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Quantitative Analysis of Continuous Symmetry in Shapes and Objects

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Abstract:

Symmetry is typically viewed as a discrete feature: an object is either symmetric or non-symmetric. However visual perception treats symmetry as a continuous feature, relating to statements such as 'one object is MORE symmetric than another' or 'an object is MORE mirror symmetric than rotational symmetric'. With this notion in mind, we view symmetry as a continuous feature and define a Continuous Symmetry Measure (CSM) to quantify the "amount" of symmetry of different shapes and the 'amount' of different symmetries of a single shape. Computational methods have been developed to compute the CSM values with respect to any point symmetry for any shape or pattern in any dimension. A preliminary study showed that the Symmetry Measure developed, is commensurate with human perceptual experience. The computational approach embeds both the hierarchical and continuous nature of symmetry of objects. Thus global and local features can be evaluated for their symmetry content. The CSM has been applied to real images and to shapes and forms extracted from measured data. The CSM has been extended to deal with symmetries of noisy and fuzzy data as well as reconstructing symmetries of occluded shapes. Additionally, the CSM is used to evaluate symmetries of 3D objects, given as measured data or evaluated from the 2D projection.

1 Introduction

Symmetry is one of the basic features of shape and form. It has been widely studied from various aspects ranging from artistic to mathematical. Symmetry is classically viewed as a binary feature - either an object is symmetric or it is not. However this view is inadequate to describe the symmetries found in the natural world. It is inconsistent with visual perception and natural behavior which treat symmetry as a continuous feature, relating to statements such as "one object is **more** symmetric than another" or "an object is **more** mirror symmetric than rotational symmetric" (see Figure 1). With this notion in mind, we view symmetry as a continuous feature. Accordingly, a Continuous Symmetry Measure (CSM) has been defined [12, 3] which quantifies the "amount" of symmetry of different shapes and the "amount" of different symmetries of a single shape.

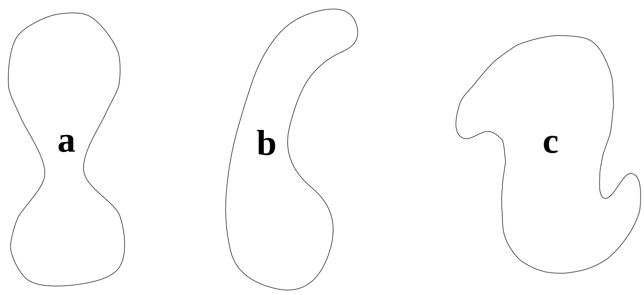

Figure 1: Symmetry as a continuous feature.
Shape **a** is "more" mirror symmetric than shape **b**.
Shape **c** is "more" rotationally-symmetric than mirror symmetric.

2 The Continuous Symmetry Measure (CSM)

The **Continuous Symmetry Measure** (CSM) is defined as the minimum 'effort' required to transform a given shape into a symmetric shape. This 'effort' is measured as the mean squared distance moved by points of the original shape in order to create the symmetric shape. Note that no a priori symmetric reference shape is assumed.

A shape P is represented by a sequence of n points $\{P_i\}_{i=0}^{n-1}$. We define a distance between every two shapes P and Q:

$$d(P,Q) = d(\{P_i\},\{Q_i\}) = \frac{1}{n}\sum_{i=1}^{n} ||P_i - Q_i||^2$$

We define the **Symmetry Transform** $\hat{P}$ of P as the symmetric shape closest to P in terms of the distance d. The **Continuous Symmetry Measure** (CSM) of P denoted $S(P)$ is defined as the distance to the closest symmetric shape:

$$S(P) = d(P, \hat{P})$$

The CSM of a shape $P = \{P_i\}_{i=0}^{n-1}$ is evaluated by finding the symmetry transform $\hat{P}$ of P and computing: $S(P) = \frac{1}{n}\Sigma_{i=0}^{n-1}||P_i - \hat{P}_i||^2$. This definition of the CSM implicitly implies invariance to rotation and translation. Normalization of the original shape prior to the transformation additionally allows invariance to scale.

