cado $\boldsymbol{O}$ determined by a triple (δ, Nat, Sat) of Π_n^0 (resp. Σ_n^0, Δ_n^0) formulas of $\boldsymbol{P}$. (A Δ_n^0 formula of $\boldsymbol{P}$ is one that is equivalent in $\boldsymbol{P}$ to a Σ_n^0 and to a Π_n^0 formula.)

The only Σ_1^0 or Π_1^0 cado is the empty operator. The proof of this is similar to the proof of Tennenbaum's theorem that there are no recursive nonstandard models of $\boldsymbol{P}$ [EK] and will be omitted.

2.2. NOTATION. Let the formula $Q_n(t)$ of $L(\boldsymbol{P})$, for $n \geq 1$, be some standard definition of a complete Π_n^0 set. To be specific assume Q_n defines satisfaction of Π_n^0 formulas, so that (i) $Q_n(x)$ is Π_n^0, (ii) for each Π_n^0 formula $\varphi(x)$, $\boldsymbol{P} \vdash \forall x(\varphi(x) \leftrightarrow Q_n(\langle \ulcorner\varphi\urcorner, x \rangle))$.

By formalizing Post's theorem, that $\Delta_{n+1}^0 =$ recursive in Π_n^0, in $\boldsymbol{P}$ we see that if $\varphi(x_1, \ldots, x_k)$ is a Δ_{n+1}^0 formula of $\boldsymbol{P}$ ($k \geq 0$) then there exists (a Gödel number of) an oracle Turing machine t such that $\boldsymbol{P} \vdash$ "the characteristic function of the relation φ is computed by machine t using the set Q_n as oracle".

2.3. DEFINITION. (1) $\varphi(x_1, \ldots, x_k)$ is said to be a bounded Δ_{n+1}^0 formula of $\boldsymbol{P}$ when it is Δ_{n+1}^0 and the machine t can be chosen so that, in addition to the above, for each $m \in N$ there exists $l \in N$ such that $\boldsymbol{P} \vdash$ "$(\forall x_1, \ldots, x_k \leq m)$ [the length of computation of t on input $x_1, \ldots, x_k$ using the oracle Q_n is $\leq l$]".

(2) $\varphi(x_1, \ldots, x_k)$ is said to be a primitive Δ_{n+1}^0 formula of $\boldsymbol{P}$ when it is Δ_{n+1}^0 and machine t can be chosen as the natural realization of a primitive recursive functional. In other words, there is a primitive recursive functional Φ (of k number variables and one set variable) such that $\boldsymbol{P} \vdash$ "for all $x_1, \ldots, x_k$, if $\varphi(x_1, \ldots, x_k)$ then $\Phi(x_1, \ldots, x_k, Q_n) = 1$; else $\Phi(x_1, \ldots, x_k, Q_n) = 0$".

REMARKS. (1) It is easily verified by induction on the definitions of primitive recursive functionals that every primitive Δ_{n+1}^0 functional is bounded Δ_{n+1}^0.

(2) A *sentence* φ is bounded Δ_{n+1}^0 in $\boldsymbol{P}$ iff φ is equivalent in $\boldsymbol{P}$ to a Boolean combination of Π_n^0 sentences. The "if" direction is trivial while "only if" follows (from 2.3(1) in the case $k = 0$) by noting that the truth value of φ is determined by the answers given by the oracle Q_n in the course of the finitely many possible computations (with no input).

2.4. DEFINITION. A bounded (primitive) Δ_{n+1}^0 cado is a cado $\boldsymbol{O}$ determined by a triple (δ, Nat, Sat) of bounded (resp. primitive) Δ_{n+1}^0 formulas.

All examples of operators mentioned in §1 are primitive Δ_2^0 cados, hence are bounded Δ_2^0 and the next theorem applies to them.

2.5. THEOREM. *If $\boldsymbol{O}$ is a bounded Δ_2^0 cado then $d_O(M) < \infty$ for all M.*

PROOF. Let (δ, Nat, Sat) be a triple of bounded Δ_2^0 formulas determining $\boldsymbol{O}$ and let t_1, t_2, t_3 be suitable Turing machines so that the conditions in 2.3 (1) hold with $n = 1$. Let χ be an alternating sentence for $\boldsymbol{O}$.

