

Continuous symmetry: a model for human figural perception

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Abstract—Symmetry is usually viewed as a discrete feature: an object is either symmetric or non-symmetric. In this presentation, symmetry is treated as a continuous feature and a continuous measure of symmetry (the *Symmetry Distance*) is defined. This measure can be easily evaluated for any shape or pattern in any dimension. A preliminary study presented here shows that the Symmetry Distance is commensurate with human perceptual experience. Good correlation is found between the continuous symmetry values and the perceived goodness of figures.

1. INTRODUCTION

William Blake's classic poem sings of the 'fearful symmetry' of his celebrated 'tyger'. Why is the symmetry fearful? A possible reason is that although we perceive the biological symmetry, it is not perfect. The tiger still looks like a tiger when you view it in a mirror — just as does a human face — but the left-hand side of the tiger or your face is not precisely the same as the reflection of the right-hand side. Symmetries abound in the biological world and in our visual world; yet most of these symmetries are not perfect. In the visual world, loss of symmetry is further enhanced: Even perfectly symmetric objects lose their exact symmetry when projected onto the image plane or the retina due to occlusion, perspective transformations, or digitization. Clearly, the popular binary conception of symmetry (i.e. an object is either symmetric or is not symmetric) and exact mathematical definitions thereof (Weyl, 1952; Miller, 1972) are inadequate to describe and quantify the symmetries found in the natural world. Needed is a continuous measure of symmetry, one that would be able to quantify the 'amount' of symmetry entailed in Blake's 'tyger'. The development of

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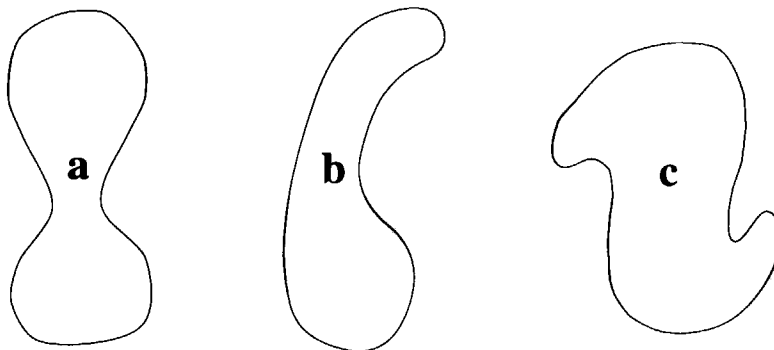


Figure 1. The continuous symmetry measure can compare the ‘amount’ of symmetry of different shapes and can compare the ‘amount’ of different symmetries of a single shape. Thus the notion that **a** is ‘more’ mirror symmetric than **b** can be quantified. Similarly, the notion that **c** is ‘more’ rotationally symmetric than mirror symmetric, can be quantified.

such a measure, and its evaluation against actual perception, form the theme of this article.

In this paper, a ‘Symmetry Distance’, capable of measuring and quantifying all types of continuous symmetries of objects is introduced. Our definition of the Symmetry Distance produces a versatile and simple tool that can supply a set of measures for any object reflecting the amount of different types of symmetry (rotational, reflectional, etc.) possessed by the object. The generality of this symmetry measure allows one to compare the symmetry distance of several objects relative to a single symmetry type and to compare the symmetry distance of a single object relative to various symmetry types. Thus the intuitive notion that the shape of Fig. 1a is ‘more’ mirror-symmetric than the shape of Fig. 1b, can be quantified. Similarly, the intuitive notion that the shape of Fig. 1c is ‘more’ rotationally symmetric (of order two) than mirror symmetric can be quantified.

This concept of continuous symmetry is commensurate with perceptual behavior. To demonstrate this we tested visual evaluation of figural goodness as mediated by symmetry. In Section 2, we define the Symmetry Distance and in Section 3 we briefly describe a method for evaluating the Symmetry Distance. In Section 4 we discuss continuous symmetry in terms of human perception and present some preliminary studies that show a good correlation with human perception of figural goodness.

2. A CONTINUOUS SYMMETRY MEASURE — DEFINITION

We define the *Symmetry Distance* (SD) as the minimum effort required to transform a given object into a symmetric object. This effort is measured by the mean of the square distances taken to move each point of the object from its location in the original object to its location in the symmetric object. Note that no a priori symmetric reference shape is assumed. In effect, we measure the distance from a given object to the set of all symmetric objects.

