LINEAR SYSTEMS WITH LARGE OFF-DIAGONAL ELEMENTS AND DISCONTINUOUS COEFFICIENTS

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OUTLINE

• Linear systems with large off-diagonal elements
• Linear systems with discontinuous coefficients
• Large off-diagonal elements and discontinuous coefficients
• Sample results
• Conclusions
LARGE OFF-DIAGONAL ELEMENTS

Some examples:
1. Convection-diffusion(-reaction) PDEs with LARGE convection terms
2. Helmholtz equation with LARGE wave numbers
3. Some circuit problems

Simple solution methods:
- Scaling and/or reordering – may leave some large off-diagonal elements
- Robust but not efficient: Kaczmarz (sequential), and the block-parallel CARP (G&G 2005), type 1 problems
- Robust and efficient results: CGMN algorithm (Björck & Elfving 1979), type 1 problems (G&G 2008)
- Parallel setting: similarly good results with CARP-CG (G&G 2009)
- Good preliminary results also on type 2 problems
\( L_2 \)-SCALING AND THE NORMAL EQUATIONS

- Given a system \( Ax = b \), let \( G = \text{diag}(1/\|a_i\|_2) \)
- Consider the system \( GAx = Gb \) – call it GRS preconditioning
- GRS = Geometric Row Scaling – iterative results depend only on the hyper-planes and not any particular algebraic representation
- Denote \( C = GA, \ b' = Gb \implies Cx = b' \)
- Consider the normal system \( CC^Ty = b', \ x = C^Ty \)
- Theorem: Let \( D = CC^T \), then \( |d_{ii}| = 1 \), and \( |d_{ij}| < 1 \) for \( i \neq j \)
- Proof: immediate (assuming \( A \) is full rank)
- Significance: A method to control large off-diagonal elements
GRS + NORMAL EQUATIONS ARE INHERENT IN:

- Kaczmarz (SOR on normal equations) – inherently sequential
- Cimmino – inherently parallel
- CGMN: CG acceleration of Kaczmarz (Björck & Elfving, 1979)
- CG-acceleration of Cimmino, equivalent to CGNR+GRS
- CARP: a block-parallel version of Kaczmarz (G&G 2005)
- CARP-CG: CG acceleration of CARP (G&G 2009)
DISCONTINUOUS COEFFICIENTS

• “Discontinuous coefficients” – linear systems with large differences between the coefficients of different equations

• Typically arise when PDEs model certain physical phenomena in heterogeneous media, e.g. flow through different materials

• Common approach: Domain Decomposition (DD)

• Problem: May be difficult to implement when:
  – Unstructured grid
  – Complicated boundaries between subdomains
ALTERNATIVE APPROACH: ROW AND/OR COLUMN SCALING

- Not always useful
- Theoretical results, e.g., van der Sluis 1969
- Widlund 1971: “well-scaled ADI methods give good rates of convergence when the coefficients of elliptic problems vary very much in magnitude”
- Duff & van der Vorst 1998: “on vector machines, diagonal scaling is often competitive with other approaches”
- Graham & Hagger 1999: “diagonal scaling has been observed in practical computations to be very effective as a preconditioner for problems with discontinuous coefficients”
- Gambolati et al. 2003: use the least square logarithm scaling on rows and columns for geomechanics problems with discontinuous coefficients
PREVIOUS WORK ON SCALING (G&G 2009):

- Problems considered: nonsymmetric systems w/discontinuous coefficients, from PDEs with small to moderate convection terms
- Considered $L_2$ scaling of the equations (GRS)
- Results:
  1. improved convergence behavior of Bi-CGSTAB and (restarted) GMRES, both with and without ILU(0)
  2. Improved eigenvalue distribution (large concentration “pushed” away from origin)
  3. Degradation of usefulness of GRS as the convection was increased
- Note: $L_1$ scaling produced similar results
PROBLEM 1

• Based on a 3D (symmetric) problem of Graham & Hagger 1999

• Original problem solved by DD techniques

• We added convection terms to make it nonsymmetric

• PDE: \[- \frac{\partial}{\partial x}(au_x) - \frac{\partial}{\partial y}(au_y) - \frac{\partial}{\partial z}(au_z) + du_x + eu_y + fu_z = 0\]

  where \( a(x, y, z) = \begin{cases} D & \text{if } \frac{1}{3} < x, y, z < \frac{2}{3}, \\ 1 & \text{otherwise} \end{cases} \quad D = 10^4 \text{ and } D = 10^6 \]

• Domain: \([0, 1]^3\)

• Convection terms: \( d = e = f = 100 \)

• Dirichlet boundary conditions (\( u = 1 \) on \( z = 0 \), \( u = 0 \) elsewhere)

• Discretizations: \( 40 \times 40 \times 40 \) and \( 80 \times 80 \times 80 \)
PROBLEM 1: EIGENVALUE DISTRIBUTION

Problem 4: Eigenvalue distribution of original matrix

Problem 4: Eigenvalue distribution of matrix with L2 scaling
PROBLEM 1: RUNTIMES (SEC.)

\[ D = 10^6, \text{ grid: } 80 \times 80 \times 80 \]
PROBLEM 1: DEGRADATION WITH INCREASED CONVECTION