Suppose $d_O(M) = \infty$ for some M. Let $M_0 = M$ and for each n let M_{n+1} be an end extension of M_n isomorphic to $\boldsymbol{O}(M_n)$. Consider the computation of machine t_3 on the arguments $\ulcorner\chi\urcorner$ and $\langle\ \rangle$. The result of this computation in M_n (with oracle $Q_1^{(M_n)}$) determines if $M_n \models \mathrm{Sat}(\ulcorner\chi\urcorner, \langle\ \rangle)$, i.e., if $M_{n+1} \models \chi$. If the computation terminates without making any call for the oracle then the truth value of χ is the same in all models $\boldsymbol{O}(M_n) = M_{n+1}$, in particular in M_1 and M_2, contradicting the fact that χ is an alternating sentence for $\boldsymbol{O}$. Thus the oracle is called at least once. Let ε_n^1 be the value (YES or NO) supplied by the oracle on the first call when the computation is carried out in M_n. Then, if $\varepsilon_m^1 = \mathrm{NO}$ for some m then $\varepsilon_n^1 = \mathrm{NO}$ for all $n \geq$ m because Q_1 is a Π_1^0 formula and hence its negation is preserved under end extension (the value fed to the oracle on the first call is, of course, the same in all models M_n). Thus the sequence $(\varepsilon_n^1)_{n=n_1}^{\infty}$ is a constant for some $n_1 \in N$.

Now continue the computation. If no second call to the oracle occurs in M_{n_1} (hence in M_n for all $n \geq n_1$) then the truth value of χ is the same in M_{n+1} for all $n \geq n_1$, again a contradiction. If there is a second call to the oracle in M_{n_1} then by the same argument as above there exists $n_2 \geq n_1$ such that ε_n^2 (the value applied by the oracle on the second call) is fixed for all $n \geq n_2$. Continuing in this way we see that there must be an unbounded number of calls to the oracle, contradicting the boundedness of $\boldsymbol{O}$. $\square$

[Alternatively, we could argue that since Sat is a bounded Δ_2^0 formula, Sat ($\ulcorner\chi\urcorner$, $\langle\ \rangle$) is a bounded Δ_2^0 sentence, hence is equivalent in $\boldsymbol{P}$ to a Boolean combination of certain Π_1^0 sentences $\psi_1, \ldots, \psi_k$. In the sequence $(M_n)_{n=0}^{\infty}$ of models, the truth value of each ψ_i is eventually constant (as $M_n \subseteq_{\mathrm{end}} M_{n+1}$ for each n) hence the truth value of Sat($\ulcorner\chi\urcorner$, $\langle\ \rangle$) is eventually constant (say constant for all $n \geq n_0$), and so the truth value of χ is the same in M_{n_0+1} and M_{n_0+2}, contradiction. We included the longer proof above as it illustrates what happens more clearly.]

2.6. COROLLARY. *If $\boldsymbol{O}$ is a bounded Δ_2^0 cado then* $\sup_M(d_0(M)) < \infty$.

PROOF. By an application of the compactness theorem to the set of sentences $\{\delta_n : n \geq 1\}$ where δ_n expresses the fact that $\boldsymbol{O}$ can be iterated at least n times. $\square$

The above theorem suggests that in the case of a bounded Δ_2^0 operator, say H, the model $\boldsymbol{O}(M)$ is somehow poorer (satisfies fewer consistency statements) than the model M. This led us to conjecture that $H(M)$ is never elementarily equivalent to the standard model N. This, and more, is true as can be seen from the corollary to the next proposition. We denote by $\mathrm{Th}(M)$ the complete theory of a model M and by $\mathrm{Th}_\forall(M) = \{\varphi \mid \varphi \text{ is } \Pi_1^0 \text{ and } \varphi \in \mathrm{Th}(M)\}$ the universal theory of M.

2.7. PROPOSITION. *Let $\boldsymbol{O}$ be a bounded Δ_2^0 cado.*

(1) *If* $\mathrm{Th}_\forall(M_1) = \mathrm{Th}_\forall(M_2)$ *and* $M_1 \in \mathrm{dom}(\boldsymbol{O})$ *then* $M_2 \in \mathrm{dom}(\boldsymbol{O})$ *and* $\mathrm{Th}(\boldsymbol{O}(M_1)) = \mathrm{Th}(\boldsymbol{O}(M_2))$.

(2) *If* $M \in \mathrm{dom}(\boldsymbol{O})$ *then* $\mathrm{Th}_\forall(\boldsymbol{O}(M)) \subsetneqq \mathrm{Th}_\forall(M)$.

PROOF. (1) Let (δ, Nat, Sat) be a triple of bounded Δ_2^0 formulas determining $\boldsymbol{O}$. Then δ, and all sentences of the form Sat ($\ulcorner\varphi\urcorner$, $\langle\ \rangle$) (φ a sentence of $L(\boldsymbol{P})$), are equivalent in $\boldsymbol{P}$ to Boolean combinations of Π_1^0 sentences. Thus, if $\mathrm{Th}_\forall(M_1) =$

$\mathrm{Th}_\forall(M_2)$ and $M_1 \models \delta$ then $M_2 \models \delta$ and for each φ, $M_1 \models \mathrm{Sat}(\ulcorner\varphi\urcorner, \langle\ \rangle)$ iff $M_2 \models \mathrm{Sat}(\ulcorner\varphi\urcorner, \langle\ \rangle)$, so that $\mathrm{Th}(\boldsymbol{O}(M_1)) = \mathrm{Th}(\boldsymbol{O}(M_2))$.

