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Hard Metrics From Cayley Graphs Of Abelian Groups

Ilan Newman   Yuri Rabinovich*

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Abstract: Hard metrics are the class of extremal metrics with respect to embedding into Euclidean spaces: they incur $\Omega(\log n)$ multiplicative distortion, which is as large as it can possibly get for any metric space of size n. Besides being very interesting objects akin to expanders and good error-correcting codes, and having a rich structure, such metrics are important for obtaining lower bounds in combinatorial optimization, e. g., on the value of MinCut/MaxFlow ratio for multicommodity flows.

For more than a decade, a single family of hard metrics was known (see London et al. (Combinatorica 1995) and Aumann and Rabani (SICOMP 1998)). Recently, a different family was found by Khot and Naor (FOCS 2005).

In this paper we present a general method of constructing hard metrics. Our results extend to embeddings into negative type metric spaces and into ℓ_1.

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1 Introduction

A famous theorem of Bourgain [4] states that every metric space (X,d) of size n can be embedded into a Euclidean space with multiplicative distortion[1] at most $O(\log n)$. We call a metric space *hard* if any embedding of V into a Euclidean space (of any dimension) has a multiplicative distortion $\Omega(\log n)$.

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[1]The precise definition appears in Section 2.

When studying a special class of metric spaces, perhaps the most natural first question is whether this class contains hard metrics. Many fundamental results in the modern Theory of Finite Metric spaces may be viewed as a negative answer to this question for some special important class of metrics. E. g., Arora et al. [2] (improving on Chawla et al. [5]) show this for Negative Type metrics, Rao [12] for planar metrics, and Gupta et al. [6] for doubling metrics. For a long time (since Linial, London and Rabinovich [10] and Rabani and Aumann [3]), the only known family of hard metrics was, essentially, the shortest-path metrics of constant-degree expander graphs. It was even conjectured that in some vague sense this is always the case. Recently, however, Khot and Naor [9] constructed a different family of hard metrics by considering certain quotient spaces of $\mathbb{Z}_2^n$ equipped with the Hamming distance.

The starting point of the current research was a plausible conjecture that a *circular* metric cannot be hard, where by circular we mean a metric on the underlying space $\mathbb{Z}_n$, such that $d(a,b)$ depends solely on $((a-b) \mod n)$. Rather surprisingly, the conjecture turns out to be false, and, moreover, it fails not only for $\mathbb{Z}_n$, but for *any* Abelian group H. More precisely, it is always possible to choose a set A of generators for H, so that the shortest-path metric of the corresponding Cayley graph $G(H,A)$ is hard. In the special case of $\mathbb{Z}_2^n$, good sets of generators are closely related to error-correcting codes of constant rate and linear distance.

Our construction is both simple to describe and easy to analyze. It differs from that of [10, 3], as the degree of such Cayley graphs is necessarily non-constant. It is more general than the construction of [9], since the latter, despite very different description and analysis, can be shown to produce the same metric space as does our construction in the special case of $\mathbb{Z}_2^n$.

Note: Although in what follows we restrict the discussion to Euclidean spaces, the same method assert the hardness of the metrics that we construct also with respect to the much richer metric space NEG, and consequently to ℓ_1.

2 Definitions

Let (X,d) be a metric space which one wants to embed into another metric space $A = (H,\nu)$. The *multiplicative distortion*, or simply the *distortion* of embedding (X,d) into A is defined as

$$c_A(d) \;=\; \mathrm{dist}(\mathrm{d} \hookrightarrow \mathrm{A}) \;=\; \min_{\phi:\mathrm{X}\to\mathrm{H}} \max_{\mathrm{x,y}\in\mathrm{X}} \frac{\nu(\phi(\mathrm{x}),\phi(\mathrm{y}))}{\mathrm{d}(\mathrm{x,y})} \cdot \max_{\mathrm{x,y}\in\mathrm{X}} \frac{\mathrm{d}(\mathrm{x,y})}{\nu(\phi(\mathrm{x}),\phi(\mathrm{y}))}\,.$$

We use the terms Euclidean metrics and ℓ_2-metrics interchangeably. The Negative Type metrics, NEG, mentioned in the Introduction, are the squares of Euclidean metrics that satisfy the triangle inequality.

