Computational Aspects of Pattern Characterization - Continuous Symmetry

Thesis submitted for the degree “Doctor of Philosophy”

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Dedicated to my father.
Abstract

One of the basic features of shapes and objects is symmetry. Symmetry is considered a pre-attentive feature which enhances recognition and reconstruction of shapes and objects. Symmetry in its mathematical form is a binary feature of an object (either the object is symmetric or it is not), however symmetry in nature and the visual world seldom follow such strict mathematical definitions, a classic example is that of faces which are actually asymmetric. When dealing with the projection of the visual world onto an image plane (the retina or a digital image) additional deviation from exact symmetry occurs, due to perspective projection, digitization, occlusion etc.

In this work we view symmetry as a continuous feature rather than a binary feature and develop a measure - the Symmetry Distance to quantify imperfect symmetries of shapes, objects and images with respect to all symmetry classes. The Symmetry Distance of an object is defined as the minimum distortion required to transform the object into a symmetric one. This measure is invariant to rigid and similarity transformations and allows comparisons between the “amount” of symmetry of different shapes and between “amounts” of different symmetries of a single shape. An efficient geometrical algorithm is developed and proven, to evaluate the Symmetry Distance. This method first finds the symmetry transform of the object, which is the symmetric object closest to the original in terms of minimal distortion. The distance between the object and its symmetry transform is determined as the Symmetry Distance.

The Symmetry Distance gives rise to a method of dealing with occluded shapes and is extended to deal with symmetries of noisy and fuzzy data. Considering grayscale images, the Symmetry Distance is used to find the orientation of symmetric objects from their images, and to find locally symmetric regions in images. Symmetry, being a characteristic of objects is exploited in reconstruction of 3D structure from 2D data. The Symmetry Distance techniques are extended to deal with projected symmetry and are used in improving 3D reconstructions.
Contents

1 Introduction .................................................. 1
   1.1 Introduction .............................................. 1
   1.2 Definitions of Symmetry ............................... 4
   1.3 Symmetry in Computer Vision ......................... 5
   1.4 Advantages of the Symmetry Distance ............... 7
   1.5 Road Map ................................................ 8

2 Symmetry in Computer Vision .......................... 9
   2.1 Symmetry of Convex Sets ............................. 9
   2.2 Symmetry in Computational Geometry .............. 10
   2.3 2D Symmetry in Digital Images ..................... 12
      2.3.1 Direct Approach .................................. 12
      2.3.2 Voting Schemes .................................. 13
      2.3.3 Global vs. Local Symmetry ....................... 13
      2.3.4 Global Symmetry ................................ 14
      2.3.5 Local Symmetry .................................. 17
   2.4 2D Projections of 3D Symmetries .................. 22
      2.4.1 Direct Detection of Skew and Parallel Symmetries ................ 23
      2.4.2 Detecting Skew-Symmetry by Recovering 3D Structure ............ 25
   2.5 Additional Approaches to 3D Symmetry Reconstruction from 2D Projections 30
   2.6 Applications Using Symmetry ......................... 31
2.6.1 2D Applications ....................................................... 31
2.6.2 3D Applications ....................................................... 32
2.7 Measuring Symmetry .................................................... 33

3 Measuring Symmetry ..................................................... 35
  3.1 A Continuous Symmetry Measure - Definition ....................... 35
  3.2 Evaluating the Symmetry Transform .................................. 37
  3.3 Point Selection for Shape Representation ............................ 41
  3.4 Symmetry of Occluded Shapes - Center of Symmetry ................ 44
  3.5 Symmetry of Points with Uncertain Locations ....................... 48
    3.5.1 The Most Probable Symmetric Shape ............................ 48
    3.5.2 The Probability Distribution of Symmetry Values ................. 51
  3.6 Application to Images ................................................ 54
    3.6.1 Finding Orientation of Symmetric 3D Objects ................... 54
    3.6.2 Using a Multiresolution Scheme ................................. 55
    3.6.3 Finding Locally symmetric Regions .............................. 57

4 Mathematical Proofs ................................................... 63
  4.1 Mathematical Proof of the Folding Method ............................ 63
    4.1.1 Background ..................................................... 63
    4.1.2 Proof of the Folding Method .................................... 65
    4.1.3 Finding the Optimal Orientation in 2D ............................ 67
  4.2 Uncertain Point Locations ............................................ 68
    4.2.1 The Most Probable $C_n$-Symmetric Shape ....................... 68
    4.2.2 The Most Probable Mirror Symmetric Shape ...................... 71
    4.2.3 Probability Distribution of Symmetry Values ..................... 72
5 3D Mirror Symmetry

5.1 Background .................................................. 75
5.2 Closed Form Method for 3D Mirror Symmetry ............... 76
  5.2.1 Topological Stage ....................................... 77
  5.2.2 Geometrical Stage ....................................... 79
5.3 Examples ..................................................... 81
5.4 Extensions - Chirality ....................................... 83

6 3D Symmetry from 2D Data

6.1 Previous Work ............................................... 85
6.2 Reconstruction of 3D Mirror-Symmetric Structures from 2D Projections ........................................... 86
6.3 2D Symmetrization Procedure ................................ 87
6.4 Closed Form Method for finding the Closest Projected Mirror-Symmetric Configuration ................................ 88
6.5 The Pairing ..................................................... 91
6.6 Experiments ................................................... 91
  6.6.1 Reconstruction from Invariants ......................... 92
  6.6.2 Simulation Results ...................................... 93
6.7 Real data ..................................................... 98

7 Conclusion

Bibliography

A Dividing Points of a Shape into Sets

B The Bounds of S Values

C Applications in Chemistry
Chapter 1

Introduction

1.1 Introduction

One of the basic features of shapes and objects is symmetry. Symmetry is considered a pre-attentive feature which enhances recognition and reconstruction of shapes and objects [17, 48, 8]. Symmetry is also an important parameter in physical and chemical processes and is an important criterion in medical diagnosis.