(2) Since $\boldsymbol{O}(M)$ is isomorphic to an end extension of M, $\mathrm{Th}_\forall(\boldsymbol{O}(M)) \subseteq \mathrm{Th}_\forall(M)$. If $\mathrm{Th}_\forall(\boldsymbol{O}(M)) = \mathrm{Th}_\forall(M)$ then by (1), $\boldsymbol{O}(M) \in \mathrm{dom}(\boldsymbol{O})$ and $\mathrm{Th}(\boldsymbol{O}^2(M)) = \mathrm{Th}(\boldsymbol{O}(M))$, contradicting the existence of an alternating sentence for $\boldsymbol{O}$. Hence the inclusion is proper. □

2.8. Corollary. *Let $\boldsymbol{O}$ be bounded Δ_2^0. If $M \in \mathrm{dom}(\boldsymbol{O})$ then $\mathrm{Th}_\forall(\boldsymbol{O}(M)) \subsetneqq \mathrm{Th}_\forall(N)$, hence $\boldsymbol{O}(M)$ is not elementarily equivalent to N. (This is generalized in* 2.10.)

Proof. $\mathrm{Th}_\forall(\boldsymbol{O}(M)) \subsetneqq \mathrm{Th}_\forall(M) \subseteq \mathrm{Th}_\forall(N)$. □

The results obtained so far have a simple interpretation in terms of trees. Let $\boldsymbol{O}$ be a fixed bounded Δ_2^0 cado. Define a relation $<$ on the class $\mathrm{Mod}(\boldsymbol{P})$ by: $M_1 < M_2$ iff $M_1 = \boldsymbol{O}^i(M_2)$ for some $i \geq 1$. By Theorem 2.5 this is a wellfounded relation, and in fact it is a partial ordering in which the set of predecessors of every element is finite. Thus we have a tree (or rather a forest) whose nth level consists of all M such that $d_O(M) = n$. By Corollary 2.6 the height of the tree is finite. Since the (universal) theory of $\boldsymbol{O}(M)$, for $M \in \mathrm{dom}(\boldsymbol{O})$, is uniquely determined by the (universal) theory of M, and $\mathrm{Th}_\forall(\boldsymbol{O}^i(M)) \neq \mathrm{Th}_\forall(M)$ for $i \geq 1$, $\boldsymbol{O}$ induces a similar tree structure on the set $\mathrm{Com}(\boldsymbol{P}) = \{\mathrm{Th}(M) \mid M \models \boldsymbol{P}\}$ of completions of $\boldsymbol{P}$ as well as on $\mathrm{Com}_\forall(\boldsymbol{P}) = \{\mathrm{Th}_\forall(M) \mid M \models \boldsymbol{P}\}$. These trees are homomorphic images of the tree of models. Corollary 2.8 shows that $\mathrm{Th}(N)$ and $\mathrm{Th}_\forall(N)$ are maximal nodes in the respective trees (it is trivial that N is a maximal node in the tree of models because N is not a proper end extension of any model of $\boldsymbol{P}$).

It would be interesting to know more about the structure of those trees, at least for particular operators like H. For example, since $\mathrm{Th}(\boldsymbol{O}(M))$ depends only on $\mathrm{Th}_\forall(M)$ every nonmaximal node in the tree of completions of $\boldsymbol{P}$ (determined by $\boldsymbol{O}$) has $2^{\aleph_0}$ immediate successors. Is the same true for the tree on $\mathrm{Com}_\forall(\boldsymbol{P})$? Is it true that $d_O(N) \geq d_O(M)$ for all M? Many more such questions can be posed. Here is one that seems particularly interesting.

2.9. *Problem.* Let $S \in \mathrm{Com}_\forall(\boldsymbol{P})$, $S \neq \mathrm{Th}_\forall(N)$. Is there a bounded Δ_2^0 operator $\boldsymbol{O}$ such that S is a nonmaximal node in the corresponding tree, that is $S = \mathrm{Th}_\forall(M)$ for some $M \in \mathrm{dom}(\boldsymbol{O})$?

Another, more elementary, question that we have not yet settled is whether the domain of a bounded Δ_2^0 operator can contain models of $\neg\mathrm{Con}(\boldsymbol{P})$ (compare Proposition 3.1).

We conclude this section by showing that Corollary 2.8 is valid for any Δ_2^0 cado $\boldsymbol{O}$, bounded or unbounded. This is clearly implied by the following proposition, which was essentially observed by the referee.