A geometrical algorithm was developed to find the Symmetry Transform $\hat{P}$, and the CSM of a shape (see [3, 12, 6] for details). Given a finite point-symmetry group G (having n elements) and given a shape P represented by $m = qn$ points, the symmetry transform of the shape with respect to G-symmetry is obtained as follows:

Algorithm for finding the G-symmetry transform:

1. Divide the points into q sets of n points.

2. For every set of n points:

(a) **Fold** the points by applying the elements of the G-symmetry group.

(b) **Average** the folded points, obtaining a single averaged point.

(c) **Unfold** the averaged point by applying the inverse of the elements of the G-symmetry group. A G-symmetric set of n points is obtained.

Perform this procedure over all possible orientations of the symmetry axis and planes of G. Select that orientation which minimizes the Symmetry Distance value. This minimization is analytic in 2D (see [3, 12]) but requires iterative minimization in 3D (except for the 3D mirror-symmetry group where a closed form solution has been derived [6]).

As an example, the Symmetry Transform for rotational symmetry of order n is calculated as follows (see Figure 2):

Algorithm for finding the rotational-symmetry transform of order n:

1. *Fold* the points $\{P_i\}_{i=0}^{n-1}$ by rotating each point P_i counterclockwise about the centroid by $2\pi i/n$ radians (Figure 2b).

2. Let $\hat{P}_0$ be the average of the points $\{\tilde{P}_i\}_{i=0}^{n-1}$ (Figure 2c).

3. *Unfold* the points, obtaining the rotationally symmetric points $\{\hat{P}_i\}_{i=0}^{n-1}$ by duplicating $\hat{P}_0$ and rotating clockwise about the centroid by $2\pi i/n$ radians (Figure 2d).

The set of points $\{\hat{P}_i\}_{i=0}^{n-1}$ is the symmetry transform of the points $\{P_i\}_{i=0}^{n-1}$. i.e. they are the rotationally symmetric configuration of points closest to $\{P_i\}_{i=0}^{n-1}$ in terms of the metric d defined above (i.e. in terms of the average distance squared). Proof is given in [3, 12].

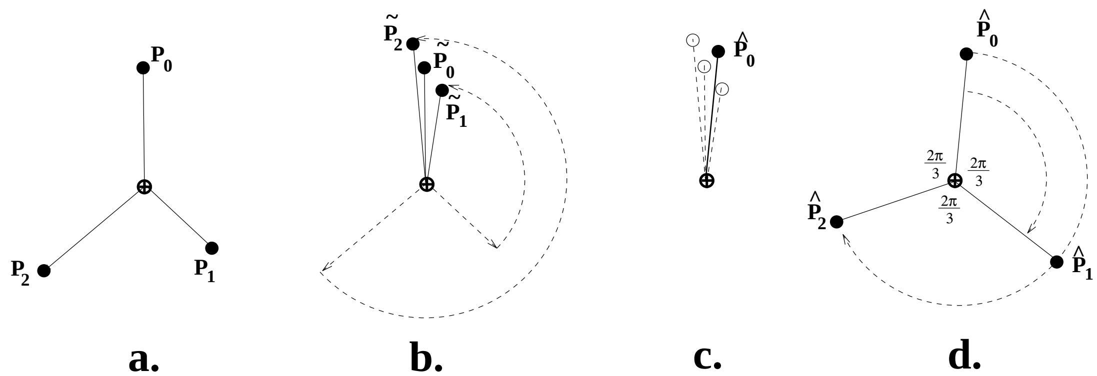

Figure 2: The Symmetry Transform of 3 points with respect to rotational symmetry of order 3.
10.5 a) Original 3 points $\{P_i\}_{i=0}^{2}$ b) Fold $\{P_i\}_{i=0}^{2}$ into $\{\tilde{P}_i\}_{i=0}^{2}$ c) Average $\{\tilde{P}_i\}_{i=0}^{2}$ obtaining $\hat{P}_0 = \frac{1}{3}\sum_{i=0}^{2}\tilde{P}_i$ d) Unfold the average point obtaining $\{\hat{P}_i\}_{i=0}^{2}$
The centroid ω is marked by $\oplus$.