![Figures](image1.png)

Figure 1.1: Faces are not perfectly symmetrical.

a) Original image.

b) Left half of original image and its reflection.

c) Right half of original image and its reflection.

However, the exact mathematical definition of symmetry [66, 112] is inadequate to describe and quantify the symmetries found in the natural world nor those found in the visual world. A classic example is that of faces - Figure 1.1a shows an image of a face which is generally described as “symmetric”. However, replacing the right half of the image with a reflection of the left half we obtain an image (Figure 1.1b) which is not the same as that obtained by replacing the left half of the image with the reflection of the right half (Figure 1.1c). In the visual world, loss of exact symmetry is further enhanced;
Figure 1.2: Perceiving continuous symmetry.

a) A shape perceived as perfectly symmetric (the oblique mirror axis passing through the vertex).
b) Shortening one arm, the shape is perceived as “almost” symmetric.
c-d) Further shortening of the arm, the shape is perceived as having “less” symmetry.
e) When the arm is eliminated the the shape is again perfectly symmetric (with a mirror axis perpendicular to the existing arm).

even perfectly symmetric objects lose their exact symmetry when projected onto the image plane or the retina due to occlusion, perspective transformations, digitization, etc.

Thus, although symmetry is usually considered a binary feature, (i.e. an object is either symmetric or it is not symmetric), we view symmetry as a continuous feature where intermediate values of symmetry denote some intermediate “amount” of symmetry. This concept of continuous symmetry is in accord with our perception of symmetry as can be seen, for example, in the shapes of Figure 1.2. Figure 1.2a is perceived as perfectly mirror-symmetric with the mirror axis passing through the vertex and bisecting the angle between the two arms. Shortening one of the arms (Figure 1.2b), the shape is perceived as being “almost” symmetric. Further shortening of the arm, the shape becomes “less and less” symmetric (Figure 1.2c-d). Note that completely eliminating the arm (Figure 1.2e), the shape becomes perfectly symmetric again (the mirror axes aligning with and perpendicular to the remaining arm).

Considering symmetry as a continuous feature, the need arises for a method of evaluating and quantifying the continuous symmetry, specifically, a measure of continuous symmetry. In this work a “Symmetry Distance”, that can measure and quantify all types of continuous symmetries of objects is introduced. We define the **Symmetry Distance** (SD) as the minimum effort required to transform a given object into a symmetric object. This effort is measured by the mean of the square distances each point of the object is moved from its location in the original object to its location in the symmetric object. In effect, we measure the distance from a given object to the set of all symmetric objects. This definition of symmetry distance produces a versatile and simple tool to quantify the distance of a given shape from any chosen element of symmetry. The generality of this symmetry measure allows one to compare the symmetry distance of several objects relative to a single symmetry element and to compare the symmetry distance of a single

\[ 	ext{Figure 1.2: Perceiving continuous symmetry.} 
\]

\[ \text{a.} \quad \text{b.} \quad \text{c.} \quad \text{d.} \quad \text{e.} \]

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object relative to various symmetry elements. Thus the intuitive notion that the shape of Figure 1.3a is “more” mirror-symmetric than the shape of Figure 1.3b, can be quantified. Similarly, the intuitive notion that the shape of Figure 1.3c is “more” rotationally symmetric (of order two) than mirror symmetric can also be quantified. In this thesis we describe the SD as applied to computer vision and pattern recognition, however, quantifying imperfect symmetry is important in other various areas of science:

- **Chemistry and Physics** -
  Many chemical and physical processes are symmetry dependent [7]. Classical textbooks describe these processes as dependent on perfect symmetry of the system or molecule (thus any deviation from perfect symmetry terminates the process). However this is not true in practice: experiments show that the reaction rate of symmetry dependent processes decreases (but continues) as the symmetry of the system decreases (see for example in [31]). Thus there is a need for a model of symmetry which allows for intermediate values, and a measure which allows for quantifying these symmetries. A measure of continuous symmetry can be used to measure symmetry of molecules and atoms and to model chemical reactions.

- **Biological systems** -
  Biological systems are rich in symmetries (though not necessarily exact) [100]. Disorder and disturbances during the development process can degrade the symmetry of the biological systems. Quantify symmetry of biological system can be used as a measure of the disorder and disturbances in the system. For example, in neurosystems, a measure of symmetry is important for studying the influence of disorder.
during growth period of neurons, on the amount of symmetry found in the bifurcating 
structure of it's axons and dendrites [52]. In botanical systems the disorder 
during growth periods, can be expressed in symmetry loss in the vein patterns of 
leaves [29].

• Medicine -
Medical diagnostics often use symmetry. For example cancerous tissues are quite 
often non symmetric and asymmetric organs may imply some abnormality or can-
cerous growth. Using symmetry measures these imperfect symmetries can be quan-
tified and used to assist in medical diagnosis. A specific case is that of skin cancer 
where the skin spot is determined to be cancerous, as a function of the “amount” 
of symmetry of the spot [63]. A symmetry measure can be directly applied to 
evaluating the asymmetries and thus the abnormalities of cancerous organs.

1.2 Definitions of Symmetry

Following are the definitions of symmetry as will be referred to in this paper. For further 
details and an excellent review see [112].

An n-dimensional object has mirror-symmetry if it is invariant under a reflection about 
a hyperplane of dimension \((n - 1)\) passing through the center of mass of the object. Thus 
a 2D object is mirror-symmetric if it is invariant under a reflection about a line (called 
the axis of mirror-symmetry) and a 3D object is mirror-symmetric if it is invariant under 
a reflection about a plane.

A 2D object has rotational-symmetry of order \(n\) if it is invariant under rotation of \(\frac{2\pi}{n}\) 
radians about the centroid of the object.