2.10. Proposition. *For each Δ_2^0 cado $\boldsymbol{O}$ there is a Π_1^0 sentence which is true in N and false in $\boldsymbol{O}(M)$ for all $M \in \mathrm{dom}(\boldsymbol{O})$.*

Proof. Let $(\delta, \mathrm{Nat}, \mathrm{Sat})$ be a triple of Δ_2^0 formulas of $\boldsymbol{P}$ that defines $\boldsymbol{O}$ and let χ be an alternating sentence for $\boldsymbol{O}$. We claim that there is a Π_1^0 sentence φ such that $N \models \varphi$ and for all M_1, if $M_1 \models \varphi$ then N and M_1 are equivalent w.r.t. the two sentences δ and $\mathrm{Sat}(\ulcorner\chi\urcorner, \langle\ \rangle)$ (i.e. each one of these sentences has the same truth value in the two models). Such a sentence φ exists because δ, $\neg\delta$, $\mathrm{Sat}(\ulcorner\chi\urcorner, \langle\ \rangle)$ and $\neg\,\mathrm{Sat}(\ulcorner\chi\urcorner, \langle\ \rangle)$ are equivalent in $\boldsymbol{P}$ to Σ_2^0 sentences, and each Σ_2^0 sentence true in N is logically implied by some Π_1^0 sentence true in N. [Alternatively, let φ be the

conjunction of the Π_1^0 sentences corresponding to YES answers from the oracle $Q_1^{(N)}$ during the (terminating) computations of the truth values in N of δ and $\mathrm{Sat}(\ulcorner\chi\urcorner, \langle\ \rangle)$.]

So let φ be as above and suppose, for contradiction, that $M \in \mathrm{dom}(\boldsymbol{O})$ and $\boldsymbol{O}(M) \models \varphi$. Then $M \models \varphi$ by Lemma 1.6, hence the three models N, M and $\boldsymbol{O}(M)$ are equivalent w.r.t. δ and $\mathrm{Sat}(\ulcorner\chi\urcorner, \langle\ \rangle)$. But $M \in \mathrm{dom}(\boldsymbol{O})$ so $M \models \delta$, hence all three models are in the domain of $\boldsymbol{O}$. Now, $\boldsymbol{O}(M) \models \chi$ iff $M \models \mathrm{Sat}(\ulcorner\chi\urcorner, \langle\ \rangle)$ iff $\boldsymbol{O}(M) \models \mathrm{Sat}(\ulcorner\chi\urcorner, \langle\ \rangle)$ iff $\boldsymbol{O}^2(M) \models \chi$, contradicting the choice of χ as an alternating sentence. □

§3. **Examples.** We show now that the good behavior of bounded Δ_2^0 cados is not shared by arbitrary cados, even Δ_2^0 ones. The examples are

(1) A Δ_2^0 cado $\boldsymbol{O}_1$ such that $\mathrm{dom}(\boldsymbol{O}_1) = \mathrm{Mod}(\boldsymbol{P})$.

(2) A *strict* Δ_2^0 cado $\boldsymbol{O}_2$ such that $d_{\boldsymbol{O}_2}(M) = \infty$ for some M (see remarks following Definition 1.4).

(3) A primitive Δ_3^0 cado $\boldsymbol{O}_3$ such that $\mathrm{dom}(\boldsymbol{O}_3) = \mathrm{Mod}(\boldsymbol{P})$ and every completion T of $\boldsymbol{P}$ is of the form $\mathrm{Th}(\boldsymbol{O}_3(M))$ for some M. When the domain of $\boldsymbol{O}_3$ is restricted so as to make it strict, the range still contains models for all completions of $\boldsymbol{P}$.

Every cado $\boldsymbol{O}$ determines a pseudo-tree structure on $\mathrm{Mod}(\boldsymbol{P})$ and on $\mathrm{Com}(\boldsymbol{P})$ just as a bounded Δ_2^0 cado does. [A pseudo-tree is a partially ordered set in which the predecessors of each element are linearly ordered.] To see this it suffices to observe that if $\boldsymbol{O}$ is a cado, $i \geq 1$ and $\boldsymbol{O}^i(M)$ is defined then $\mathrm{Th}(\boldsymbol{O}^i(M)) \neq \mathrm{Th}(M)$. The inequality is an immediate consequence of the fact that $\boldsymbol{O}^i$ is a cado as well, and has an alternating sentence. Since $\mathrm{Th}(\boldsymbol{O}(M))$ depends only on a part of bounded complexity of $\mathrm{Th}(M)$, more precisely on the truth value in M of the sentences $\mathrm{Sat}(n, \langle\ \rangle)$ $(n = 0, 1, 2, \ldots)$, it follows that every nonmaximal node in the pseudo-tree of complete theories has $2^{\aleph_0}$ immediate successors. In the case of $\boldsymbol{O}_3$, for example, we obtain a pseudo-tree of complete theories where for each node x, $\{y \mid y < x\}$ is ordered like the negative integers and x has $2^{\aleph_0}$ immediate successors. Thus, for any $T_1, T_2 \in \mathrm{Com}(\boldsymbol{P})$ there is an automorphism of the pseudo-tree taking T_1 to T_2, and no theory is distinguished.