The general definition of the CSM enables evaluation of a given shape for different types of symmetries (mirror-symmetries, rotational symmetries etc) in any dimension. Moreover, this generalization allows comparisons between the different symmetry types, and allows expressions such as “a shape is more mirror-symmetric than rotational-symmetric”.

An additional feature of the CSM is that we obtain the symmetric shape which is ‘closest’ to the given one, enabling visual evaluation of the CSM.

The CSM approach to measuring symmetry allows the hierarchical nature of symmetry to be expressed and quantified, as will be discussed below. Additionally, the CSM method can deal with noisy and occluded data, also discussed below.

3 Measuring the CSM of Shapes, Images and 3D Objects

The versatility of the CSM method has induced its use in various fields such as Chemistry [10, 5], Psychology [4], Archaeology [2] and more. Underlying these studies is the ability of the CSM to evaluate continuous symmetry in shapes, images and objects. Previous approaches to evaluating symmetry in shapes, images and objects which mainly rely on the binary concept of symmetry are reviewed in [12, 3].

3.1 CSM of a Set of Points

Given a set of points, possibly with connectivities between these points, the CSM is calculated using the geometrical algorithm previously developed and described above [12, 3]. Given a shape such as a polygon, the vertices can be used as the input to the algorithms. Figure 3 shows an example of a shape and its Symmetry Transforms and CSM values with respect to several types of symmetries.

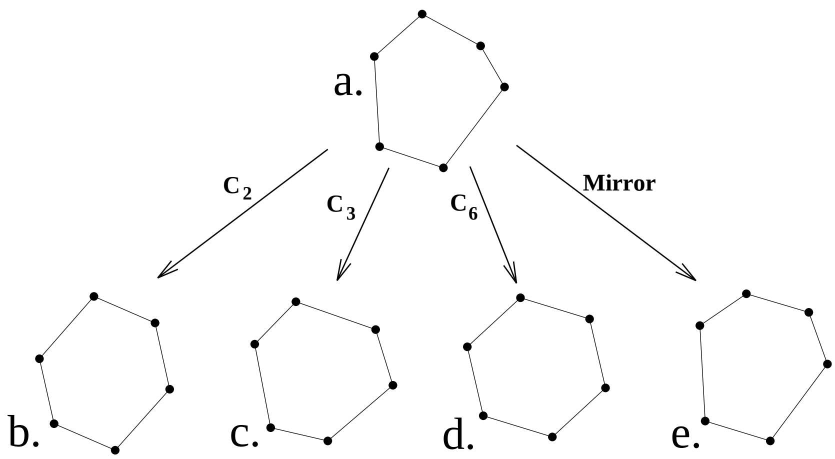

Figure 3: Symmetry Transforms of a 2D polygon.
a) 2D polygon and its Symmetry Transform with respect to b) Rotational symmetry of order 2 (S = 1.87). c) Rotational symmetry of order 3 (S = 1.64). d) Rotational symmetry of order 6 (S = 2.53). e) Mirror-symmetry (S = 0.66).

3.2 CSM of a Continuous Shape

Prior to measuring the CSM of a general continuous shape, the shape must be represented as a collection of points. Figure 4 shows an example of a general shape whose contour has been sampled, producing a collection of representative points for which the Symmetry Transform and CSM value were calculated with respect to mirror symmetry.