A 3D object has rotational-symmetry of order \(n\) if it is invariant under rotation of \(\frac{2\pi}{n}\) 
radians about an axis passing through the centroid of the object. This axis is the

![Diagram](image-url)

Figure 1.4: Examples of symmetries: a) \(C_\sigma\)-symmetry b) mirror-symmetry 
c) \(D_\sigma\)-symmetry (radial symmetry of order 8). d) circular symmetry (\(C_\infty\-
symmetry)).
rotational symmetry axis.

Rotational symmetry of order $n$ is denoted **C$_n$-Symmetry**.

Radial symmetry (referred to in Chapter 2) is the symmetry of a 2D object having both mirror-symmetry and $C_n$-symmetry (note that such objects have $2n$ axes of mirror-symmetry). Radial symmetry of order $n$ is denoted **D$_n$-Symmetry**. Circular symmetry (referred to in Chapter 2) is $C_{\infty}$-symmetry (see Figure 1.4).

### 1.3 Symmetry in Computer Vision

A detailed review of symmetry studies in computer vision is given in the next chapter.

As an intrinsic characteristic of objects and shapes, symmetry can be used to describe and recognize objects and can be exploited for reconstruction and efficient encoding of symmetric objects.

Early studies in geometry and theoretical mathematics dealt with evaluation of symmetry of convex sets [42]. These studies approach symmetry evaluation from the theoretical point of view and no method has been suggested to efficiently evaluate these measures. Furthermore, these methods are limited to measuring inversion-symmetry (rotational symmetry of order 2) of convex sets.

Transformation of the symmetry detection problem to a pattern matching problem introduces efficient algorithms for detection of mirror and rotational symmetries and location of symmetry axis [28, 6, 45, 113, 3]. These algorithms assume noise free input and detect symmetry, if it exists, in collections of features such as points and line segments. Slight perturbation of the input will fail symmetry detection. However, upper bounds on the complexity of symmetry detection with a limited error tolerance has been presented [3].

When dealing with symmetry in 2D, several approaches can be taken. The direct approach for determining if a given object is mirror or rotationally symmetric is to apply the symmetry transformation (i.e. reflection or rotation) to the object and then compare it to the original object [108, 58, 22, 57]. These methods assume that the object is either perfectly symmetric or it is not at all. These methods are highly sensitive to noise and occlusion. Furthermore, they do not serve as satisfactory measures of symmetry (see discussion in Chapter 2).

A different approach to symmetry detection uses a voting scheme. The voting scheme is based on the fact that the symmetry axis or point of rotation is determined by two points in the object. In the voting scheme pairs of points are tested and vote for their preferred symmetry axis. The oriented line with highest vote is selected as the symmetry axis of the object [76, 72, 60, 83, 81, 71, 128]. These voting schemes have high complexity. Several
methods have been suggested to reduce complexity by grouping points into regions or into curve-segments, thus reducing the number of possible pairs involved in the voting [90, 38].

These voting schemes are robust, to a certain degree, under noise and occlusion in the input image, however, they have high complexity. These methods usually assume the existence of symmetry axes and know the number of such axes (although thresholding heuristics could provide these variables, the process is image and noise dependent and generally unstable). These studies generally approach symmetry as a binary feature, where thresholding is performed to overcome noise in the input. Even though a measure of certainty can be associated with a voted symmetry axis, it does not necessarily correctly measure the quality and “amount” of symmetry in the object (and most probably does not reflect the intuition and perception of continuous symmetry).

Symmetry in 2D can be discussed as a global feature where all points in the object contribute to determining the symmetry [116, 11, 49, 64, 24, 103], or as a local feature where every symmetry element is supported locally by some subset of the object [14, 77, 19, 18, 23, 11, 85]. The global symmetry methods are much more efficient in run time, usually having a linear time complexity however they are generally sensitive to noise and occlusion. The local symmetry methods are more robust to noise and occlusion, and they are easily parallelized, however they have high time complexity.

Symmetry in 3D is usually approached from the 2D projections of the 3D objects. The symmetry can be determined directly from the projections [83, 71, 38, 73, 82, 117, 72, 34] or the 3D symmetric object can be reconstructed from its projection and 3D symmetry determined [53, 97, 114, 106, 70, 33]. Additional approaches to detection of 3D symmetry involve deformable contours [99], tomography [27], structure from motion techniques [80, 67, 88] and physical mirrors [74, 51, 1, 54].

In most of the above mentioned techniques, symmetry is treated as a binary feature: either it exists or it does not exist in an object. The notion of quantifying symmetry has been seldomly discussed; The early work in [42] reviews methods of geometrically measuring symmetry of convex sets. In [116], information theory is used to evaluate the distribution of symmetries in a pattern. In [64], a coefficient of mirror-symmetry with respect to a given axis is presented. In [37, 44, 10], the idea of a Measure of Chirality (a measure of deviation from mirror-symmetry) is presented.

The symmetry detection and evaluation methods mentioned in this section are each limited to a certain type of symmetry (mirror or circular symmetry) and most generally assume that the object or shape is perfectly symmetric or it is not at all. In this thesis, the notion of symmetry distance is introduced and a general continuous measure of symmetry is presented for evaluating all types of symmetries in any dimension. We presented this work in [122, 121, 123, 126, 127, 125, 124, 118, 120].
1.4 Advantages of the Symmetry Distance

In this thesis symmetry is considered as an inherently continuous property rather than a binary “yes/no” feature. Accordingly, dealing with symmetry in the real world and in projected images, should be based on continuous symmetry values. The Symmetry Distance measure has the following advantages:

- Defining the Symmetry Distance (SD) as the minimum distortion required to turn a given object into a symmetric object produces a versatile and simple tool to quantify continuous symmetry. A simple geometric algorithm is presented and proven for efficiently calculating the symmetry distance. The SD is presented in a mathematical framework and forms a metric in the space of all shapes of a given dimension.

- The generality of this symmetry measure allows one to compare the symmetry distance of several objects relative to a single symmetry element. Additionally, in contrast with most previous studies, the SD is not limited to a specific type of symmetry (rotational, mirror, etc) but is applicable to any symmetry group in any dimension. This allows one to compare the symmetry distance of a single object relative to various symmetry elements.