3 General Abelian Groups

Let $G = (V,E)$ be a d-regular connected graph on n vertices, and let μ_G be its shortest-path metric. Our first step is to get a general lower bound on distortion of embedding μ_G into an Euclidean space. We use a standard (dual) method of comparing the so-called Poincare forms (see, e. g., [10, 11], with further details therein). For $\Delta : V(G)\times V(G) \to \mathbb{R}^+$, consider the following projective quadratic form:

$$F(\Delta) \;=\; \frac{\sum_{(i,j)\in E(G)} \Delta^2(i,j)}{\sum_{i<j\in V(G)} \Delta^2(i,j)} \tag{3.1}$$

Then,

$$F(\mu_G) \;=\; \frac{|E|}{\binom{n}{2}\mathrm{avg}(\mu_G^2)}\,,$$

where $\mathrm{avg}(\mu_G^2)$ is the average value of $\mu_G^2(i,j)$ over all pairs of vertices of G.

Consider now a Euclidean metric on $V(G)$, $\delta \in \ell_2$, namely, a metric of the form

$$\delta(i,j) \;=\; \|x^i - x^j\|_2\,,\quad \left\{x^i\right\}_{i\in V(G)} \subset \mathbb{R}^m\,.$$

If $F(\delta)$ is much larger than $F(\mu_G)$ for *every* such δ, one immediately concludes that any such δ must significantly distort μ_G. Formally,

Proposition 3.1.

$$\mathrm{dist}^2(\mu_G \hookrightarrow \ell_2) \;\geq\; \min_{\delta\in\ell_2} F(\delta)/F(\mu_G)\,.$$

By a standard argument (see e. g., [11], Sect. 15.5), the minimum of $F(\delta)$ over all such δ's is precisely γ_G/n, where γ_G is the *spectral gap* of G, that is, $(d-\lambda_G)$ where λ_G is the second largest eigenvalue of the adjacency matrix of G. Thus, Proposition 3.1 implies,

Proposition 3.2.

$$\mathrm{dist}^2(\mu_G \hookrightarrow \ell_2) \;\geq\; \frac{n-1}{n}\cdot\frac{\gamma_G}{d}\cdot \mathrm{avg}(\mu_G^2)\,.$$

In particular,

Corollary 3.3. *If a graph G has a constant normalized spectral gap γ_G/d, and* $\mathrm{avg}(\mu_G^2) = \Omega(\log^2 n)$*, then the above method yields a $\Omega(\log n)$ lower bound on the distortion of embedding μ_G into a Euclidean space.*

In what follows we shall deal solely with graphs for which $\mathrm{avg}(\mu_G^2) \approx \mathrm{Diam}(G)^2$ (here the approximation is by any constant factor). We note that, in particular, any vertex-transitive graph has this property. Indeed, let r be the smallest radius such that the corresponding r-ball in μ_G contains at least $n/2$ vertices. Clearly, $\mathrm{avg}(\mu_G^2) \geq r^2/2$, while $\mathrm{Diam}(G) \leq 2r+1$. For such graphs, it suffices to ensure a constant normalized spectral gap, and a $\Omega(\log n)$ lower bound on the diameter.

Turning to Cayley graphs, it is well known that for (some) non-Abelian groups, there exist Cayley graphs with *constantly many* generators, and a constant spectral gap (see, e. g., [7], the section on Cayley expander graphs). Since the constant number of generators guarantees that the diameter is $\Omega(\log n)$, this yields a graph as required in Corollary 3.3. (This is precisely the construction used in [10, 3]). For Abelian groups such construction is impossible, since in order to ensure a constant normalized gap γ_G/d, the number of generators must be at least $\Omega(\log n)$ (see, e. g., [7]). This might seem to be a problem, since, at least for general groups, that many generators may well cause the diameter to be $O(\log n/\log\log n) = o(\log n)$. For Abelian groups, however, this does not happen! While the following simple fact is well known (see, e. g., [7], proof of Prop. 11.5), it has been apparently overlooked in the context of hard metrics.

Let $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$ be the binary entropy function. For an Abelian group H, a set $A \subseteq H$ is called symmetric if $A = -A$ (we use the additive notation for the Abelian group operation).