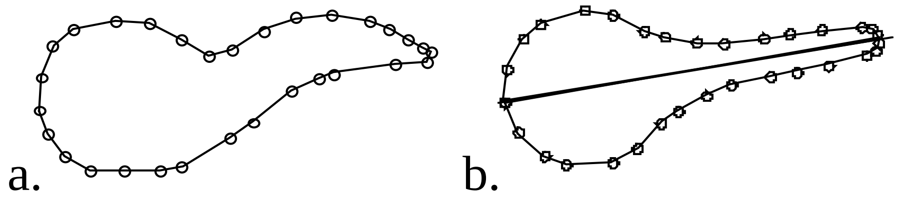

Figure 4: Symmetry Transform of a general 2D shape.
a) A general continuous shape is sampled. b) The Symmetry Transform of the shape.

Several methods have been suggested to sample the continuous contour of general shapes in order to obtain a collection of points as input to the CSM algorithm [8, 9, 12]. When contours and shapes are noisy, sampling at regular intervals does not produce satisfactory results. Instead, selection at equal angles is used where points are selected on the contour at equal angular intervals around the centroid (Figure 5). An extension of this approach was developed to deal with contours of occluded objects. In these cases, selection is at equal angles around the Center of Symmetry rather than the centroid. The Center of Symmetry is that point about which selection produces the minimal symmetry distance (intuitively, this point is proximal to the centroid of the unoccluded symmetric shape) [9, 12].

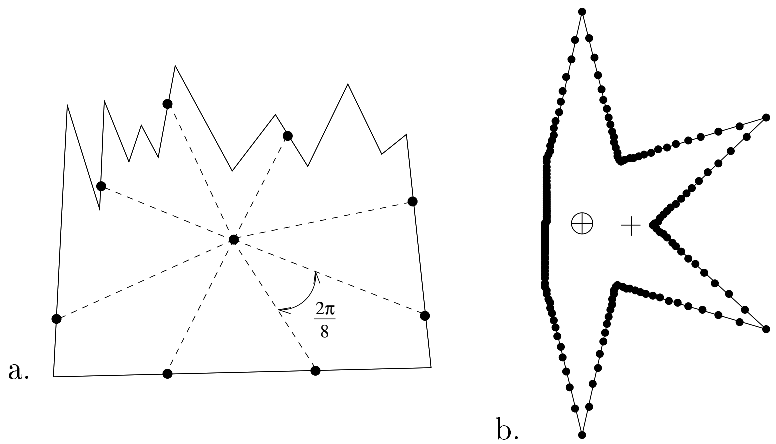

Figure 5: Symmetry Transform of a general 2D shape.
a) Selection points are distributed along the contour at regular angular intervals around the centroid. b) Selection points are distributed along the contour at regular angular intervals around the Center of Symmetry (marked by $\oplus$). The symmetry distance obtained using these points is smaller than the symmetry distance obtained using points selected at equal angles about the centroid (marked by $+$).

3.3 CSM of Gray-scale Images

Two approaches have been used in applying the CSM to gray-scale images. One approach segments areas of interest from the image and regards their contour as 2D shapes (see Figure 12 for example).

Another approach in dealing with images, lets pixel values denote elevation, and considers an image as a 3D object on which 3D symmetries can be measured. Figure 6 shows a range image and a gray-scale image for which the 3D mirror Symmetry Transform was computed, the 3D reflection plane was found and the 3D object rotated to a frontal vertical view.

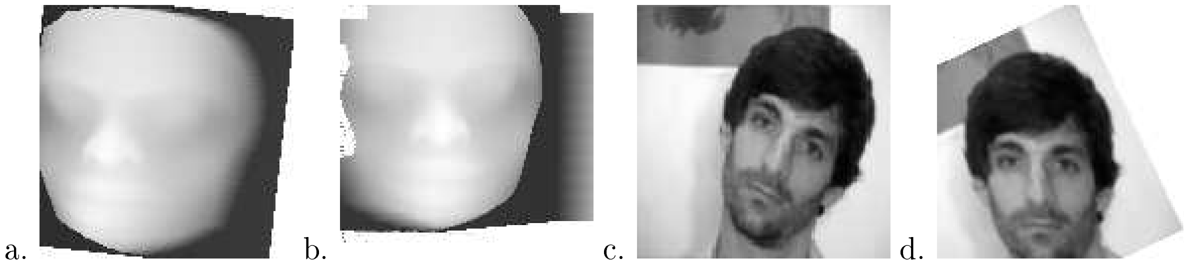

Figure 6: Applying CSM with respect to 3D mirror-symmetry to find orientation of a 3D object.
a) Original range image. c) Original Gray-scale image. b,d) The 3D symmetry reflection plane was found and the object in the image rotated to a frontal vertical view.