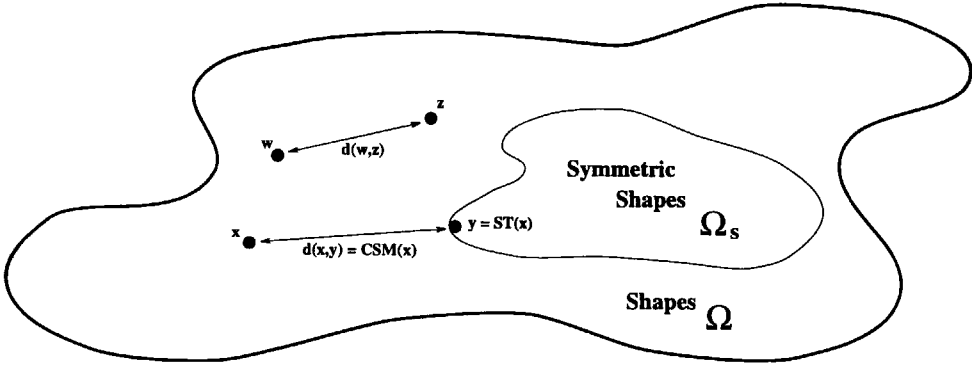


Figure 2. The space Ω of all shapes of a given dimension, where each shape P is represented by a sequence of n points. A metric d is defined on this space which serves as a distance function between every two shapes in Ω .

Denote by Ω the space of all shapes of a given dimension, where each shape P is represented by a sequence of n points $\{P_i\}_{i=0}^{n-1}$ (Fig. 2). We define a metric d on this space as follows:

$$d : \Omega \times \Omega \rightarrow R,$$

$$d(P, Q) = d(\{P_i\}, \{Q_i\}) = \frac{1}{n} \sum_{i=0}^{n-1} \|P_i - Q_i\|^2.$$

This metric defines a distance function between every two shapes in Ω .

We define the *Symmetry Transform* of a shape P , with respect to a given symmetry type G , as the shape \hat{P} which is G -symmetric and closest to P in terms of the metric d .

The *Symmetry Distance* (SD) of a shape P with respect to a given symmetry type G , is now defined as the distance between P and its Symmetry Transform \hat{P} :

$$SD = d(P, \hat{P}).$$

The SD of a shape $P = \{P_i\}_{i=0}^{n-1}$ is evaluated by finding the symmetry transform $\hat{P} = \{\hat{P}_i\}_{i=0}^{n-1}$ of P (Fig. 3d) and computing:

$$SD = \frac{1}{n} \sum_{i=0}^{n-1} \|P_i - \hat{P}_i\|^2.$$

This definition of the Symmetry Distance implicitly implies invariance to rotation and translation. Normalization of the original shape prior to the transformation allows insensitivity to size (Fig. 3). We normalize by scaling the shape so that the maximum distance between points on the contour and the centroid is a given constant (in this paper all examples are normalized to 100). The normalization presents an upper bound on the mean squared distance moved by points of the shape. Thus the SD value is limited in range, where $SD = 0$ for perfectly symmetric shapes.