Proposition 3.4. *Let H be an Abelian group of size n, and let $A \subset H$ be a symmetric set of generators of size $d = c_0 \log_2 n$. Then, for any constant c_1 such that $(c_0 + c_1) \cdot h(c_1/(c_0 + c_1)) < 1$, the diameter of the corresponding Cayley graph $G(H,A)$ is $\geq c_1 \log_2 n$ for a large enough n.*

The proposition follows from the observation that the number of distinct endpoints of paths of length l in G starting at any (fixed) vertex is at most $\binom{d+l}{l}$, since due to commutativity of G it is at most the number of partitions of a set of l (identical) elements to d (distinct) parts. Therefore, the number of points reachable by a path of length $\leq c_1 \log_2 n$ from a fixed vertex is at most

$$\sum_{l=0}^{c_1 \log_2 n} \binom{c_0 \log_2 n + l}{l} = 2^{h\left(\frac{c_1}{c_0+c_1}\right)\cdot(c_0+c_1)\cdot\log_2 n + o(\log n)} =$$

$$n^{(c_0+c_1)\cdot h\left(\frac{c_1}{c_0+c_1}\right) + o(1)} < n.$$

Thus, as long as the number of generators is $O(\log n)$, our only concern is getting a constant normalized spectral gap γ_G/d. This is summed up in the following theorem.

Theorem 3.5. *Let H be an Abelian group of size n, let $A \subset H$ be a symmetric set of generators of size $d = c_0 \log_2 n$ for a suitable universal constant c_0 (100 would certainly suffice) and let $G(H,A)$ be the corresponding Cayley graph. If the normalized spectral gap $\gamma_G/|A| = \Omega(1)$, then μ_G is a hard metric.*

It is well known that a random construction achieves this goal (see, e. g., [1], in particular the section on Abelian groups):

Proposition 3.6. *Let H be an Abelian group of size n, and let $A \subset H$ be a random symmetric set of generators of size $d = c_0 \log_2 n$ for a suitable universal constant c_0 (100 would certainly suffice). Then, the corresponding Cayley graph $G(H,A)$ is almost surely connected, and has a normalized spectral gap ≥ 0.5.*

To prove the proposition, one needs first to realize that the eigenvectors of G are the *characters* of H, i. e., functions χ from H to the unit circle in $\mathbb{C}$, such that $\chi(a+b) = \chi(a) \cdot \chi(b)$. In particular, all such functions with the exception of the constant one (that corresponds to the eigenvalue d), sum up to 0. From here it is little more than an application of the Chernoff Bound. For an efficient deterministic construction of such A's see [13].

Combining Theorem 3.5 and Proposition 3.6, we arrive at the main result of this section:

Theorem 3.7. *Let $G = G(H,A)$ be a Cayley graph obtained by taking a random symmetric set of generators $A \subset H$ of size $d = c_0 \log_2 |H|$ for a suitable universal constant c_0. Then, the shortest-path metric of G is almost surely a hard metric.*

4 When the Group is $\mathbb{Z}_2^n$

In this case the group is just an n-dimensional vector space over $\mathbb{Z}_2$. Any set of generators (vectors) A is automatically symmetric. Following the requirements of Corollary 3.3, we have to ensure three conditions: a constant normalized spectral gap, connectivity of $G(\mathbb{Z}_2^n, A)$, and $\Omega(n)$ diameter.

The construction is based on good linear codes. Let $\mathcal{C} \subset \mathbb{Z}_2^m$ be a linear code of *dimension* n, that is, $\mathcal{C}$ is generated by a set of n linearly independent m-dimensional vectors. The *distance* $D(\mathcal{C})$ of $\mathcal{C}$ is the minimum number of 1's in any $c \in \mathcal{C}$. $\mathcal{C}$ is said to be of linear distance if $D(\mathcal{C}) = \Omega(m)$. In addition, if $n = \Omega(m)$ the code is said to have a *constant rate*.

Let M be an $n \times m$ matrix whose rows form a basis for $\mathcal{C}$ (such an M is called the generator matrix of $\mathcal{C}$) and let $A \subset \mathbb{Z}_2^n$, $|A| = m$, be the set of columns M. It is easy to see that for any such linear code, the graph $G(\mathbb{Z}_2^n, A)$ is connected due to the fact that the rank of M is n.

Proposition 4.1. *Let $\mathcal{C}$ be a linear code of linear distance and let M and A be the corresponding matrix and the set of vectors as above. Then the normalized spectral gap γ_G/n of $G(\mathbb{Z}_2^n, A)$ is constant. Conversely, any A with this property is necessarily the set of columns of a generator matrix of a linear code with linear distance.*

The proposition is folklore (see e. g. [1], proof of Proposition 2). Here is a sketch of the proof.