- As would be expected of such a measure of symmetry, the SD is continuous and reflects the intuitive perception of symmetry (see Figure 3.6 for the evaluation of SD in the perceptual example of Figure 1.2). It is inherently invariant to rotation and translation (and with appropriate normalization, is invariant to scale).

- Unlike some previously suggested symmetry detection methods which compare the tested object with an a-priori chosen reference structure or shape (a perfect cube, a tetrahedron, a perfect gray-scale circle etc), the SD assumes no reference structure. In fact, finding the distance between the test object and the closest symmetric object, as required by the SD, can be viewed as comparing the test object with a reference structure which is specific for the given tested object. Formally, the SD finds the distance between the test object and the set of all symmetric objects of a given dimension.

- Using the SD paradigm the symmetric shape “closest” to the test shape can be constructed. The “closest symmetric shape” allows visual evaluation of the SD, such that similarity of a shape and its closest symmetric shape, implies low symmetry distance values. The closest symmetric shape can also be used in applications such as approximations, reconstructions of occluded shapes and reconstructions of objects from their projections.
• The SD is a robust measure and unlike previous symmetry studies, can be applied to occluded and noisy data and can be applied globally or locally.

• Finally, the SD is applicable to a wide variety of objects, ranging from point configurations, digital images, energy maps, topographic (contour) maps, and has also been applied to structured molecules and to fuzzy shapes.

1.5 Road Map

In Chapter 2 we briefly review some previous studies on symmetry in computer vision. In Chapter 3 we introduce the Symmetry Distance: in Section 3.1 we define the Symmetry Distance and in Section 3.2 describe a method for evaluating this measure. Following, in Sections 3.3-3.5 we describe features of the symmetry distance including its use in dealing with occluded objects and with noisy data. In Section 3.6 we describe the application of the symmetry distance to finding face orientation and to finding locally symmetric regions in images. In Chapter 4 we give mathematical derivations and proofs for the algorithms described in Chapter 3. In Chapter 5 we expand on determining the SD for the specific case of 3D symmetry. Finally, in Chapter 6 we deal with 3D symmetry from 2D projected data.
Chapter 2

Symmetry in Computer Vision

2.1 Symmetry of Convex Sets

The basic question of symmetry detection has been discussed in early papers in geometry and theoretical mathematics. These early studies deal with evaluating deviation from symmetry of sets in Euclidean space and are based on cord length, surface area (contour length) and internal volume (internal area). A review of geometrical evaluation of symmetry of convex sets is given in [42]. Following are several examples, described by 2D analogues for evaluating inversion-symmetry (rotational symmetry of order 2):

- **Minkowski measure of symmetry** For each point $P$ in the convex set $K$ and for each cord $D$ of $K$ passing through $P$ (Figure 2.1a) we define $g(D, P)$ as the ratio ($\leq 1$) by which $P$ divides $D$. Denoting $g(P) = \min_{D} \{g(D, P)|D \ni P\}$, the symmetry measure defined as $F_1(K) = \max_{P} \{g(P)|P \in K\}$.

- **Winternitz measure of symmetry** For each point $P$ of a convex set $K$ and for each line $H$ passing through $P$ (Figure 2.1b), we define $f(H, P)$ as the ratio ($\leq 1$) of the volumes of the two parts of $K$ determined by $H$. Denoting $f(P) = \min_{H} \{f(H, P)|H \ni P\}$, the symmetry measure is defined as $F_2(K) = \max_{P} \{f(P)|P \in K\}$. Several analogues to this measure were suggested. For example, a measure defined as above except that $f(H, P)$ is the ratio in which the contour length of $K$ is divided by $H$.

- **Kovner-Besicovitch measure of symmetry** For a point $P$ of a convex set $K$, let $L(P)$ be the maximal symmetric subset of $K$ with center at $P$ (Figure 2.1c). Denoting by $f(P)$ the ratio of volumes of $L(P)$ and $K$, the symmetry measure is defined as $F_3(K) = \max_{P} \{f(P)|P \in K\}$.
Figure 2.1: Geometrical evaluation of symmetry of convex sets.

a) Minkowski measure of symmetry: $F_1$ is defined as $\max_P \min_D g(D, P)$, where $g(D, P)$ is the ratio by which point $P$ divides cord $D$.

b) Winternitz measure of symmetry: $F_2$ is defined as $\max_P \min_H f(H, P)$ where $f(H, P)$ is the ratio of volumes of two parts of $K$ determined by $H$.

c) Kovner-Besicovitch measure of symmetry: $F_3$ is defined as $\max_P f(P)$ where $f(P)$ is the ratio of volumes of $K$ and the maximal symmetric subset of $K$ with center at $P$ (shaded region).

These early studies approach symmetry evaluation from the theoretical point of view, and upper and lower bounds on the values have been proved. No method has been suggested to efficiently evaluate these measures, nor has any application been suggested. Furthermore, these methods are limited (in their original definition) to convex sets. The definition of these measures are purely geometrical and do not always fit our intuition and perception of symmetry.

### 2.2 Symmetry in Computational Geometry

A totally different approach to symmetry definition can be found in a collection of papers in which a computational approach was taken [28, 45, 113, 3, 6]. These studies present algorithms for detecting symmetry in collections of geometrical objects such as points, line segments, circles, etc. The basic idea in these algorithms is to reformulate the problem as a 1D pattern matching problem which can be solved efficiently (using known techniques such as KMP [56]). The complexity of these algorithms are shown to be $O(n \log n)$ where $n$ is the number of geometric objects. For example, finding rotational symmetry of a set of points in 2D, \( \{P_i\}_{i=0}^{n-1} \), is performed as follows:

1. The set \( \{P_i\} \) is translated so that the centroid is at the origin.

2. Represent the points in polar coordinates \( \{\theta_i, l_i\}_{i=0}^{n-1} \) where \( \theta_i \) is the angle and \( l_i \) is the distance to the origin of point \( P_i \). The points are sorted according to angle
Chapter 2: Symmetry in Computer Vision

(w.l.g. we assume that no two points have the same angle). Assume $\{P_i\}_{i=0}^{n-1}$ are the sorted points.