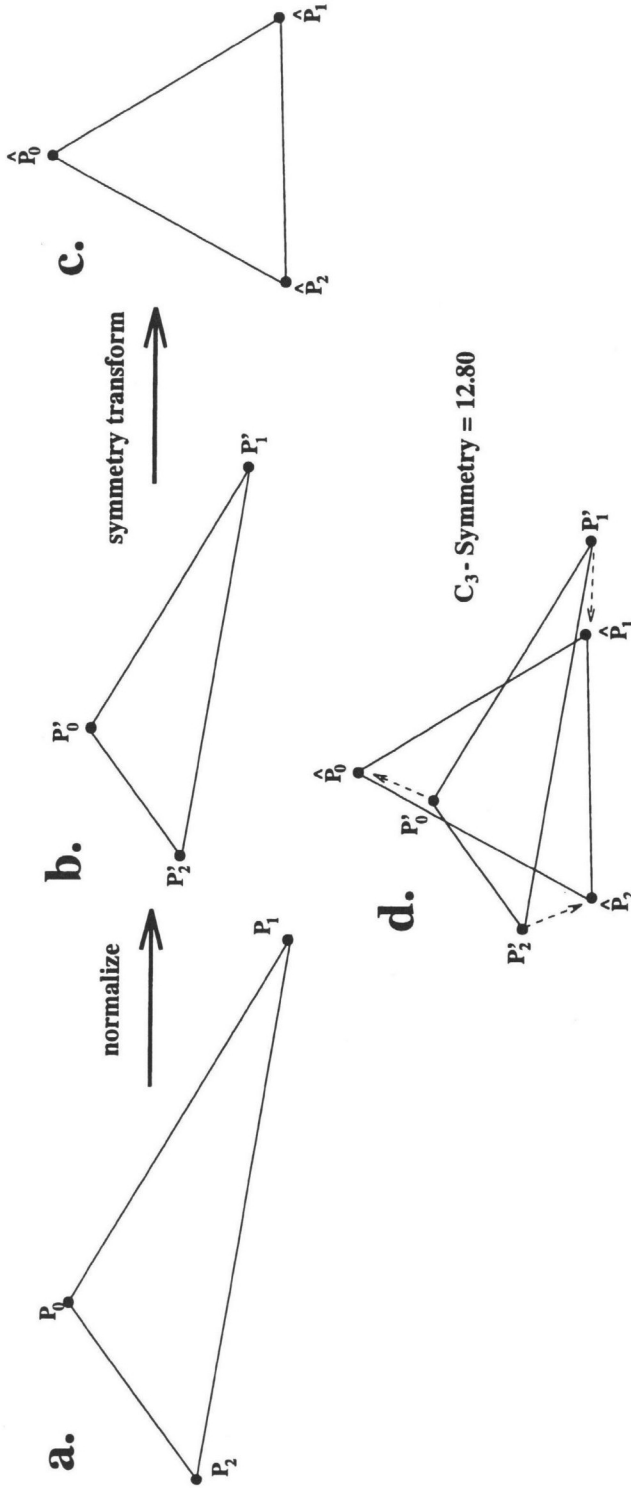


Figure 3. Calculating the Symmetry Distance of a shape: (a) Original shape $\{P_0, P_1, P_2\}$. (b) Normalized shape $\{P'_0, P'_1, P'_2\}$, such that maximum distance to the center of mass is constant (100). (c) Applying the symmetry transform to obtain a symmetric shape $\{\hat{P}_0, \hat{P}_1, \hat{P}_2\}$. (d) Applying the symmetry transform to obtain a symmetric shape $\{\hat{P}_0, \hat{P}_1, \hat{P}_2\}$ such that maximum distance to the center of mass is constant (100). (d) $SD = (\|P'_0 - \hat{P}_0\|^2 + \|P'_1 - \hat{P}_1\|^2 + \|P'_2 - \hat{P}_2\|^2)/3$.

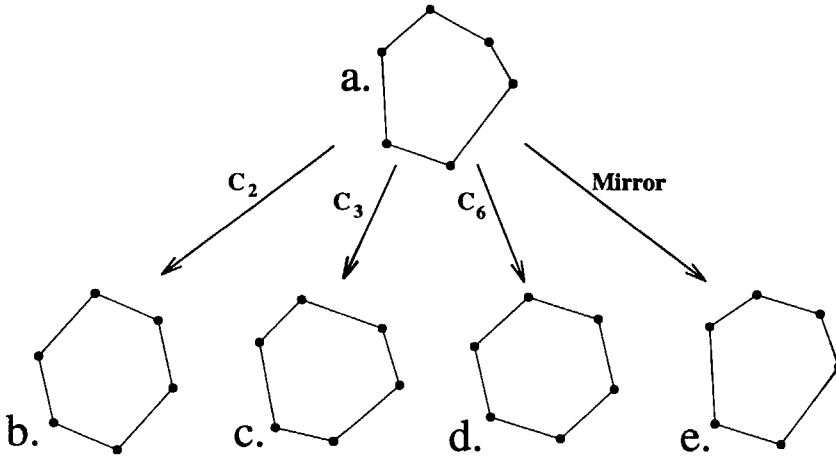


Figure 4. Symmetry Transforms and Symmetry Distances of a 2D polygon. (a) The 2D polygon. (b) Symmetry Transform of (a) with respect to C_2 -symmetry (SD = 1.87). (c) Symmetry Transform of (a) with respect to C_3 -symmetry (SD = 1.64). (d) Symmetry Transform of (a) with respect to C_6 -symmetry (SD = 2.53). (e) Symmetry Transform of (a) with respect to Mirror-symmetry (SD = 0.66).

The general definition of the Symmetry Distance enables evaluation of a given shape for different types of symmetries (mirror-symmetries, rotational symmetries, etc.). Moreover, this generalization allows comparisons between the different symmetry types, and allows expressions such as ‘a shape is more mirror-symmetric than rotationally-symmetric of order two’. An additional feature of the Symmetry Distance is that we obtain the symmetric shape which is ‘closest’ to the given one, the symmetry transform, enabling visual evaluation of SD.

An example of a 2D polygon and its symmetry transforms with respect to various symmetry types and the corresponding SD values are shown in Fig. 4. Note that shape 4e is the most similar to the original shape 4a and, indeed, its SD value is the smallest. The Symmetry Distance and symmetry transform have recently been applied to image processing and chemical applications (Zabrodsky *et al.*, 1993; Zabrodsky and Avnir, 1993).

3. EVALUATING THE SYMMETRY TRANSFORM

A simple geometric algorithm has been derived for evaluating the Symmetry Transform (and accordingly, the Symmetry Distance) of a shape represented by a sequence of points, with respect to any type of symmetry. For simplicity, an outline of the algorithm is given here for the case of rotational symmetry in 2D. For more details and extensions to other types of symmetry in higher dimensions, see Zabrodsky (1993).