3.4 CSM of 3D Objects

Measuring the CSM of 3D objects is straight forward given a collection of 3D points as representatives [12, 3, 7, 10]. Figure 7 shows a 3D object and its Symmetry Transform with respect to tetrahedral symmetry. A variation of this scheme involves measuring the symmetry of the projection of a 3D object onto the image plane [13]. Given a noisy projection of a collection of 3D points onto a 2D plane, the closest projected symmetric configuration is found based on the CSM approach (Figure 8). This approach is used in reconstruction of 3D objects from their 3D projections. Figure 9 shows an example of 3 projections of a 3D object which was reconstructed using the CSM approach.

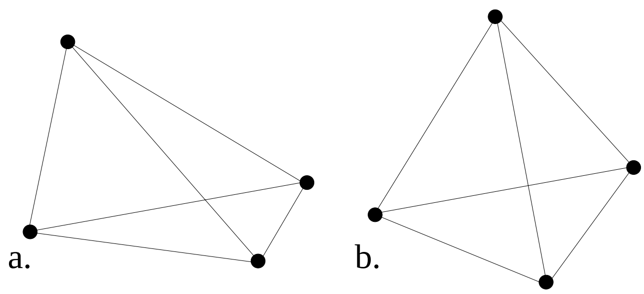

Figure 7: A 3D-object (a) and its Symmetry Transform (b) with respect to Tetrahedral-symmetry.

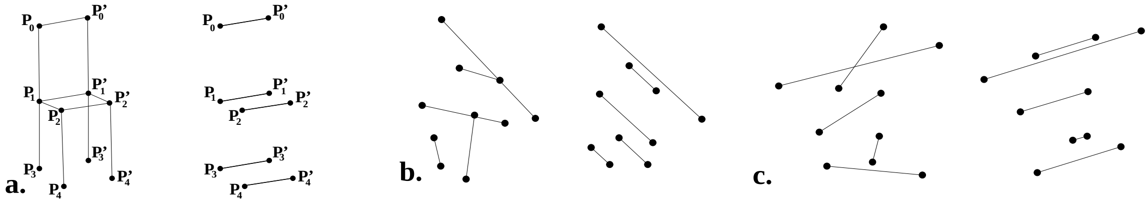

Figure 8: Finding the closest projected mirror-symmetric configuration.
a) The projection of a $3D$ mirror-symmetric configuration, induces projected symmetry. Points P_i and P_i' are corresponding mirror-symmetric pairs in $3D$. In the 2D projection these points are on parallel segments. c-d) Examples of noisy $2D$ projections of mirror-symmetric configurations of points (left) and the closest projected mirror-symmetric configuration (right).

Figure 9: Reconstruction of a 3D-mirror-symmetric object from 2D images of different view points.

4 Exploiting the Characteristics of the CSM

The CSM approach to measuring symmetry can be embedded in a scheme that takes advantage of the multi-resolution characteristics of symmetry The symmetry transform of a low resolution version of an image is used to evaluate the Symmetry Transform of the high resolution version. This technique was used in estimating face orientation [12, 7].

In many cases the source data is noisy. The CSM method can be exploited to deal with noisy and missing data. Considering noisy data, where the collection of representative points are given as probability distributions, the CSM approach can evaluate the following properties [12, 11]:

- The most probable symmetric configuration represented by the points.
- The probability distribution of CSM values for the points.