We now present the three examples informally.

3.1. Proposition. *There is a Δ_2^0 cado $\boldsymbol{O}_1$ such that* $\mathrm{dom}(\boldsymbol{O}_1) = \mathrm{Mod}(\boldsymbol{P})$.

Proof. We define $\boldsymbol{O}_1$. $\mathrm{Dom}(\boldsymbol{O}_1) = \mathrm{Mod}(\boldsymbol{P})$ (i.e. $\delta \equiv 0 = 0$). To define $\boldsymbol{O}_1(M)$, work in M.

Using a complete Π_1^0-oracle Q_1, check if $M \models \mathrm{Con}(\boldsymbol{P})$. If so, use the Henkin operator, i.e. $\boldsymbol{O}(M) = H(M)$. If not, for every $n \in M$, let $\boldsymbol{P}_m$ denote the set of axioms of $\boldsymbol{P}$ whose Gödel number is less than m. Search to find the first $c \in M$ such that $M \models \neg\,\mathrm{Con}(\boldsymbol{P}_c)$. This involves applying the oracle. c must be nonstandard. Since $M \vdash \mathrm{Con}(\boldsymbol{P}_{c-1})$, a model of $\boldsymbol{P}_{c-1}$ may be constructed by applying the Henkin construction. Let $\boldsymbol{O}(M)$ be this model. □

Since $\boldsymbol{O}_1$ is total, it is clear that $d_{\boldsymbol{O}_1}(M) = \infty$ for each M. However, $\boldsymbol{O}_1$ is clearly not a strict cado, for the domain of a strict cado contains only models of $\boldsymbol{P} + \mathrm{Con}(\boldsymbol{P})$.

3.2. Proposition. *There is a strict Δ_2^0 cado $\boldsymbol{O}_2$ such that $d_{\boldsymbol{O}_2}(M) = \infty$ for some M.*

Proof. Let $\Gamma(x)$ be a Π_1^0 formula of $L(\boldsymbol{P})$ expressing in a natural way $\mathrm{Con}^x(\boldsymbol{P})$

(where $\mathrm{Con}^1(\boldsymbol{P}) \equiv \mathrm{Con}(\boldsymbol{P})$, $\mathrm{Con}^{n+1}(\boldsymbol{P}) \equiv \mathrm{Con}(\boldsymbol{P} + \mathrm{Con}^n(\boldsymbol{P}))$. The domain of $\boldsymbol{O}_2$ is $\{M \mid M \models \delta\}$ where δ is the sentence $\mathrm{Con}(\boldsymbol{P}) \wedge \exists x \neg \Gamma(x)$. Let $M \models \delta$. To define $\boldsymbol{O}(M)$ work in M. Using the oracle Q_1^M search for the least x such that $\neg\Gamma(x)$; denote it by j ($j > 1$). Now construct a model of $\boldsymbol{P}$ by Henkin's method, placing the sentences $\mathrm{Con}(\boldsymbol{P})$, $\mathrm{Con}^2(\boldsymbol{P})$, ..., $\mathrm{Con}^{j-1}(\boldsymbol{P})$ at the head of the list (just as in Example 1.8, but now the process is described in M and j may be nonstandard).

Note that $\boldsymbol{O}_2(M) \models \min\{x \mid \neg\Gamma(x)\} = j - 1$; $\boldsymbol{O}_2^2(M) \models \min\{x \mid \neg\Gamma(x)\} = j - 2$, etc. Thus, if j is standard $d_{\boldsymbol{O}_2}(M) = j - 1$ and if j is nonstandard $d_{\boldsymbol{O}_2}(M) = \infty$. It is easy to see that there exists $M \models \delta$ where j is nonstandard. $\square$

3.3. Proposition. *There exists a strict Δ_2^0 cado $\boldsymbol{O}_2'$ such that* $\mathrm{dom}\, \boldsymbol{O}_2' = \{M \mid M \models \mathrm{Con}(\boldsymbol{P})\}$ *and* $d_{\boldsymbol{O}'_2}(N) = \infty$.