Consider the simple case where a shape P is represented by n points and its Symmetry Transform with respect to rotational symmetry of order n (C_n -symmetry) is to be found. The following algorithm finds the C_n -symmetric configuration of points which is closest to P in the mean square sense (i.e. in terms of the metric d defined in Section 2):

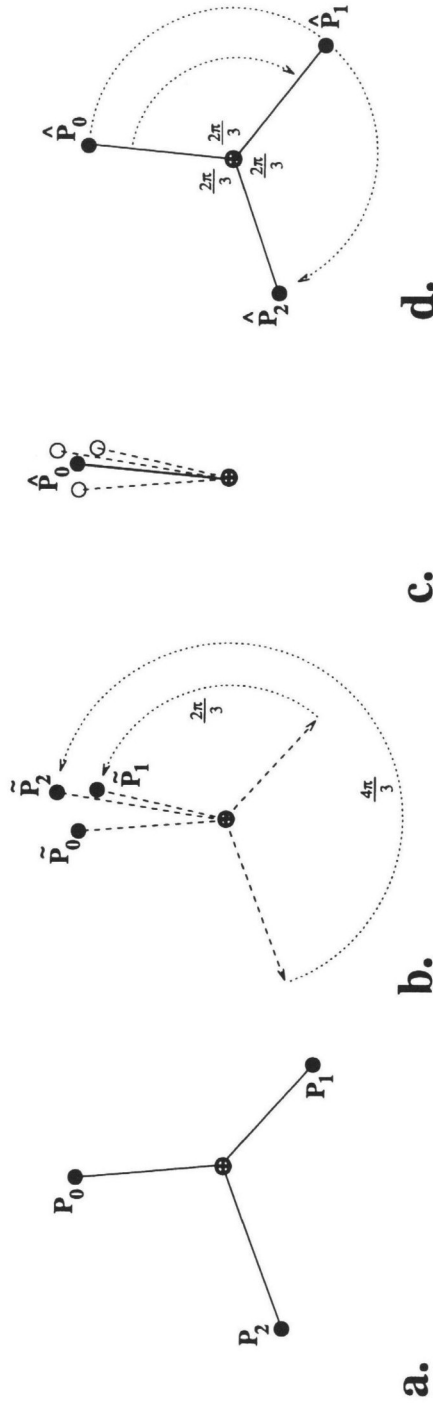


Figure 5. The Symmetry Transform of 3 points with respect to rotational symmetry of order 3. (a) Original 3 points $\{P_i\}_{i=0}^2$. (b) Fold $\{P_i\}_{i=0}^2$ into $\{\tilde{P}_i\}_{i=0}^2$. (c) Average $\{\tilde{P}_i\}_{i=0}^2$ obtaining $\hat{P}_0 = \frac{1}{3} \sum_{i=0}^2 \tilde{P}_i$. (d) Unfold the average point obtaining $\{\hat{P}_i\}_{i=0}^2$. The centroid is marked by \oplus .

Algorithm for finding the C_n -symmetry transform:

1. *Fold* the points $\{P_i\}_{i=0}^{n-1}$ by rotating each point P_i counterclockwise about the centroid by $2\pi i/n$ radians obtaining the points $\{\tilde{P}_i\}_{i=0}^{n-1}$ (Fig. 5b).
2. *Average* the points $\{\tilde{P}_i\}_{i=0}^{n-1}$ obtaining point \hat{P}_0 (Fig. 5c).
3. *Unfold* the points by duplicating \hat{P}_0 and rotating clockwise about the centroid by $2\pi i/n$ radians obtaining the C_n -symmetric points $\{\hat{P}_i\}_{i=0}^{n-1}$ (Fig. 5d).

The set of points $\{\hat{P}_i\}_{i=0}^{n-1}$ is the symmetry transform of the points $\{P_i\}_{i=0}^{n-1}$; i.e. they are the C_n -symmetric configuration of points closest to $\{P_i\}_{i=0}^{n-1}$ in terms of the average distance squared. Proof of the correctness of this algorithm can be found in Zabrodsky (1993).

The common case, however, is that shapes have more points than the order of the symmetry. For symmetry of order n , the folding method can be extended to shapes having a number of points which is a multiple of n . A 2D shape P having qn points is represented as q sets $\{S_r\}_{r=0}^{q-1}$ of n interlaced points $S_r = \{P_{iq+r}\}_{i=0}^{n-1}$. The C_n -symmetry transform of P is obtained by applying the above algorithm to each set of n points separately, where the folding is performed about the centroid of all the points (Fig. 6).

The algorithm described above is general and extends to all symmetry groups in any dimension (see Zabrodsky, 1993). Specifically, for mirror-symmetry in 2D, if the axis of reflection is not specified, the optimal axis can be determined analytically.

