A New Derandomization of Auctions

Oren Ben-Zwi, Ilan Newman, and Guy Wolfovitz

Department of Computer Science, University of Haifa,
Mount Carmel, Haifa 31905, Israel
nbenzv03|ilan|gwolfovi@cs.haifa.ac.il

Abstract. Let A be a randomized, unlimited supply, unit demand, single-item auction, which given a bid-vector $b \in [h]^n$, has expected profit $\mathbb{E}[P(b)]$. Aggarwal et al. showed that given A, there exists a deterministic auction which given a bid-vector b, guarantees a profit of $\mathbb{E}[P(b)]/4 - O(h)$. In this paper we show that given A, there exists a deterministic auction which given a bid-vector b of length n, guarantees a profit of $\mathbb{E}[P(b)] - O(h\sqrt{n \ln hn})$. As is the case with the construction of Aggarwal et al., our construction is not polynomial time computable.

1 Introduction

For our good fortune, we were hired to design a mechanism for handling the upcoming 'world cup' TV broadcasts. We are given a two sided communication with the (numerous) potential costumers, the marginal cost for adding one viewer is negligible, and our primary goal is to maximize our revenue. The classical approach for maximizing the revenue on scenarios like this is to set up a fixed price, and charge it from any viewer. However, the price can be fixed too high, causing a smaller number of viewers, or too low, causing a low price collecting from each viewer. Either way, the overall revenue might be too low.

This motivates the study of an *unlimited supply, unit demand, single item* auction. These auctions can guarantee a revenue which is a constant approximation to the *best single price revenue* (which is not necessarily known). In this paper we study the derandomization of such *truthful* auctions. Goldberg et al. [5] introduced randomized auctions that achieve on expectation a constant fraction approximation of the optimal single price revenue.[1] They named these auctions *competitive* after the notion of *competitive analysis* of *online algorithms*. They also proved that randomization is essential assuming the auction is symmetric (that is, assuming the outcome of the auction does not depend on the order of the input

[1] Actually, they looked on optimal single price where there are at least two buyers, see [5] for farther details.

bids). Aggarwal et al. [1] later showed how to construct from any randomized auction a deterministic, asymmetric auction with approximately the same revenue. More accurately, given a randomized auction A which accepts bid-vectors in $[1, h]^n$, they constructed a deterministic, asymmetric auction A_D satisfying $P_{A_D}(b) \geq \mathbb{E}[P_A(b)]/4 - O(h)$ for every $b \in [1, h]^n$; here $P_{A_D}(b)$ is the profit of A_D given a bid-vector b and $\mathbb{E}[P_A(b)]$ is the expected profit of A given a bid-vector b. The same result also holds in the more restrictive case where A accepts bid-vectors in $[h]^n$. In addition, Aggarwal et al. showed that if the bid-vectors are restricted to be vectors of powers of 2 then the multiplicative factor of 4 above can be improved to 2.

In this paper we show that in the case where the bid-vectors come from $[h]^n$, one can improve the construction of A_D above so as to guarantee a better lower bound for $P_{A_D}(b)$, for the cases where $P_{A_D}(b) = \omega(h\sqrt{n \ln hn})$. Formally we prove the following.

Theorem 1. *Let A be a randomized auction which accepts bid-vectors in $[h]^n$. Assume that A has expected profit $\mathbb{E}[P_A(b)]$ for every bid-vector $b \in [h]^n$. Then there exists a deterministic auction A_D that guarantees a profit of $P_{A_D}(b) \geq \mathbb{E}[P_A(b)] - O(h\sqrt{n \ln hn})$ for every bid-vector $b \in [h]^n$.*

The proof of Theorem 1 can be outlined roughly as follows. Given a randomized auction A, we first define a distribution over a set of deterministic auctions. We then show that if we choose a deterministic auction A_D from that distribution at random, then $\mathbb{E}[P_{A_D}(b)] = \mathbb{E}[P_A(b)]$ for every bid-vector b (where the expectancy on the left-hand side is w.r.t. the choice of A_D and the expectancy on the right-hand side is w.r.t. the coin tosses of A). In addition to that, our distribution has the property that the event Bad_b, that $P_{A_D}(b) < \mathbb{E}[P_{A_D}(b)] - t$, depends on a relatively few number of other events $Bad_{b'}$. Moreover, for every b, we have that the probability of Bad_b is sufficiently small. We then apply the Lovász Local Lemma to show that there exists a choice for A_D for which none of the events Bad_b occur. For our choice of t, this will give the theorem.

We stress the fact that the result of Aggarwal et al. [1] is more general in the sense that it deals with bid-vectors in $[1, h]^n$, while Theorem 1 only deals with discrete bid-vectors. Still, discrete bid-vectors make more sense in real life auctions, where for example, bids are being made in Dollars and Cents. We also note that the construction used in the proof of Theorem 1 is not polynomial time computable and that this is also the case in the construction of Aggarwal et al. [1].