Proof. Let $\boldsymbol{O}_1'(M) = \boldsymbol{O}_2(M)$ if $M \models \mathrm{Con}(\boldsymbol{P}) \wedge \exists x \neg \Gamma(x)$. If $M \models \forall x \Gamma(x)$ then $M \models \mathrm{Con}(T)$ where T is defined as $\boldsymbol{P} \cup \{\Gamma(n) \mid n = 1, 2, 3, \ldots\} \cup \{\exists x \neg \Gamma(x)\}$, so define $\boldsymbol{O}_2'(M)$ in M as the model obtained by applying Henkin's construction to the theory T. Thus, if $M \models \forall x \Gamma(x)$ then $M_1 = \boldsymbol{O}_2'(M)$ is a model of infinite rank for $\boldsymbol{O}_2$ (by the proof of Proposition 3.2) and $\boldsymbol{O}_2'^{k+1}(M) = \boldsymbol{O}_2'^{k}(M_1)$ for each k. Hence $d_{\boldsymbol{O}_2'}(M) = \infty$. $\square$

3.4. Theorem. *There is a cado, $\boldsymbol{O}_3$, such that* $\mathrm{dom}(\boldsymbol{O}_3) = \mathrm{Mod}(\boldsymbol{P})$ *and for every* $T \in \mathrm{Com}(\boldsymbol{P})$ *there exists a model M of* $\boldsymbol{P} \cup \mathrm{Th}_\forall(N)$ *such that* $T = \mathrm{Th}(\boldsymbol{O}_3(M))$. *$\boldsymbol{O}_3$ may be taken to be primitive Δ_3^0.*

The proof will depend on the following lemma, where all models M are assumed to be extensions of N and we denote $K_i^{(M)} = Q_i^{(M)} \cap N = \{n \in N \mid M \models Q_i(n)\}$ (see Notation 2.2 for Q_i).

3.5. Lemma. *There exists a primitive recursive functional Φ such that for each $A \subseteq N$ there exists $M \models \boldsymbol{P} \cup \mathrm{Th}_\forall(N)$ for which $A = \Phi(K_2^M)$ in the sense that $\chi_A(n) = \Phi(K_2^{(M)}, n)$ for each n (χ_A is the characteristic function of A).*

Remark. The lemma remains true upon replacing $K_2^{(M)}$ by $K_1^{(M)}$ if one requires only $M \models \boldsymbol{P}$ and not $M \models \mathrm{Th}_\forall(N)$. The proof is similar but easier.

Proof. Let the formula $U(x)$ be a Π_1^0 truth definition for universal sentences, so that $\boldsymbol{P} \vdash \varphi \leftrightarrow U(\ulcorner\varphi\urcorner)$ for each Π_1^0 sentence φ. With every finite set X of sentences of $L(\boldsymbol{P})$ one can effectively associate a sentence $\psi = \psi_X$ such that if $T = \boldsymbol{P} \cup \mathrm{Th}_\forall(N) \cup X$ is consistent then ψ is undecidable in T. Simply produce by diagonalization a sentence ψ such that

$$\begin{aligned}\psi \leftrightarrow \forall x[&(x \text{ is a proof of } \ulcorner\psi\urcorner \text{ from } \boldsymbol{P} + U + X)\\ &\rightarrow (\exists y < x)(y \text{ is a proof of } \ulcorner\neg\psi\urcorner \text{ from } \boldsymbol{P} + U + X)]\end{aligned}$$

and use the fact that U is a truth definition for universal sentences to show that ψ has the desired property. The right-hand side $(\forall x[\cdots])$ is Π_2^0 hence ψ_X is equivalent in $\boldsymbol{P}$ to a Π_2^0 sentence $Q_2(n_X)$ where n_X depends effectively (in fact, primitive recursively) on X. Compare Scott [Sc, Lemma] for a similar, more involved, result.

It is now easy to construct a full binary tree where at each node $\eta = (\eta_0, \ldots, \eta_{l-1})$ ($\eta_i = 0$ or 1) there is a sentence $Q_2(m_\eta)$ which is undecidable in the consistent theory $\boldsymbol{P} \cup \mathrm{Th}_\forall(N) \cup \{\eta_j \cdot Q(m_{\eta|j}) \mid j < l\}$ (where $1 \cdot \varphi = \varphi$, $0 \cdot \varphi = \varphi$). m_η depends primitive recursively on η.