As presented here, the input to the algorithm is a set of points; however, given a contour, an image or a 3D volume, the measure of symmetry can be evaluated by selecting points to represent these objects. The selection of the points must be such that the division into sets, as described in the above algorithm, can be performed. This requires every point in the collection of representation points to be matched, under symmetry, to other points in the collection. This is analogous to the correspondence

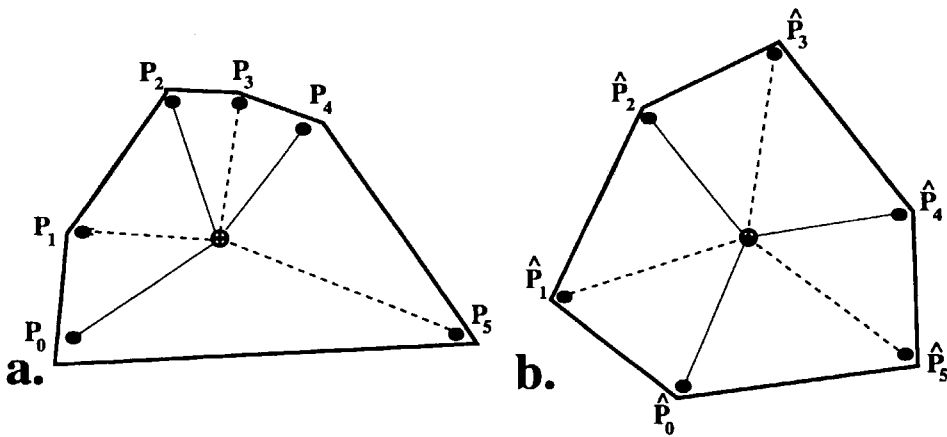


Figure 6. Geometric description of the C_3 -symmetry transform for 6 points. The centroid of the points is marked by \oplus . (a) The original points shown as two sets of 3 points: $S_0 = \{P_0, P_2, P_4\}$ and $S_1 = \{P_1, P_3, P_5\}$. (b) The obtained C_3 -symmetric configuration.

problem in stereo matching and motion determination. We deal with the selection of points by considering an object in two possible ways for the symmetry procedure with the underlying assumptions that (a) all matches of points are in the final collection of representation points, i.e. no points are missing nor are points redundant, and (b) the order of the points (on a contour for example) are preserved under the symmetry transform.

The two ways of considering an object for symmetry are:

1. The shape is a contour with no 'special' points, i.e. points at corners or high curvature points are not considered salient. Thus under the symmetry transform these points are not preserved (corners may round off, protrusions may disappear etc.). In these cases a contour sampling procedure is required. There are several ways to select a sequence of points to represent continuous 2D shapes; one such method is sampling at *equal distances*; however, contour length is not always meaningful, as in noisy or occluded shapes. In such cases, we sample points on a smoothed version of the contour and then project the sampled points back onto the original contour. The level of smoothing can vary and, for a high level of smoothing, the resulting shape becomes almost circular about the centroid in which case the sampling is reduced to sampling the original shape at *equal angles* about the centroid of the shape. (For details and examples of the various sampling procedures see Zabrodsky, 1993.) For any of the approaches mentioned above, it can be shown empirically that, as the density of sampled points increases, the symmetry distance obtained asymptotes to some value.
2. Salient points are considered and should be matched to salient counterparts. In this case the special points (vertices of polygons, corner points, etc.) are all chosen to represent the object. However, because of the above-mentioned assumptions, no spurious points (or unmatched points) are assumed to exist. Future work would extend the topological stage of the symmetry procedure to allow points to be discarded or added into the collection of representation points, in analogy to the correspondence problem, where unmatched feature points would be allowed (see Zabrodsky *et al.*, 1992 where this is partially done by deleting the extraneous points of occluding boundaries).

4. CONTINUOUS SYMMETRY AND HUMAN PERCEPTION

The notion of symmetry was introduced to psychological theorizing and research by Mach and later by Gestalt psychologists in the first half of this century. It has since been associated with the Gestalt constructs of 'good Gestalt' or 'figural goodness', referring to perception of the simplest, most stable figure possible. Given the same number and kind of physical components, some stimuli appear to be unique, simple, regular, or better organized than other stimuli. Those impressions are captured by the quality of 'goodness', the end product of inherent organizational principles that govern the perception of form. In that scheme, symmetry contributes to figural goodness. The contingency between the subjective experience of goodness and the objective feature

of symmetry could not be pursued further, however, because the Gestalt psychologists have made no attempt towards an objective, quantitative analysis of figural goodness.

In a series of seminal papers in the 1950s, Attneave, Hochberg and their associates (e.g. Hochberg and McAlister, 1953; Attneave, 1954, 1955; Attneave and Arnoult, 1957) applied concepts of information theory by way of a more quantitative analysis of figural goodness. Good forms contain portions that are repetitive and predictable from other portions. Cast in informational terms, good forms are redundant, containing little information. A symmetric figure is redundant because parts of it can be predicted from other parts. Of all the possible forms of redundancy, symmetry is the easiest to quantify (cf. Attneave, 1959). Significantly, as Palmer (1991) has pointed out, the informational definition of 'goodness' fully conveys the original Gestalt notions of simplicity, order, and singularity.