3. Denote by $\alpha_i$ the angle difference $\theta_{i+1} - \theta_i$. Defining the $2n$-tuple $L(P)$ as $L(P) = l_0, \alpha_0, \ldots, l_{n-1}, \alpha_{n-1}$, create the $4n$-tuple $L(P)L(P)$.

4. Using pattern matching techniques, find the pattern $L(P)$ in the string $L(P)L(P)$. The number of matches found is equal to the order of the rotational symmetry.

Extending this algorithm to detect mirror symmetry is obtained by replacing the pattern in Step 4 with the pattern $\hat{L}(P)$ which is the inverse sequence to $L(P)$ ($\hat{L}(P) = \alpha_{n-1}, l_{n-1}, \ldots, \alpha_0, l_0$). Similar algorithms are described for line segments, circles and polygons. These algorithms are simple and efficient, but the major drawback is that they are highly sensitive to noise. In fact, slight perturbations of the elements location, or slight computer precision errors, will cause the algorithm to fail in finding symmetry.

This approach for finding symmetry in collections of geometric elements is extended to higher dimensions in [3], where it is described for the specific case of points in $R^d$. The outline of the approach is as follows:

1. The points are translated so that the centroid is at the origin.

2. The points are projected onto the unit sphere and each projected point is labeled with its original distance from the origin.

3. The convex hull of the projected points is created (which necessarily includes all the projected points as vertices).

4. For each vertex in the convex hull, a $2n$-tuple is created similar to the 2D algorithm described above: the neighboring vertices are ordered according to the angle and the distance and angle differences are concatenated into a $2n$-tuple. The lexicographically smallest cyclic shift of the tuple is concatenated to the label of the vertex.

5. The convex hull with its labeled vertices is considered as a planar labeled graph $A$. The Hopcroft isomorphism algorithm [2] is applied to the pair $A, A$ of graphs. The isomorphism algorithm partitions the graph-vertices into equivalent classes from which symmetry can be easily deduced.

This method is efficient ($O(n \log n)$), however, as in the 2D case, it is not robust to noise, perturbation of points, nor to numerical imprecisions.

The matter of imprecision and approximate symmetry in these cases is described in [3] in terms of time complexity.
2.3 2D Symmetry in Digital Images

2.3.1 Direct Approach

Detection of 2D symmetry in digital images has been widely studied and numerous approaches suggested. The most basic method for determining if a given image is mirror or rotationally symmetric (see definitions in Section 1.2) is to apply the symmetry transformation (i.e. reflection or rotation) to the image and then compare it to the original image. Such an approach is assumed in [108] where an optical-mechanical system is described to optically determine 2D symmetry in images. In [58], comparison of an image and its reflection is used for detection of vehicles.

In [22] this approach is combined with a recursive strategy using a multiresolution representation of an image - specifically, an image pyramid. This method recursively tests for perfect symmetry, starting at low resolution where the test is inexpensive, and continuing to high resolution images.

In [57], the overlap method is used to determine rotational symmetry by performing autocorrelation of the image in polar coordinates.

![Figure 2.2: Two images which give low symmetry content values using overlap techniques, though they are intuitively “almost” symmetric.

a) Folding across the mid horizontal axis, the region of overlap is minimal.

b) Rotating about the center by π/2 (obtaining the shape with the dotted outline), the overlap is minimal.](image)

These overlap methods assume that the image is either perfectly symmetric or it is not at all. These methods are intolerant to noise and occlusion. Furthermore, they do not serve as satisfactory measures of symmetry: although in [108] one may suggest that the area of overlap between an object and its reflection can be used as a measure of symmetry, this would not always support our intuition of the symmetry measure value. For example, the images in Figure 2.2 are intuitively evaluated as highly symmetric (almost symmetric) however the suggested overlap symmetry measure would give very low values (see also [111]).
2.3.2 Voting Schemes

A different approach to symmetry detection uses a voting scheme. These methods are based on the fact that for mirror-symmetry, mirror-symmetric points of a figure determine the symmetry axis uniquely - the axis is the perpendicular bisector of the segment connecting these two points. In the same manner, for rotational symmetry, $2$ symmetric points determine the line on which the center of rotation must lie. The general idea of the voting scheme is to test pairs of points in the given image and allow each pair to vote for their preferred symmetry axis. The oriented line with highest vote is selected as the symmetry axis of the image.

In [76], a digital polygon is represented by its chain code. Given $n$ pixel points of the polygon, $n/2$ possible mirror-symmetry axes are tested for. In this method, the direction of the axis is voted for, but not its position, and the assumption that the axis divides the polygon points into $n/2$ pixel points on each side, is often incorrect.

In [72, 60] the Hough transform is used for the voting scheme. For every pair of points in the image, the midpoint and the direction perpendicular to the segment connecting the pair are determined and voted for in the Hough space. The line with maximal votes is selected as the mirror symmetry axis. A variant of this method is used in [117] for detecting rotational symmetry.

A variant of the Hough transform voting scheme is the projection scheme [83, 81, 71] described below in Section 2.4.1. Another variant of the voting scheme can be found in [128] where a feed-forward network is used to detect and enhance edges which are symmetric in terms of edge orientation.

The voting schemes describe above suffer from high complexity. Several methods have been suggested to reduce complexity by grouping points into regions or into curve-segments, thus reducing the number of possible pairs involved in the voting [90, 38].

2.3.3 Global vs. Local Symmetry

Dealing with the problem of symmetry detection in 2D, two types of approaches can be considered:

- the global approach - where the image or shape is assumed to be symmetric on a global scale, i.e. every symmetry element (rotational symmetry or mirror-symmetry) concerns the whole image or shape (Figure 2.3a).