Now let $A \subseteq \omega$. Then the set of sentences on the branch corresponding to χ_A is

consistent with $\boldsymbol{P} + \mathrm{Th}_\forall(N)$. Thus there is a model M of $\boldsymbol{P} \cup \mathrm{Th}_\forall(N) \cup \{\chi_A(j) \cdot Q_2(m_{\chi_A|j}) \mid j \in N\}$. From $K_2^{(M)}$ we can now recover A. For example, to determine if $\chi_A(0)$ is 1 or 0 check if $m_{\langle\,\rangle}$ does or does not belong to $K_2(M)$. Assuming it does, you can determine if $\chi_A(1)$ is 1 or 0 by seeing if $m_{\langle 1\rangle}$ is in $K_2^{(M)}$; if $m_{\langle 1\rangle} \notin K_2^{(M)}$ then $\chi_A(2) = 1$ iff $m_{\langle 1,0\rangle} \in K_2^{(M)}$, and so on. This gives a uniform primitive recursive way of computing $\chi_A(n)$ with the help of an oracle for $K_2^{(M)}$. $\square$

PROOF (OF THEOREM 3.4). Let $\boldsymbol{O}_3(M) = \boldsymbol{O}_1(M)$ if $M \models \neg\mathrm{Con}(\boldsymbol{P})$. To define $\boldsymbol{O}_3(M)$ where $M \models \mathrm{Con}(\boldsymbol{P})$ we are going to define in $\boldsymbol{P}$ a predicate $\mathrm{True}(x)$ such that $\boldsymbol{P} + \mathrm{Con}(\boldsymbol{P}) \vdash$ "$\{x \mid \mathrm{True}(x)\}$ is a complete extension of $\boldsymbol{P}$". Once this is done, define (in $\boldsymbol{P}$) Nat as the set of all terms built up from $0, ', +, \cdot$ and the minimum operator μ and define Sat in $\boldsymbol{P}$ as follows. Let $\varphi(v_1, \ldots, v_n)$ be a formula of $L(\boldsymbol{P})$ and let $t_1, \ldots, t_n$ be μ-terms. Then $\mathrm{Sat}(\varphi, \langle t_1, \ldots, t_n\rangle)$ iff True (ψ) where ψ is the sentence obtained from $\varphi(t_1, \ldots, t_n)$ by eliminating the μ-terms. (We ignore the small problem of interpreting "$=$" as the identity relation rather than merely by a congruence relation.) It is easy to see, under the above assumption on the formula $\mathrm{True}(x)$, that with these definitions of Nat and Sat, $\boldsymbol{O}_3$ is a strict cado on models of $\boldsymbol{P} + \mathrm{Con}(\boldsymbol{P})$, and is primitive Δ_3^0 if the formula True is. Moreover, for $M \models \mathrm{Con}(\boldsymbol{P})$ and φ a sentence of $L(\boldsymbol{P})$ we have $\boldsymbol{O}_3(M) \models \varphi$ iff $M \models \mathrm{True}(\ulcorner \varphi \urcorner)$.

The remaining task, therefore, is to define a primitive Δ_3^0 predicate True such that

(1) $\boldsymbol{P} + \mathrm{Con}(\boldsymbol{P}) \vdash$ "$\{x \mid \mathrm{True}(x)\}$ is a complete extension of $\boldsymbol{P}$".

(2) For every complete extension T of $\boldsymbol{P}$ there is a model M of $\boldsymbol{P} + \mathrm{Th}_\forall(N)$ such that for all φ, $\varphi \in T$ iff $M \models \mathrm{True}(\ulcorner \varphi \urcorner)$.

Here is the definition of True. If x is not a sentence of $L(\boldsymbol{P})$ then $\neg\mathrm{True}(x)$. If x is a sentence make a list $\varphi_0 < \varphi_1 < \cdots < \varphi_k = x$ of all sentences up to and including x (as we are working in $\boldsymbol{P}$ sentences are identified with their Gödel numbers). For $0 \le i \le k$ let $t_i = \Phi(Q_2, \varphi_i)$ where Φ is the functional of Lemma 3.5, so each t_i is 1 or 0. It is at this step (computing $t_0, \ldots, t_k$) that the main use of the oracle Q_2 occurs. Now check (again using Q_2, though Q_1 is enough) whether the sentence $t_0 \cdot \varphi_0 \wedge \cdots \wedge t_k \cdot \varphi_k$ is consistent with $\boldsymbol{P}$. If it is consistent then $\mathrm{True}(x) \Leftrightarrow t_k = 1$. If it is not consistent find $j \le k$ so that $\boldsymbol{P} + \bigwedge_{i=0}^{j-1} t_i \cdot \varphi_i$ is consistent but $\boldsymbol{P} + \bigwedge_{i=0}^{j} t_i \cdot \varphi_i$ is inconsistent. Then decide whether $\mathrm{True}(x)$ by finding the truth value that x gets in the Henkin construction of a model of $\boldsymbol{P} + \bigwedge_{i=0}^{j-1} t_i \cdot \varphi_i$.