Nevertheless, for all its originality and heuristic value, the approach espoused by theorists like Attneave, Hochberg or Berlyne (e.g. Berlyne, 1957, 1966) is limited in a crucial aspect. These theorists applied the informational analysis to components of single stimuli. However, as Palmer notes, such piecewise analysis does not fit well with the Gestalt style of explanation, emphasizing global properties and analyzable wholes. More seriously, in mathematical information theory itself, redundancy applies to sets of stimuli, and is not characteristic of unique stimuli (Garner, 1962). As Garner (1974) observed, 'information is a function not of what the stimulus is, but rather of what it might have been, of its alternatives' (p. 194). This is the major contribution of information theory to psychology: Presented with a stimulus, our perception is influenced by other stimuli that could have been presented. Here, too, lies the challenge to the theorists; namely, specifying the set of those other stimuli or identifying the alternatives that did not appear in any given trial. Garner's theory of *Rotation and Reflection (R & R) Subsets* was suggested to accomplish that goal in a truly Gestalt spirit. Along with Palmer, we believe that Garner's work has laid a solid foundation for future studies of pattern goodness and symmetry, including the present endeavor.

Garner suggested that good patterns are those which have few alternatives. Good patterns produce less variance than do 'bad' patterns when spatial transformations are applied to the figures. Quantitatively, the perceived goodness of a pattern is inversely related to the size of a subset of equivalent patterns that are obtained by applying the transformations. In Garner's view, the observer associates the pattern in question with the subset of its transformational variants. Good patterns come from small inferred subsets and poor patterns come from large ones (cf. Dember and Warm, 1979).

To test this theory, Garner and Clement (1963) prepared dot patterns, similar to those illustrated in Fig. 7 by placing 5 dots in an imaginary 3×3 matrix. They applied four rotations (angles of 0, 90, 180 and 270 deg) and four reflections (horizontal, vertical and two diagonals) to define the *R & R set* for each pattern. The subset of distinguishably different patterns within each set defines the *R & R subset* (cf. Palmer, 1991). Garner and Clement had one group of subjects rate the patterns for 'goodness', and another arrange the patterns into sets on the basis of perceived similarity. The sets did not need to have equal numbers of patterns; indeed, the total number of patterns placed in a set was one parameter of interest. Garner and Clement found that the

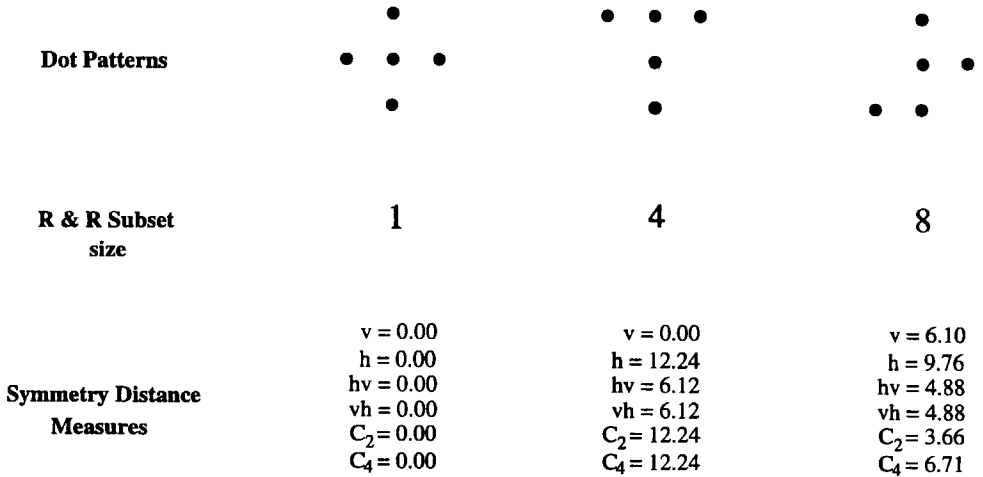


Figure 7. Analysis of figural goodness for dot patterns in terms of R & R subset size, and in terms of Symmetry Distance measures developed here. *v*: vertical mirror symmetry, *h*: horizontal mirror symmetry, *hv*: diagonal top-left to bottom-right mirror symmetry, *vh*: diagonal top-right to bottom-left mirror symmetry, *C*₂: rotational symmetry of order two (120 deg), *C*₄: rotational symmetry of order four (90 deg).

higher the rating for goodness, the smaller is the similarity set. In many additional experiments (summarized in Garner, 1974), Garner and his associates confirmed that goodness of a figure was inversely related to the size of its R & R subset.

Garner did not couch his theory in terms of symmetry because he felt that subsets of equivalent patterns served better for perceptual explanation than 'the more restrictive concept of symmetry' (Garner, 1974, p. 17). In point of fact, however, Garner's R & R sets conform fully to the mathematical definition of symmetry including rotational and mirror symmetries (see also Palmer, 1991). Indeed, Palmer developed an alternative theory of figural goodness focusing on the transformations over which the patterns remain invariant. A figure is symmetrical with respect to a given transformation, if that transformation leaves the figure unchanged. One can list all such transformations, thus specifying the resulting *symmetry subgroups*. Garner's R & R theory only defines the *number* of the transformations, whereas Palmer's symmetry subgroup theory also refers to the *identity* of those transformations. The latter, as research has shown, does matter to perception (e.g. Royer, 1981; Rock, 1983). For instance, it has been repeatedly demonstrated that vertical symmetry influences perception to a greater extent than does horizontal symmetry (Rock and Leaman, 1963; Chipman, 1977; Palmer and Hemenway, 1978).