2 Preliminaries

Definition 1 (Unlimited supply, unit demand, single item auction). *An* unlimited supply, unit demand, single item auction *is a mechanism in which there is one* item *of unlimited supply to sell by an* auctioneer *to n* bidders*. The bidders place* bids *for the item according to their* valuation *of the item. The auctioneer then sets prices for every bidder. If the price for a bidder is lower than or equal to its bid, then the bidder is considered as a* winner *and gets to buy the item for its price. A bidder with price higher than its bid does not pay nor gets the item. The auctioneer's* profit *is the sum of the winners prices.*

For a natural number k, let $[k]$ denote the set $\{1, 2, ..., k\}$. A bid-vector $b \in [h]^n$ is a vector of n bids in $[h]$. For $b \in [h]^n$ we denote by b_{-i} the vector which is the result of replacing the ith bid in b with a question mark; that is, b_{-i} is the vector $(b_1, b_2, \ldots, b_{i-1}, ?, b_{i+1}, \ldots b_n)$.

A *truthful auction* is an auction in which every bidder bids its true valuation for the item. It is well known that truthfulness can be achieved through *bid-independent* auctions (see for example [4]). A *bid-independent* auction is an auction for which the auctioneer computes the price for bidder i using only the vector b_{-i} (that is, without the ith bid).

2.1 A structural lemma

Let A be a randomized truthful auction that accepts bid-vectors from $[h]^n$. We can view A's execution in the following manner. The auction maintains a set of nm functions $\{g_{i,j}\colon i \in [n], j \in [m]\}$, where $g_{i,j}$ is a function from vectors in $([h] \cup \{?\})^n$ with exactly one question mark to $[h]$. On a bid-vector $b \in [h]^n$, the auction tosses some coins, and chooses accordingly an integer $j \in [m]$. We let p_j be the probability that $j \in [m]$ was chosen. The auction then offers bidder i the price $g_{i,j}(b_{-i})$. Let $\text{accept}_{i,j}(b)$ be 1 if $g_{i,j}(b_{-i}) \leq b_i$ and 0 otherwise. The expected profit of the auction on input b is then:

$$\mathbb{E}[P_A(b)] = \sum_j p_j \sum_i \text{accept}_{i,j}(b) \cdot g_{i,j}(b_{-i}).$$

One can define the following randomized auction A', which is equivalent to the above randomized auction A with respect to expected profits. First, A' maintains the exact same list of functions as A. On a bid-vector $b \in [h]^n$, the auction performs the following independently for every $i \in [n]$: it tosses the same coins that A does, chooses accordingly an integer $j \in [m]$

and then offers the ith bidder price $g_{i,j}(b_{-i})$. The expected profit of A' on input b is given by:

$$\mathbb{E}[P_{A'}(b)] = \sum_i \sum_j p_j \cdot \text{accept}_{i,j}(b) \cdot g_{i,j}(b_{-i}).$$

We call A' the *bidder-self-randomness-dual* of A. The following clearly follows from the discussion above.

Lemma 1. *Let A be a randomized auction and A' be its bidder-self-randomness-dual auction. Then A and A' have the same expected profit on every bid-vector.*

2.2 Probabilistic Tools

The proof of Theorem 1 makes use of the Lovász Local Lemma [3]. We need the following version of the lemma [2].

Lemma 2 (The local lemma; symmetric case). *Let Bad_i, $1 \le i \le N$, be events in an arbitrary probability space. Suppose that each event Bad_i is mutually independent of a set of all the other events Bad_j but at most d, and that $\Pr[Bad_i] \le p$ for all $1 \le i \le N$. If $ep(d+1) \le 1$, where e is the base of the natural logarithm, then $\Pr[\bigwedge_{i=1}^{N} \neg Bad_i] > 0$.*

Let X be the average of n independent random variables X_i, where $X_i \in [a_i, b_i]$ for all i. We will need the following inequality [6]:

Lemma 3 (Hoeffding). $\Pr[X < \mathbb{E}[X] - t] \le 2\exp\left(\frac{-2n^2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$

3 Proof of Theorem 1

Let A be a randomized auction which accepts bid-vectors in $[h]^n$. Let $\{g_{i,j}\colon i \in [n], j \in [m]\}$ be the set of functions that A maintains. We construct a deterministic auction A_D. To do that, we first define a certain tripartite graph $G = (\mathit{Lft}, \mathit{Cntr}, \mathit{Rght}, E)$. Using this tripartite graph, we define a distribution over deterministic auctions. We will then obtain a deterministic auction A_D by choosing an auction at random according to that distribution. The proof of Theorem 1 will follow by showing that A_D satisfies the conclusion in the theorem with positive probability.