Figure 10 shows the effect of varying the probability distribution of the object points on the resulting symmetric shape.

Figure 11 displays a fuzzy image of points (a Laue photograph which is an interference pattern created by projecting X-ray beams onto crystals). and the probability distribution of the CSM values obtained for the pattern.

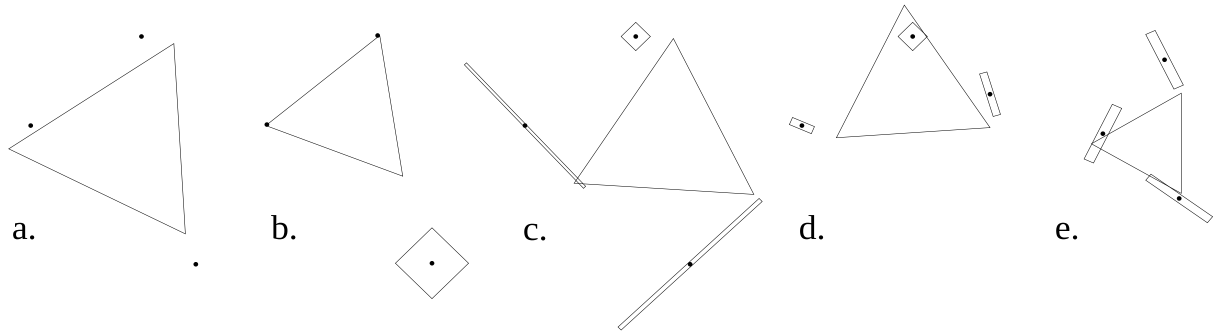

Figure 10: The most probable rotationally symmetric shape of order 3 for a set of measurements after varying the probability distribution and expected locations of the measurements.

a-c) Changing the uncertainty (s.t.d.) of the measurements.

d-e) Changing both the uncertainty and the expected location of the measurements.

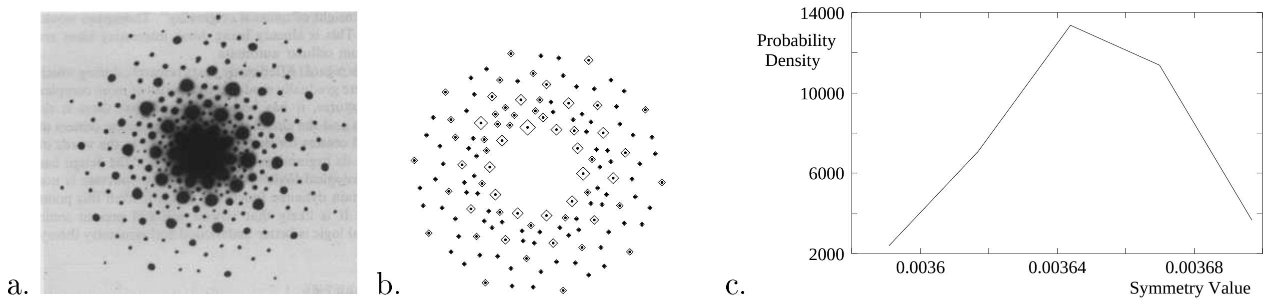

Figure 11: Probability distribution of symmetry values
a) Interference pattern of crystals. b) Probability distribution of point locations corresponding to a. c) Probability distribution of symmetry distance values with respect to Rotational symmetry of order 10. Expectation value = 0.003663.

Finally, the CSM approach to measuring symmetry allows the hierarchical nature of symmetry to be expressed and quantified thus, global and local features can be evaluated for their symmetry content. Figure 12 shows an example where the CSM approach measures local mirror symmetry to find faces in an image.