Our theory of continuous symmetry follows in the footsteps of Garner's and Palmer's contributions. Palmer's construct of symmetry subgroups is well taken, and our analysis provides a rich variety of symmetries, both rotational and reflectional (see again Fig. 7). At the same time, we also share Garner's concern with the limited use of traditional *binary* notion of symmetry. Although Palmer's analysis relates to different types of symmetry, for any given transformation, a figure is still either symmetric or asymmetric. In contradistinction, our theory employs a continuous measure of

symmetry, and it applies to any figure or pattern in two or three dimensions. Indeed, generality of application is the hallmark of our approach.

To serve a useful role, of course, one must show that the present definition of symmetry is commensurate with perceptual experience. The following study provided a preliminary observation using Symmetry Distance for specification of the stimuli. We created 4 sets of 9 'random shapes', of which 9 stimuli (comprising a single set) are shown in Fig. 8. Within a set, each shape had different values of Symmetry Distance, with respect to mirror-symmetry and to rotational symmetry of order 2. The shapes were constructed by randomly choosing one of four possible radius lengths for each of the twelve points forming the shape, and spacing the points at $2\pi/12$ deg angles. Twenty observers provided ratings of goodness for all 36 shapes on a 20-point scale (with 20 and 1 standing for the best and the worst figures, respectively). Following Garner, we left the definition of goodness to the subject's discretion. Despite the completely arbitrary fashion of creating the figures and the small number of subjects, the results are encouraging. As the scatter plots in Fig. 9 show, the continuous measures of mirror symmetry and rotational symmetry covaried with the judgement of goodness. The correlation coefficients were -0.689 and -0.570 , respectively, for reflectional and rotational symmetries. The multiple correlation of these two measures with experienced goodness equaled 0.90. In other words, the continuous measures

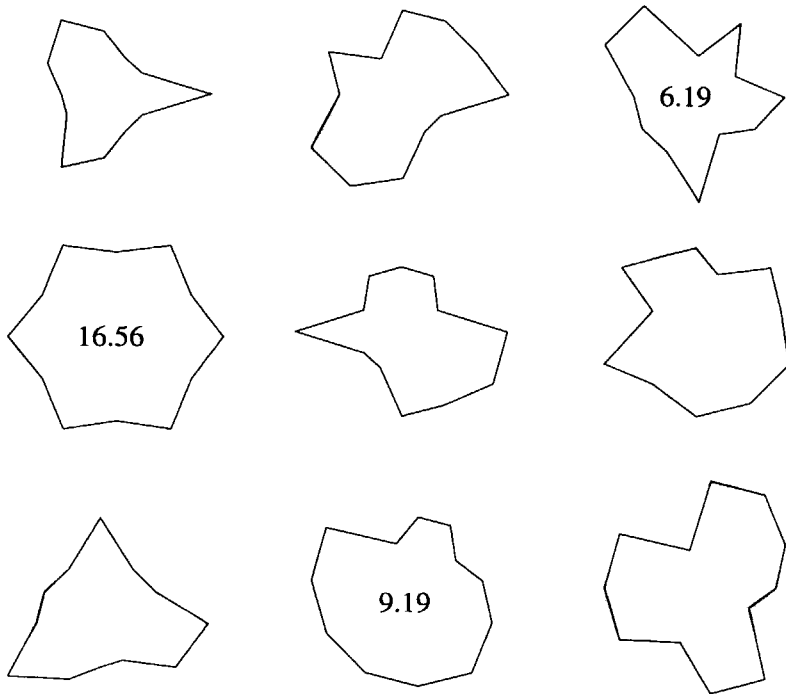


Figure 8. Nine of the thirty-six random shapes for which observers provided ratings of goodness. The shapes were constructed by randomly choosing one of four possible radius lengths for each of the twelve points of the shape and spacing the points at $2\pi/12$ deg angles. The highest, lowest and middle goodness ratings, averaged over all observers, are shown.