<table>
<thead>
<tr>
<th>Method / Convection:</th>
<th>100</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bi-CGSTAB</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>with GRS</td>
<td>1.2E-12</td>
<td>1.7E-11</td>
<td>3.4E-13</td>
<td>—</td>
</tr>
<tr>
<td>Bi-CGSTAB+ILU(0)</td>
<td>2.5E-14</td>
<td>3.2E-5</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>with GRS</td>
<td>2.5E-14</td>
<td>4.1E-5</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>GMRES</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>with GRS</td>
<td>1.8E-5</td>
<td>2.7E-5</td>
<td>5.6E-5</td>
<td>—</td>
</tr>
<tr>
<td>GMRES+ILU(0)</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>with GRS</td>
<td>1.2E-5</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Notes: ‘—’ means no convergence.
Numbers are the best relative error obtained.
# Problem 1: Solutions for Large Convection

<table>
<thead>
<tr>
<th>Convection:</th>
<th>100</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convergence goal:</td>
<td>$10^{-4}$ $10^{-7}$ $10^{-10}$</td>
<td>$10^{-4}$ $10^{-7}$ $10^{-10}$</td>
<td>$10^{-4}$ $10^{-7}$ $10^{-10}$</td>
<td>$10^{-4}$ $10^{-7}$ $10^{-10}$</td>
</tr>
<tr>
<td>Bi-CGSTAB+GRS</td>
<td>1.0 2.8 3.6</td>
<td>2.8 7.2 9.9</td>
<td>9.7 21.0 38.2</td>
<td>-- -- --</td>
</tr>
<tr>
<td>Bi-CGSTAB+ILU(0) with GRS</td>
<td>1.4 1.6 1.7</td>
<td>2.3 -- --</td>
<td>-- -- --</td>
<td>-- -- --</td>
</tr>
<tr>
<td>GMRES+GRS</td>
<td>2.1 -- --</td>
<td>2.2 -- --</td>
<td>2.8 -- --</td>
<td>-- -- --</td>
</tr>
<tr>
<td>GMRES+ILU(0)+GRS</td>
<td>0.8 -- --</td>
<td>-- -- --</td>
<td>-- -- --</td>
<td>-- -- --</td>
</tr>
<tr>
<td>CGNR+GRS (note 2)</td>
<td>5.1 10.4 11.3</td>
<td>4.8 9.5 10.2</td>
<td>6.0 12.4 13.4</td>
<td>7.3 15.0 16.2</td>
</tr>
<tr>
<td>CGNR+GCS (note 3)</td>
<td>9.4 10.4 11.3</td>
<td>8.7 9.4 10.2</td>
<td>11.3 12.4 13.4</td>
<td>13.7 14.9 16.1</td>
</tr>
<tr>
<td>CGMN (note 4)</td>
<td>2.0 4.3 4.6</td>
<td><strong>1.9</strong> <strong>4.6</strong> <strong>5.0</strong></td>
<td><strong>2.0</strong> <strong>5.8</strong> <strong>6.3</strong></td>
<td><strong>2.5</strong> <strong>6.9</strong> <strong>7.6</strong></td>
</tr>
</tbody>
</table>

Notes:

1. Times in seconds; minimal times in boldface
2. CGNR+GRS is CG-accelerated Cimmino
3. GCS = $L_2$ Column Scaling
4. CGMN is also CARP-CG on one processor
PROBLEM 2

- System(s) with large off-diagonal elements and discontinuous coefficients.
- Solved originally by DD techniques
- PDE: \(-\text{div}(\nu(x)\nabla u) + \vec{b} \cdot \nabla u + u = 0\), with

\[
\nu(x) = \begin{cases} 
10^{-1} & \text{if } x < \frac{1}{2}, \\
10^{-5} & \text{otherwise}.
\end{cases}
\]

- \(\vec{b} = (1,0,0) / (0,1,1) / (1,3,5) / (0,1,0)\)
- Domain: unit cube
- Dirichlet boundary conditions \((u = 1 \text{ on } z = 0, u = 0 \text{ elsewhere})\)
- Discretizations: \(40 \times 40 \times 40\) and \(80 \times 80 \times 80\)
PROBLEM 2 – SUMMARY OF RESULTS

• When \( \vec{b} = (1, 0, 0) \) or \( (0, 1, 0) \) (only one large convection term):
  – Bi-CGSTAB+ILU(0) and GMRES+ILU(0) very efficient
  – GRS helped very little
  – CGMN relatively slow

• When \( \vec{b} = (0, 1, 1) \) or \( (1, 3, 5) \) (two or more large convection terms):
  – GMRES converged very slowly, GRS was quite useful
  – CGMN was very efficient

• Conclusion: the difficulties with large convection terms occur when there are more than two large off-diagonal elements (each convection term contributes two off-diagonal elements with second-order finite difference schemes)
SUMMARY

- Two tools for nonsymmetric systems with discontinuous coefficients:

- Small to moderate off-diagonal elements: $L_2$-scaling with your favorite algorithm/preconditioner

- Also: one or two large off-diagonal elements

- Three or more large off-diagonal elements: CGMN (sequential), CARP-CG (parallel), CGNR+GRS (CG-accelerated Cimmino, sequential or parallel)
SOME LITERATURE


THANK YOU