- the local approach - where symmetry of an image or shape is assumed to be local, i.e. every symmetry element is supported locally by some subset of the image or shape (Figure 2.3b).
Chapter 2: Symmetry in Computer Vision

Figure 2.3: Global vs. Local Symmetry.

a) Global Symmetry - the mirror-symmetry axis reflects symmetry in the whole shape.
b) Local Symmetry - the symmetry axes (curves) are supported locally by some subset of the shape.

2.3.4 Global Symmetry

In the case of global symmetry, all points (or pixels) in the object contribute to determining the symmetry. The voting schemes described above fall into this category. Two other prominent global approaches are the methods based on basis functions and methods based on moments.

Basis Function Methods

In [116], the Walsh functions are used as basis functions for evaluating mirror-symmetry (horizontal, vertical or both) and rotational-symmetry (of order 2 - $C_2$). The 2D basis functions (Figure 2.4a) denoted $W_{n,m}$ (with integer $n, m$ values) are equally divided into 4 sets according to the type of symmetry they represent:

- vertical mirror symmetry: $m$-even, $n$-odd.
- horizontal mirror-symmetry: $m$-odd, $n$-even.
- doubly mirror symmetric: $m$-even, $n$-even.
- rotational-symmetry: $m$-odd, $n$-odd.

Applying the Walsh transform to a digital image and summing the coefficients according to the above classification, a vector of 4 values is obtained representing the symmetries in the image. An overall evaluation of symmetry is obtained for the image by taking the entropy of these 4 values (accordingly, this single value is termed “symmentropy”).

In [11, 12], basis functions are used to detect rotationally symmetric images. The ba-
Chapter 2: Symmetry in Computer Vision

Double Symmetry  
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<tr>
<th>w_{0,0}</th>
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Horizontal Symmetry  
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Vertical Symmetry  
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Rotational Symmetry  
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Figure 2.4: Basis functions for global symmetry detection.

a) The Walsh basis functions divided into 4 classes according to symmetry, are used to detect horizontal, vertical and double mirror-symmetry and to detect rotational symmetry (of order 2).

b) The radial basis functions are used to detect rotational and circular symmetry in images.

Basis functions are spiral-like with varying number of “arms” and variable curvature (see Figure 2.4b). Given an image of radius $R$, the orthogonal basis functions are given by $\psi_{m,n}(r, \theta) = e^{i(m\omega r + n\theta)}$ where $\omega = 2\pi/R$ and $(r, \theta)$ are polar coordinates. Thus $n$ represents the number of “arms” and $m$ represents the degree of curvature. Since, these basis functions are similar to the Fourier transform basis function, but in polar coordinates, the advantages of the Fourier transform can be exploited. Thus, the time complexity is $O(n \log n)$, and the power spectrum of the transform is invariant to rotation.

Transforming a given image according to these polar basis functions, the coefficients $C_{m,n}$ are obtained. The rotational symmetry is evaluated by computing the weighted averages:

$$m_d = \left( \frac{\sum \sum \|C_{m,n}\|^2 m^2}{\sum \sum \|C_{m,n}\|^2} \right)^{1/2} \quad n_d = \left( \frac{\sum \sum \|C_{m,n}\|^2 n^2}{\sum \sum \|C_{m,n}\|^2} \right)^{1/2}$$

The pair $(m_d, n_d)$ reflects the rotational symmetry in the image. Additionally an associated uncertainty value is computed. Low values of uncertainty reflect high symmetry content of type $(m_d, n_d)$ with center of image as center of rotation.
Moments Based Methods

Another global symmetry approach is that based on moments. The $M_{pq}$ moment of a 2D shape is defined as

$$M_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p y^q I(x, y) \, dx \, dy$$

where $I(x, y)$ is 1 in side the shape and 0 elsewhere. The central moment $\mu_{pq}$ is defined as

$$\mu_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})^p (y - \bar{y})^q I(x, y) \, dx \, dy$$

where $\bar{x} = M_{10}/M_{00}$ and $\bar{y} = M_{01}/M_{00}$. For a general shape, these moments define properties of the shape, thus $\bar{x} = M_{10}/M_{00}$ and $\bar{y} = M_{01}/M_{00}$ are the coordinates of the center of mass of the shape. Defining the inertia tensor as the matrix of second order moments: $\begin{pmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{22} \end{pmatrix}$, the eigen vectors of this matrix are the principle axes of the shape. If a shape is perfectly mirror symmetric, the mirror axis aligns with one of the principle axes. However, for a rotationally symmetric shape (including shapes which have $n$ axes of symmetry, $n > 1$), the principle axes are not uniquely determines since for these shapes $\mu_{11} = 0$ and $\mu_{20} - \mu_{22} = 0$.

In [49], moment invariants are developed for pattern recognition. As a specific case, an invariant was developed to distinguish between “mirror-image” (i.e. a formulation which is invariant to rotations but changes sign under reflections). Regarding circular symmetries and rotational symmetries with mirror axis, constraints on possible invariants are discussed derived from lack of unique principle axes, as mentioned above.

In [64], a method is presented for finding the $n$-mirror axes of a 2D shape which is rotationally symmetric of order $n$ and having $n$ mirror-axes (i.e. $D_n$ symmetric). The method is based on the fact that all mirror-axes of a shape pass through the centroid of the shape $(\bar{x}, \bar{y})$.

In [24, 103] a method is suggested for overcoming the problem of inexistence of unique principle axes for rotationally symmetric shapes. In [103] a method was suggested for generalizing the principle axes to rotationally symmetric shapes. The rotationally symmetric shape is transformed into a shape which generally has no rotational symmetry and thus the principle axes can be computed. The computed principle axes are then transformed back as generalized principle axes of the original rotational symmetric shape.