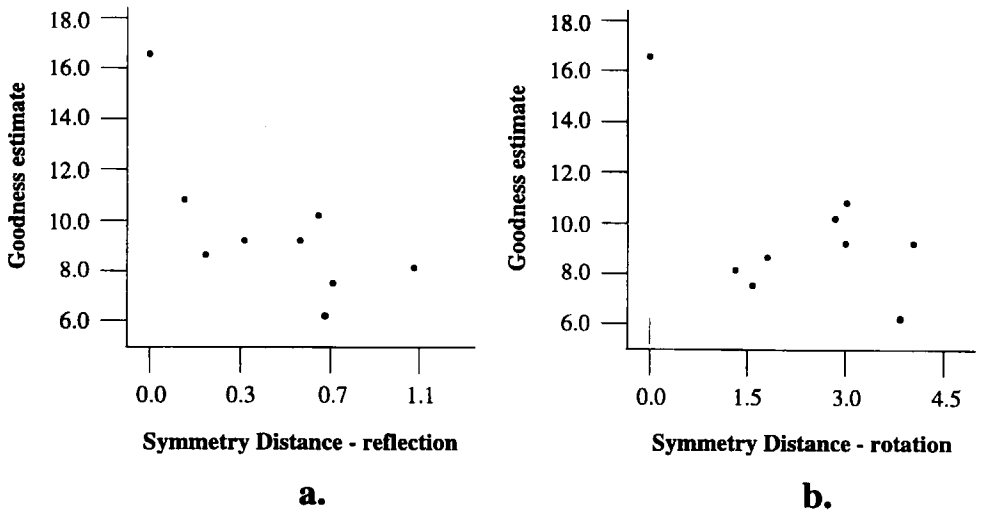


Figure 9. The dependence of perception of goodness on (a) reflectional and (b) rotational measures of Symmetry Distance. The plot displays the result for the set of shapes shown in Fig. 8. The judgements of goodness for each shape are averaged over all observers.

of symmetry accounted for over 80% of the variance of the goodness judgements, although much of that proportion is attributable to one highly symmetric figure.

Again, despite the moderating effect of the small number of subjects, these results are impressive. Equipped with the present measures, the Garnerian notion of inferred equivalence sets readily generalizes to many forms, not just to simple patterns created for the laboratory. So does Palmer's analysis in terms of symmetry subgroups. The results provide powerful support for a general perceptual principle: Figures are processed in terms of their uniqueness measured in units of symmetry distance.

REFERENCES

- Attneave, F. (1954). Some informational aspects of visual perception. *Psychol. Rev.* **61**, 183–193.
- Attneave, F. (1955). Symmetry information and memory for patterns. *Am. J. Psychol.* **68**, 209–222.
- Attneave, F. (1959). *Applications of Information Theory to Psychology*. Holt, Rinehart, and Winston, New York.
- Attneave, F. and Arnoult, M. D. (1957). The quantitative study of shape and pattern perception. *Psychol. Bull.* **53**, 452–471.
- Berlyne, D. E. (1957). Uncertainty and conflict: A point of contact between information-theory and behavioural-theory concepts. *Psychol. Rev.* **64**, 329–339.
- Berlyne, D. E. (1966). Curiosity and exploration. *Science* **153**, 25–33.
- Chipman, S. (1977). Complexity and structure in visual patterns. *J. Exp. Psychol.: General* **106**, 269–301.
- Dember, W. N. and Warm, J. S. (1979). *Psychology of Perception*. Holt, Rinehart, and Winston, New York.
- Garner, W. R. (1962). *Uncertainty and Structure as Psychological Concepts*. Wiley, New York.
- Garner, W. R. (1974). *The Processing of Information and Structure*. Erlbaum, Potomac, MD.
- Garner, W. R. and Clement, D. E. (1963). Goodness of pattern and pattern uncertainty. *J. Verbal Learning and Verbal Behavior* **2**, 446–452.

- Hochberg, J. and McAlister, E. (1953). A quantitative approach to figural 'goodness'. *J. Exp. Psychol.* **46**, 361–364.
- Miller, W. (1972). *Symmetry Groups and their Applications*. Academic Press, London.
- Palmer, S. E. (1991). Goodness, Gestalt, groups, and Garner: Local symmetry subgroups as a theory of figural goodness. In: *The Perception of Structure*. G. R. Lockhead and J. R. Pomerantz (Eds). American Psychological Association, Washington, DC.
- Palmer, S. E. and Hemenway, K. (1978). Orientation and symmetry: Effects of multiple rotational and near symmetries. *J. Exp. Psychol: Human Percept. Perform.* **4**, 691–702.
- Rock, I. (1983). *The Logic of Perception*. MIT Press, Cambridge, MA.
- Rock, I and Leaman, R. (1963). An experimental analysis of visual symmetry. *Acta Psychologica* **21**, 171–183.
- Royer, F. (1981). Detection of symmetry. *J. Exp. Psychol: Human Percept. Perform.* **7**, 1186–1210.
- Weyl, H. (1952). *Symmetry*. Princeton University Press.
- Zabrodsky, H. (1993). *Computational Aspects of Pattern Characterization — Continuous Symmetry*. PhD Thesis, Dept of Computer Science, Hebrew University, Jerusalem, Israel.
- Zabrodsky, H. and Avnir, D. (1993). Measuring symmetry in structural chemistry. In: *Advanced Molecular Structure Research*, Vol. 1. I. Hargittai (Ed.). (In press).
- Zabrodsky, H., Peleg, S. and Avnir, D. (1993). Completion of occluded shapes using symmetry. In: *IEEE Conference on Computer Vision and Pattern Recognition*. New York, pp. 678–679.