The verification of (1) is left to the reader. To prove (2), let $T \in \mathrm{Com}(\boldsymbol{P})$ and let $A \subseteq N$ be the set of Gödel numbers of theorems of T. Choose M as in Lemma 3.5. Then $\varphi \in T$ iff $\ulcorner \varphi \urcorner \in A$ iff $\Phi(K_2^{(M)}, \ulcorner \varphi \urcorner) = 1$. Let $\varphi_0 < \cdots < \varphi_k = \varphi$ be the list of all (Gödel numbers of) sentences of $L(\boldsymbol{P})$ up to φ, and let $t_i = \Phi(K_2^{(M)}, \varphi_i)$. Then $t_i = 1$ if $\varphi_i \in T$, $t_i = 0$ if $\neg\varphi_i \in T$.

Thus, to show that $\varphi \in T$ iff $M \models \mathrm{True}\ (\ulcorner \varphi \urcorner)$ it suffices to show that $M \models$ "$\boldsymbol{P} + \bigwedge_{i=0}^{k} t_i \cdot \varphi_i$ is consistent". However, the consistency of any theorem of T with $\boldsymbol{P}$ is a true universal sentence hence is true in M as $M \models \mathrm{Th}_\forall(N)$. $\square$

REMARK. By Proposition 2.10 no Δ_2^0 cado can have every completion of $\boldsymbol{P}$ in its range as $\boldsymbol{O}_3$ does. In fact, it cannot have $\mathrm{Th}(N)$ in its range. But we can get a Σ_2^0-cado $\boldsymbol{O}_4$ which is strict and has every $T \in \mathrm{Com}(\boldsymbol{P})$ in its range. The domain of $\boldsymbol{O}_4$ is $\{M \mid M \models \mathrm{Con}(\boldsymbol{P}) \wedge \neg\mathrm{Con}(\boldsymbol{P} + U)\}$. The definition of $\boldsymbol{O}_4$ is based on a variant of Lemma 3.5 where one codes the given $A \subseteq N$ by a Δ_2^0-definable element of M. This

is achieved by a slight complication of the choice of formulas to be placed on the tree.

To conclude, we present some more problems and remarks. We have seen that the alternating sentence of a bounded Δ_2^0 cado is unique up to (provable) equivalence in $\boldsymbol{P}$. On the other hand, if $\boldsymbol{O}$ is a total operator ($\mathrm{dom}(\boldsymbol{O}) = \mathrm{Mod}(\boldsymbol{P})$) and χ is an alternating sentence for $\boldsymbol{O}$ so is $\neg\chi$, so a total cado (like $\boldsymbol{O}_1$ above) has at least two nonequivalent alternating sentences. But what about *literal* alternating sentences? Adapting the usage of Kreisel and Takeuti [KT] to our context we introduce

3.6. DEFINITION. A literal self-referential sentence for a formula $\psi(x)$ is a sentence χ such that [not only $\boldsymbol{P} \vdash \chi \leftrightarrow \psi(\ulcorner\chi\urcorner)$, but] χ is literally of the form $\psi(t)$ where t is a closed term (say of primitive recursive arithmetic) whose numerical value is $\ulcorner\chi\urcorner$. If (δ, Nat, Sat) is a triple of formulas defining a cado $\boldsymbol{O}$ then by a literal alternating sentence for $\boldsymbol{O}$, w.r.t. (δ, Nat, Sat), we mean a literal self-referential sentence for the formula $\delta \wedge \mathrm{False}(x)$ where $\mathrm{False}(x)$ is $\neg\mathrm{Sat}(x, \langle\ \rangle)$.

Note that the standard construction of a self-referential sentence for a formula $\psi(x)$ gives a literal self-referential sentence. In particular, every cado has a literal alternating sentence.

3.7. *Problem.* Is the literal alternating sentence of a cado $\boldsymbol{O}$, w.r.t. a given definition (δ, Nat, Sat) of $\boldsymbol{O}$, unique up to equivalence in $\boldsymbol{P}$?

The problems discussed in this paper have obvious analogues where Peano's number theory $\boldsymbol{P}$ is replaced by a system of analysis or set theory. G. Kreisel has pointed out to us that the main result of Friedman [F] may be interpreted as an analogue of our finiteness result (Theorem 2.5), where models of $\boldsymbol{P}$ are replaced by ω-models of analysis and the complete diagram of $\boldsymbol{O}(M)$ is defined in M by a fixed analytical definition. There is no restriction on the complexity of the three formulas defining the domain of $\boldsymbol{O}$, the collection of sets of $\boldsymbol{O}(M)$ and the satisfaction relation of $\boldsymbol{O}(M)$. (The natural numbers of $\boldsymbol{O}(M)$ are the same as those of M—the standard natural numbers. But the sets of $\boldsymbol{O}(M)$ are supposed to be represented by natural numbers in M, so that the *free* variables x and y of $\mathrm{Nat}(x)$ and $\mathrm{Sat}(x, y)$ are number variables.)

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