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High order schemes for the Helmholtz equation in 3D

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We consider the Helmholtz equation in 3D: $\Delta u + k^2 u = F$, where $u(x, y, z)$ is the displacement at point (x, y, z), $\Delta u = u_{xx} + u_{yy} + u_{zz}$ and k is the wave number. This equation has applications in many fields, such as acoustics and electromagnetism. In geophysics, methods of full wave inversion reconstruct the structure of some volume of the earth from data received by many sensors recording the reflection of sound waves. The frequency domain approach to this reconstruction requires the repeated solution of the Helmholtz equation, in which the RHS function F represents an impact source, so it has a small finite support.

Due to dispersion errors (also known as the pollution effect), it is well known that problems with wave propagation are solved more efficiently by using high order accurate methods. In addition to high order accuracy it is also useful to consider only compact stencils (3×3×3 in 3D). This has several advantages. Foremost, is that no non-physical boundary conditions need be considered. In addition this reduces the bandwidth of the matrix to be inverted. Another practical advantage is that huge 3D domains require parallel solvers, and compact schemes lend themselves more easily to parallel domain decomposition techniques, such as the CARP-CG algorithm [2] used in this work.

The main difficulty in applying a compact high order finite difference scheme to solve the Helmholtz equation on a geophysical problem is that large geophysical domains consist of different types of materials with different acoustic properties, i.e., the wave number k varies within the domain. In this work we use the compact 6th order scheme of [4], which was developed for variable k and shown to be far superior to lower order schemes.

However, high order finite difference schemes are not enough for correctly simulating the wave equation. The numerical solution is carried out in a finite domain, so in order simulate an infinite domain, we need to impose so-called absorbing boundary conditions (ABCs) that will inhibit reflections from the boundaries. One of the first ABCs was formulated by Engquist and Majda. An equivalent BC was later formulated by Higdon

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[3]. For spherical domains, a series of absorbing boundary conditions was developed in [1]. We have developed several compact high order schemes based on these ABCs.

In order to evaluate the various ABCs, we ran experiments on a domain with a constant k. The source was simulated by a function of the type $F = A(\cos(\pi x/L)\cos(\pi y/L)\cos(\pi z/L))^n$ for $-L/2 \leq x, y, z \leq L/2$ and zero elsewhere. A is the amplitude, $n \geq 5$, and L determines the size of the support. Experiments were ran with various values of the frequency f.

It is well known that for constant k, the analytic solution is obtained by a convolution of F and G, where G is the Green's function for 3D. The convolution integral was calculated numerically at a finer grid than the one used for the domain, and compared with the solutions obtained by various ABCs. Our results show that the solutions with the ABCs of [1] are significantly better than those with of [3].

In order to determine the quality of the ABCs in absolute terms, we compared the results with the results when Dirichlet boundary conditions were applied to the exterior boundaries. The values on the boundaries were determined by the values of the convolution on the boundaries.

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