We first describe the tripartite graph G. We let Lft be the set of all nh^{n-1} vectors b_{-i}, where $b \in [h]^n$ is a bid-vector and $i \in [n]$. We let Cntr be the set of all pairs $\{(b_{-i}, g_{i,j}(b_{-i})) : i \in [n], j \in [m]\}$. We let

Rght be the set of all possible h^n bid-vectors in $[h]^n$. The edges E are defined as follows. A vertex $b_{-i} \in Lft$ is connected to all the vertices in the set $\{(b_{-i}, g_{i,j}(b_{-i})) \in Cntr\colon j \in [m]\}$. A vertex $(b_{-i}, g_{i,j}(b_{-i})) \in Cntr$ is connected to all bid-vectors $r \in Rght$ for which it holds that $b_{-i} = r_{-i}$ and $\mathrm{accept}_{i,j}(r) = 1$.

Observe that every subgraph G' of G in which every $b_{-i} \in Lft$ has exactly one adjacent edge induces naturally a deterministic auction A_D. To see that this is indeed the case, consider such a subgraph G' of G. The deterministic auction A_D behaves as follows: on a bid-vector $b \in [h]^n$, the price offered to the ith bidder is $g_{i,j}(b_{-i})$ if and only if $\{b_{-i}, (b_{-i}, g_{i,j}(b_{-i}))\}$ is an edge in G'.

Let G' be a subgraph of G chosen in the following way. Independently, for every $b_{-i} \in Lft$, choose a random edge $\{b_{-i}, (b_{-i}, g_{i,j}(b_{-i}))\}$ according to the distribution $\{p_j\}_{j=1}^{m}$. Let A_D be the deterministic auction that is naturally induced by G'. Note that for every bid-vector $b \in [h]^n$,

$$\mathbb{E}[P_{A_D}(b)] = \sum_i \sum_j p_j \cdot \mathrm{accept}_{i,j}(b) \cdot g_{i,j}(b_{-i}),$$

which by Lemma 1, is equal to $\mathbb{E}[P_A(b)]$.

Let Bad_b be the event that $P_{A_D}(b) < \mathbb{E}[P_{A_D}(b)] - t$, where we define $t := 10h\sqrt{n \ln hn}$. We need the following two claims.

Claim 1. For all $b \in [h]^n$, $\Pr[Bad_b] < 1/(10hn)$.

Proof. Fix $b \in [h]^n$ and let X_i be the profit extracted from bidder i, that is, $X_i = \mathrm{accept}_{i,j}(b) \cdot g_{i,j}(b_{-i})$ (recall that j is determined by A_D). Note that $X_i \in [1, h]$ for all i and that the X_i's are independent random variables. Let X be the average of the X_i's. We have

$$\Pr[Bad_b] = \Pr[P_{A_D}(b) < \mathbb{E}[P_{A_D}(b)] - t] = \Pr[X < \mathbb{E}[X] - t/n],$$

which by Lemma 3 is at most $2\exp\left(\frac{-2t^2}{h^2 n}\right)$. The claim now follows since $t = 10h\sqrt{n \ln hn}$. □

Claim 2. For all $b \in [h]^n$, Bad_b depends on at most hn other events $Bad_{b'}$.

Proof. Fix $b \in [h]^n$. It is enough to show that there are at most hn vertices $b' \in Rght$ with the following property: there is a vertex $b_{-i} \in Lft$ such that there is a path of length 2 from b_{-i} to b and from b_{-i} to b'. Indeed, for the vertex $b \in Rght$, there are at most n vertices $b_{-i} \in Lft$ which are at distance 2 from b. In addition, for every $b_{-i} \in Lft$ there are at most h vertices $b' \in Rght$ which are at distance 2 from b_{-i}. □

Combining the two claims above with the Lovász Local Lemma, we get that with positive probability Bad_b does not occur for all $b \in [h]^n$. Hence, with positive probability, for every bid-vector $b \in [h]^n$, $P_{A_D}(b) \geq \mathbb{E}[P_{A_D}(b)] - t$. This proves the theorem.

References

1. G. Aggarwal, A. Fiat, A. V. Goldberg, J. D. Hartline, N. Immorlica, and M. Sudan. Derandomization of auctions. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*, pages 619–625, 2005.
2. N. Alon and J. Spencer. *The Probabilistic Method (3rd edition)*. Wiley, New-York, 2008.
3. P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. *Infinite and Finite Sets (A. Hajnal et al., eds.)*, pages 609–628, 1975.
4. A. Fiat, A. V. Goldberg, J. D. Hartline, and A. Karlin. Competitive generalized auctions. In *Proceedings of the 34th Annual ACM Symposium on Theory of Computing (STOC)*, ACM Press, New York, 2002.
5. A. V. Goldberg, J. D. Hartline, and A. Wright. Competitive auctions and digital goods. In *Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 735–744, 2001.
6. Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30.