Proof. The characters $\{\chi_u\}$ of $\mathbb{Z}_2^n$, indexed by the group elements $u \in \mathbb{Z}_2^n$, are of the form

$$\chi_u(x) \;=\; (-1)^{\langle u,x\rangle}\,,$$

where the inner product (with a slight abuse of notation) is (mod 2). Let $A \subset \mathbb{Z}_2^n$, $|A| = m$, be a set of generators (vectors), and let M_A be an $n \times m$ matrix over $\mathbb{Z}_2$ whose columns are the vectors of A. For a vector in $v \in \mathbb{Z}_2^m$ let $w(v)$ be the number of 1's in v. Let 0 denote the n-dimensional 0-vector. Keeping in mind that the eigenvectors of $G(\mathbb{Z}_2^n, A)$ are the characters, we conclude that the second largest eigenvalue λ_G of $G(\mathbb{Z}_2^n, A)$ is

$$\lambda_G \;=\; \max_{u \in \mathbb{Z}_2^n - \{0\}} \sum_{a \in A} (-1)^{\langle u,a\rangle} \;=\; \max_{u \in \mathbb{Z}_2^n - \{0\}} \left\{ m - 2w(u^T M_A) \right\}\,.$$

Let $\mathcal{C} \subseteq \mathbb{Z}_2^n$ be a linear code generated by M_A, that is, all linear combinations of the rows of M_A. Then $\mathcal{C} = \{u^T M_A\}_{u \in \mathbb{Z}_2^n} \subset \mathbb{Z}_2^m$ and hence $\lambda_G = m - 2D(\mathcal{C})$. Since $\gamma_G = m - \lambda_G$, it follows that $\gamma_G \;=\; 2D(\mathcal{C})$. Therefore, $\gamma_G = \Omega(m)$ if and only $\mathcal{C}$ is a linear code of linear distance. □

It remains to ensure that the diameter of $G(\mathbb{Z}_2^n, A)$ is $\Omega(n)$. By Proposition 3.4, this condition will necessarily hold provided $m = O(n)$, that is, if $\mathcal{C}$ is of constant rate. Thus,

Theorem 4.2. *Let $\mathcal{C}$ be a linear code of constant rate and linear distance, and* $\dim(\mathcal{C}) = n$. *Let M be an $n \times m$ matrix whose rows form a basis for $\mathcal{C}$, and let $A \subset \mathbb{Z}_2^n$, be the set of M's columns. Then the metric of $G(\mathbb{Z}_2^n, A)$ is hard.*

Such codes are at the core of the Coding Theory and they have received considerable attention. Their existence has been established by numerous randomized and deterministic efficient constructions, with the first explicit construction due to Justesen [8].

We conclude this section with a comparative study of the construction of hard metrics due to Khot and Naor [9]. Let $\mathcal{C} \subset \mathbb{Z}_2^m$ be a linear code of constant rate and linear distance, of dimension n. Let $\mathcal{C}^\perp$ be the dual code, i. e., $\mathcal{C}^\perp = \{u | Mu = 0\}$ where M is the generator matrix of $\mathcal{C}$. Define an equivalence relation on $\mathbb{Z}_2^m$ by $x \equiv y$ if $(x - y) \in \mathcal{C}^\perp$. Now, let X be a quotient metric space of Z_2^m equipped with the Hamming metric, with respect to $\equiv$. That is, the distance between two points a and b in X is the Hamming distance between the two corresponding cosets $A, B \subset \mathbb{Z}_2^m$. Khot and Naor show that X with the induced metric is hard.

Proposition 4.3. *The above construction is isometric to the construction described in Theorem 4.2.*

Proof. Let M be a matrix as in Theorem 4.2. Then X can be viewed as the image of $\mathbb{Z}_2^m$ under the linear mapping $\phi : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$, $\phi(x) = Mx$. Define the *edges* of X as the images of Hamming edges of $\mathbb{Z}_2^m$ under ϕ. Clearly, the quotient metric of X is precisely the shortest-path metric of the resulting graph. The images of the Hamming edges are, however, precisely the column vectors of M, and the isometry follows. □

5 Additional Remarks

The constructions of Cayley graphs with hard shortest-path metric as described in Theorem 3.7 and Theorem 4.2, yield graphs of degree logarithmic in the number of vertices. It is natural to ask whether this must hold for all Cayley graphs of Abelian groups that induce a hard metric. Here we partially answer this question and show that the degree can be anything between $\Omega(\log n)$ and $O(n^{1-\varepsilon})$ for any fixed $1 > \varepsilon > 0$.