The general use of moments either as invariants for shape description and recognition or as determinants for shape localization and orientation, are very sensitive to noise and occlusion, thus they are usually impractical for use in digital images. The method in [64] as presented in its theoretical formulation, must assume perfect n-fold symmetry. However, the method is extended to deal efficiently with imperfectly symmetric images,
thus becoming more robust to noise and occlusion. Additionally, this method finds the number of reflection axes of the image. The method in [103] is apriori defined for all shapes including imperfectly symmetric shapes. However, whereas the method in [64] can be defined as a measure of symmetry, the study in [103] deals with finding the principle axes and does not extend as a measure. Additionally, the methods in [24, 103] assume a priori knowledge of the order $n$ of the rotational symmetry.

### 2.3.5 Local Symmetry

In the case of local symmetry, only part of a shape or a subset of its points is symmetric with respect to any given symmetry (rotational symmetry or mirror-symmetry). The subset relevant to a given symmetry is usually taken as a continuous section of the shape’s contour or as a continuous neighborhood around and image point.

It should be noted that the global symmetry approached described above, can be implemented for detecting local symmetry by segmenting the shape or image into parts or regions and applying the global symmetry method to each region independently. However, in contrast with the global symmetry approaches, the methods described here as local symmetry methods, are inherently local by definition.

When dealing with 2D shapes defined by their bounding contour, the notion of local symmetry is generally used for shape description, either in terms of compact coding, or in terms of shape recognition. Shape description using local symmetry, termed axial description of shape is a region-based description which involves spines and curves for representing 2D shapes. It is discussed in [87] and is briefly reviewed in the following.

**Symmetry Axis Transform - SAT**

In [14], the *Symmetry Axis Transform* SAT is presented (also termed Medial axis transform - MAT). The SAT of a given 2D shape is the loci of the centers of all maximal

![Figure 2.5: Symmetry Axis Transform (SAT).](image)

(a-c) The SAT is the loci of all maximal disks enclosed in the shape (dashed lines). The SAT description of a shape is very sensitive to noise (c).
disks, where a maximal disk is entirely contained in the shape and no other contained disk subsumes it (see example in Figure 2.5a). The SAT is shown to be piecewise smooth [16] and forms a graph-like structure (tree structure if there are no holes in the shape). Associating with every point of the SAT, the radius of the maximal disk, the original shape can be perfectly reconstructed. A weakness of the SAT description is that it is very sensitive to noise. Thus a small dent in the bounding curve of the shape, creates drastic changes in the SAT (see Figure 2.5b-c). The curves of the mat reflect various properties of the shape, thus some curves reflect the general shape and orientation and represent the major axis or spine of the shape (for example the horizontal axis in the SAT of Figure 2.5b). Other curves in the SAT reflect local boundary formation (for example the oblique segments of the SAT in Figure 2.5b, reflect the shape of the corners).

In [78], the sensitivity to noise and the variance in saliency of SAT parts are dealt with by extending the SAT to a hierarchical description. Smoothing a shape, simplifies its SAT graph-structure and eliminates non-salient limbs. Thus an hierarchy of saliency of the SAT limbs can be obtained.

A further extension in [36] builds a hierarchical SAT description of gray-scale images by thresholding the image at successive levels obtaining a sequence of binary images which can be considered 2D shapes, and on which the standard 2D SAT is applied. These 2D SATs are combined to create a 3D SAT of the intensity image.

In [69] the SAT is extended to proper 3D where the SAT description is a 3D graph structure representing the loci of all maximal balls bounded in the 3D object. This 3D SAT description is used to decompose 3D objects into primitives.

In digital images, the SAT has been extended to deal with graylevel images in [77, 59].

**Generalized Ribbons - GR**

In [19], a class of shapes called *Generalized Ribbons - GR* is described. These shapes are generated from a 2D curve serving as a spine and from a generator segment which

![Figure 2.6: Generalized Ribbons (GR).](image)

The GR is defined as the shape swept out by a segment moving along a 2D curve or spine. The 2D curve is the GR representation of a shape.

a-b) The GR is not uniquely defined.

b) The GR may be curved thus representing local symmetry of the shape.
Chapter 2: Symmetry in Computer Vision

moves along the spine at a constant angle (and possibly changing length). The segment is usually taken to move along the spine so that its centroid is on the spine. The area swept out by the segment is a generalized ribbon (see Figure 2.6). The spine of the GR can be taken as a description of the ribbon shape. This definition of an axial shape description is more flexible than the SAT description [14], but is not uniquely defined (see Figure 2.6a-b). Note that the spine of a GR need not be straight and may be curved. Thus, the spine represents local symmetry of the shape (Figure 2.6c). The GR are 2D versions of the generalized cylinders (cones) [71] which are generated by moving a planar surface along a 3D curve (spine) at a constant angle to the spine and allowing changes of size in the planar shape (see also [94]).

Smoothed Local Symmetries - SLS

Figure 2.7: Smoothed Local Symmetries (SLS).

a) The SLS is defined between two points \( A, B \) on a bounding contour of a shape if the two angles between the segment \( AB \) and the two normals to the curves \( n_A, n_B \) at points \( A \) and \( B \), are equal. The midpoint \( P \) of a segment connecting a pair of locally symmetric points is a symmetry locus.

b) In general a point \( A \) may have local symmetries with several points.

An extension to the GR is the study of Smoothed Local Symmetries - SLS [18]. This description of 2D shapes is defined in two parts: the determination of local symmetry and the formation of maximal smooth loci of these local symmetries. A local symmetry exists between points \( A \) and \( B \) on the contour of a shape if the two angles between the segment \( AB \) and the two normals to the curves at points \( A \) and \( B \) are equal (see Figure 2.7a). In general a point may have local symmetries with several points (see Figure 2.7b). The midpoint of a segment connecting a pair of locally symmetric points is a symmetry locus. These loci are connected into smooth curves (spines) creating the SLS representation of the shape (Figure 2.8a-b). This definition of SLS gives rise to multiple

\(^1\text{An additional constraint is that the difference between the directions of the tangents is greater than a given threshold. Thus two points on a straight edge are not considered locally symmetric}\)
limbs of the SLS in shapes that have deep concavities or protrusions (see Figure 2.8c). This happens when the loci of local symmetries of the inner and outer pairs of curves are close to each other.