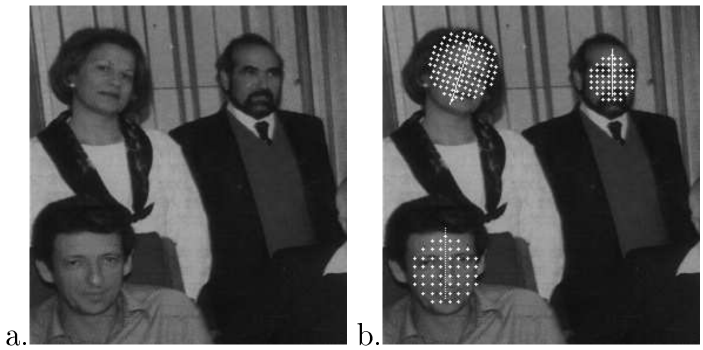

Figure 12: Multiple mirror-symmetric regions in images.
a) Original image. b) Faces are found as locally symmetric regions.

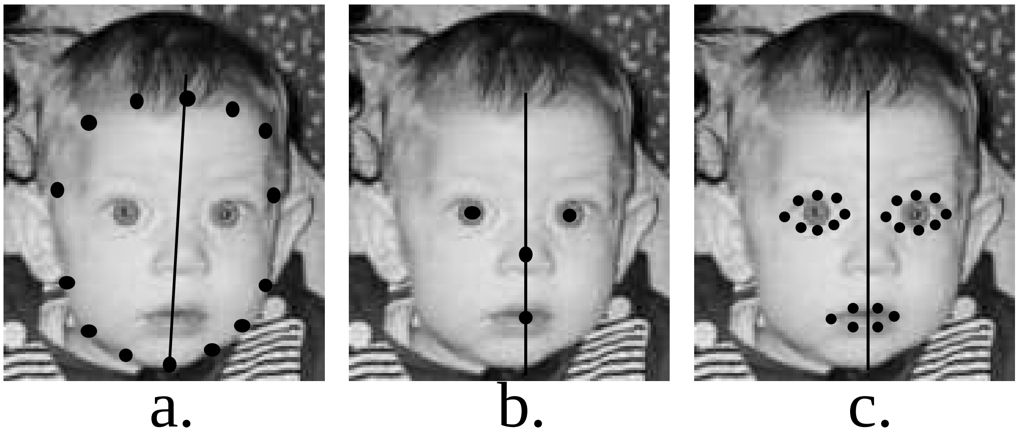

Figure 13: Measuring symmetry of human faces at 3 levels:
a) Face Contour. b) Facial Features Centroids c) Facial Features Contours

This issue of global vs local symmetry is important in the case of measuring symmetry of human faces.

5 Symmetry of Human Faces

The human face is a complex structure comprising of a number of facial features. It can be modeled at several levels of hierarchy: at the top level is the global structure of the face (the face contour) and at lower levels facial details (such as facial features) are revealed. Accordingly, symmetry of the human face should be considered hierarchically. The CSM approach to measuring symmetry can be employed to evaluate mirror-symmetry of a face at several different levels (Figure 13):

- Face Contour - symmetry is evaluated by considering the contour of the face alone.
- Facial Features Centroids - the centroids of each of the facial features serve as the set of input points for symmetry evaluation.
- Facial Features Contours - the face contour and the contours of all the facial features are considered in the evaluation of symmetry.

Recent studies have proposed that there is a positive correlation between symmetry of faces and physical attraction [1]. However the methods used for evaluating symmetry do not capture the complexity of this characteristic. We propose that using the CSM approach in a hierarchical scheme will provide a more flexible reliable and meaningful measure of symmetry of human faces.

6 Conclusion

We view symmetry as a continuous feature and adopt the Continuous Symmetry Measure (CSM) to evaluate it. The CSM can evaluate any symmetry in any dimension and has been applied to shapes, images and 3D objects. The CSM can deal with noisy data, such as fuzzy and occluded data. The method of evaluating symmetry using the CSM can be applied to global and local symmetries. This can be extended to deal with symmetry in a hierarchical manner as in the case of measuring symmetry of human faces. The CSM is currently being used in numerous fields including Chemistry, Physics, Archaeology, Botany and more.

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