We start with the following simple fact:

Proposition 5.1. *Let H be an Abelian group, and let $m < |H|$ be a natural number. Then, there exists a symmetric set $B \subseteq H$ of size $\Theta(m)$ such that for every natural r the size of $rB = \{\sum_{i=1}^{r} b_i \mid b_i \in B\}$ is at most $r \cdot |B|$.*

Proof. Any finite Abelian group is a direct product of cyclic groups. Let $H = C_1 \times C_2 \times \ldots \times C_t$, and assume that $s_j = |C_1| \cdot |C_2| \cdot \ldots |C_j| < m$, while $s_{j+1} = |C_1| \cdot |C_2| \cdot \ldots |C_j| \cdot |C_{j+1}| \geq m$. Let a be a generator of C_{j+1}. Define $K_{j+1} = \{i \cdot a \mid i \in [-k, k] \subset \mathbb{N}\}$ where k is the smallest natural number such that $s_j \cdot (2k + 1) \geq m$. Finally, define $B = C_1 \times C_2 \times \ldots C_j \times K_{j+1} \times \{0\} \times \{0\} \ldots$ It is easy to verify that B has the required properties. □

Theorem 5.2. *Let H be an Abelian group of size n, and let $1 > \varepsilon > 0$ be fixed. Then there exists a symmetric set of generators A of size $\Theta(n^{1-\varepsilon})$ such that the metric μ_G of the Cayley graph $G = G(H, A)$ is hard.*

Proof. Let $G' = G'(H, A)$ be a Cayley graph as in Theorem 3.7 (or Theorem 4.2), with $|A| = c_0 \log_2 n$. Assume for simplicity that A is augmented by $\{0\}$. Let $B \subseteq H$ be as in Proposition 5.1 with $m = n^{1-\varepsilon}$. We claim that the Cayley graph $G = G(H, A \cup B)$ has the desired properties. To see that, we employ the Poincare form $F'(\Delta)$ similar to $F(\Delta)$ of (3.1), where in the numerator we use the edges of G' instead of the edges of G. Arguing as in Proposition 3.2, we conclude that

$$\mathrm{dist}^2(\mu_G \hookrightarrow \ell_2) \;\geq\; \frac{n-1}{n} \cdot \frac{\gamma_{G'}}{d'} \cdot \mathrm{avg}(\mu_G^2)\,. \tag{5.1}$$

We already know that the normalized spectral gap of G' is constant. Thus, it will suffice to show that the diameter of G is logarithmic. A closer examination of the proof of Proposition 3.4 reveals that the number of distinct endpoints of paths of length $\le c_{\varepsilon/2}\log_2 n$ starting at a fixed vertex in G', is at most $n^{\varepsilon/2+o(1)}$, provided that $(c_0+c_{\varepsilon/2})\cdot h(c_{\varepsilon/2}/(c_0+c_{\varepsilon/2}))\le \varepsilon/2$. Therefore, the number of points reachable by a path of length at most $r = c_{\varepsilon/2}\log_2 n$ in G is

$$|r\cdot\{A\cup B\}| \;\le\; |rA|\cdot|rB| \;\le\; |rA|\cdot r|B| \;\le\; n^{\varepsilon/2}\cdot n^{1-\varepsilon}\cdot\Theta(n^{o(1)}) \;<\; n\,.$$

(*At most* r rather than *exactly* r since both A and B contain 0.) Thus, the diameter of G is at least $c_{\varepsilon/2}\log_2 n$. This concludes the proof. □

The last issue we would like to address in this paper is the following. As the proof of Theorem 5.2 shows, the hardness of μ_G can be deduced from the hardness of $\mu_{G'}$, where G' is a sparse subgraph of G. It is natural to ask whether the hardness of μ_G itself, where G is constructed as in Theorem 3.7, can also be traced to a simple hard subgraph G' of G. What is the "core" of the hardness? It turns out that G does indeed contain such a subgraph! To avoid technicalities, we bring here only a broad outline of the argument.

First, a purely graph-theoretic argument implies that

Proposition 5.3. *G contains as a subgraph an expander of a bounded degree (say ≤ 500) and size $\Omega(n)$.*

Indeed, by Cheeger's Inequality, see e.g. [7], which relates the normalized spectral gap to the normalized edge-expansion, G has edge expansion $\ge d/4$ where d is the degree of G. Choosing each edge of G randomly and independently from the others with probability $100/d$, we obtain a graph $\tilde{G}$ which almost has the required properties. Recall that d is large enough ($\approx 100\log n$), and hence, by Chernoff inequality, almost surely all sufficiently large subsets of vertices S, say $|S|\ge n/4$, will have an edge-boundary of size at least $c_2\cdot 100\cdot|S|$ in $\tilde{G}$, for some constant c_2 (e.g., $c_2 = 1/8$ suffices). Let D be the set of all vertices of degree more than 500. By Chernoff, this set is of size at most $10^{-100}\cdot n$ and has edge boundary of size at most $10^{-50}n$. Thus, in $G'' = \tilde{G}-D$ all sets S of vertices, of size, say $|S|\ge n/3$, will have an edge-boundary of size at least $c_3\cdot 100\cdot|S|$, where c_3 could be taken to be e.g., $1/9$. Next, remove one by one the subsets of vertices U that have less than $c_3\cdot 100\cdot|S|$ outgoing edges in the remaining part of the graph. Since the union W of all removed U's has less than $c_3|W|$ outgoing edges, we conclude that the size of W is at most $n/3$. Thus, the the graph $G' = G''\setminus W$ is almost surely a subgraph of G of size $\ge n/2$, degree ≤ 500, and a edge expansion $\ge c_3\cdot 100$. Of course, G' is not a Cayley graph anymore, it is not even regular.

Next, having such a large subgraph G' in G we get,

Proposition 5.4. *The existence of such G', combined with the property of G (as in Proposition 3.4), that the radius of a μ_G-ball of size $n^{\Omega(1)}$ in G is at least $\Omega(\log n)$, implies the hardness of μ_G.*

Indeed, let μ' denote the restriction of μ_G to $V(G')$. The hardness of μ' can be asserted by employing the Poincare form $F_{G'}(\Delta)$ as in Equation (3.1), and using the expansion of G' to get, via Cheeger's inequality, a lower bound on the first eigen value of the Laplacian of G'.

Now, using the same form $F_{G'}(\Delta)$ as in Equation (3.1), this time for μ_G, we conclude that the square of the distortion of μ_G is at least $\mathrm{dist}^2(\mu_{G'}\hookrightarrow \ell_2)\cdot\frac{\mathrm{avg}(\mu_G^2)}{\mathrm{avg}(\mu_{G'}^2)}$. Since both $\mathrm{avg}(\mu_G^2),\mathrm{avg}(\mu_{G'}^2)$ are $\theta(\log^2 n)$, the hardness of μ_G follows.

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AUTHORS[2]

Ilan Newman [About the author]
Department of Computer Science
Haifa University, Haifa, 31905, Israel
ilan@cs.haifa.ac.il
http://cs.haifa.ac.il/~ilan

Yuri Rabinovich [About the author]
Department of Computer Science
Haifa University, Haifa, 31905, Israel
yuri@cs.haifa.ac.il
http://cs.haifa.ac.il/~yuri

ABOUT THE AUTHORS

ILAN NEWMAN has been on the faculty at Haifa University since 1992. He obtained his B. S. and M. S. at the Technion. He received his Ph. D. on 1992, at the Computer Science department of the Hebrew University. The thesis title is: "On the complexity of simple Boolean functions" and was done under the supervision of Avi Wigderson. His current research interests are in finite metric embeddings, combinatorial algorithms, property testing, and computational complexity.

YURI RABINOVICH Born in Odessa, Ukraine, Yuri received his Ph. D. in the theory algorithms from the Hebrew University at 1993. The thesis title is: "Theoretical properties of genetic algorithms" and was done under the supervision of Avi Wigderson and Nati Linial. Currently he works at the Computer Science Department of the Haifa University. He is interested in geometrical-like phenomena in discrete structures in general, and in the theory of finite metric spaces in particular.

[2] To reduce exposure to spammers, THEORY OF COMPUTING uses various self-explanatory codes to represent "AT" and "DOT" in email addresses.