![Figure 2.8: Smoothed Local Symmetries (SLS).](image)

a-b) The midpoint of segments connecting pairs of locally symmetric points are connected into smooth curves (spines) creating the SLS representation of a shape (dashed lines).

c) The SLS gives rise to multiple limbs of the SLS in shapes that have deep concavities or protrusions. (only some of the SLS limbs are shown).

In [23], an extension to SLS is presented, namely *Hierarchical Local Symmetry* - HLS, were presented which eliminate these redundancies in the SLS.

In [30] the SLS is extended to circular symmetries and local rotation symmetries (LRS) were presented.

**Axial Shape Description - Discussion**

Computationally, finding these axial descriptions of 2D shapes is complex. The basic methods use voting schemes (described earlier) where pairs of contour points vote for a local symmetry axis or point. According to the highest vote, local symmetry axes are determined. In order to reduce complexity, contours are either approximated by parameterized curves (circular arcs [18], splines [90] etc) or a projection method is assumed as described above [71, 23, 81].

Additional discussion on axial representations can be found in [81] where skew symmetry (see below) is shown to be a special case of GR with a straight spine. Also, in [61], Process Inferring Symmetry Analysis (PISA) was introduced. PISA points have a 1-to-1 correspond with the SAT points but are differently located so that they are more adequate for inferring the process of shape formation.

In [87], the above described axial descriptions of 2D shapes are discussed and compared. It is shown that for the special case where the spines are straight and constant angles of
Chapter 2: Symmetry in Computer Vision

Figure 2.9: The geometry of different local symmetry points.
C - SAT.
P - SLS, HLS, GR.
Q - PISA.

the GR are set to 90°, the following relationship holds: SAT ⊂ GR ⊂ SLS. However, in [81], it is shown that in the general case the above relationship does not hold. In [23] the geometry of different symmetry points is discussed (see Figure 2.9).

Local Symmetry in Images

The above described local symmetries apply to shapes defined by bounding contours and the locality is in terms of subparts of the contour. For grey level images, locality is in terms of local regions or neighborhoods.

\[
P = (P_i + P_j)/2 \\
\Psi = (\theta_i + \theta_j)/2
\]

Figure 2.10: Local Symmetries in digital images.
a) A symmetry operator is applied to every point in the image with respect to a local neighborhood of the point (see text).
b) Dividing the angular orientations into bins, different symmetries can be detected.

In [85], the local symmetry approach is followed in order to find points of local mirror and circular symmetry. These points are defined as points of interest in the digital image. The local symmetries are found using a symmetry operator which is applied to every point in the image and with respect to a local predefined neighborhood. For every point \( P_k \) in the image, a gradient vector \( V_k = (r_k, \theta_k) \) is defined where \( r_k \) denotes the gradient intensity and \( \theta_k \) denotes the gradient direction. For every two points \( P_i \) and \( P_j \), \( l \) denotes the line passing through them and \( \alpha_{ij} \) denotes the angle between \( l \) and the x-axis (see Figure 2.10a). For a given direction \( \psi \), a symmetry measure \( S(P, \psi) \) is defined as follows:

\[
S(P, \psi) = \sum_{P_i, P_j \in \Gamma(P)} \frac{r_ir_j(1 - \cos(\theta_i + \theta_j - 2\alpha_{ij}))(1 - \cos(\theta_i - \theta_j))}{\|P_i - P_j\|}
\]
where $\Gamma(P)$ is a neighborhood of $P$. Points $P_i, P_j$ contribute maximally when there is good correlation between the two gradients $(r_i, r_2)$ and when $(\theta_i - \alpha_{ij}) + (\theta_j - \alpha_{ij})$ is close to $\pi$ (excluding the case where $\theta_i - \alpha_{ij} = \theta_j - \alpha_{ij} = \pi/2$ which occurs when both $P_i, P_j$ are on the same straight edge). Accumulating the values $S(P, \psi)$ in bins according to the direction $\psi$, allows detection of various symmetries. Thus, considering the two bins in Figure 2.10b, horizontal symmetry is detected when $S(P, \psi)$ is summed over $\psi$ values falling in bin 1 and vertical symmetry in bin 2. Circular symmetry is detected by summing over all $\psi$ values.

This method is easily parallelized and is applied directly on gradients of the original image with no need of edge detection or segmentation. This method has been applied to detection of facial features for normalization and recognition [86].

In [96], local mirror symmetry is found in images by autocorrelating a local region in the image with its reflection across a mirror-axis at angle $\theta$. A 1D function of $\theta$ is obtained describing the symmetry content of the region with respect to mirror symmetry at angle $\theta$. Local mirror symmetry is determined to be at the angle with highest correlation value. The correlation is simplified by first averaging the grey values along radial rays so that the angular correlations are then performed on a 1D grey scale function of the averages.

2.4 2D Projections of 3D Symmetries

![Figure 2.11: 2D projections of 3D symmetries.](image)

a-b) Parallel symmetry.

b-c) Skew symmetry.

d) 3D mirror symmetric object.

A 3D mirror symmetric object (or configuration of objects) projected onto a 2D plane, takes the form of special symmetries. Following the definition in [106] - 2D projections of 3D mirror-symmetry can be expressed in two forms of symmetry:

- **parallel symmetry** - two curves $X_1$ and $X_2$ in 2D have parallel symmetry if
  $$\theta_1(s) = \theta_2(as + b)$$
  where $a, b$ are constants and $\theta_i(s)$ is the angle between the tangent of the curve $X_i$ and the x-axis, parameterized by arc length $s$. Examples of parallel symmetric curves are shown in Figure 2.11a-b.
Chapter 2: Symmetry in Computer Vision

- **skew-symmetry** - a 2D pattern or shape is skew symmetric if two axes $x$ and $y$ can be defined (not necessarily orthogonal) such that for every $(x_i, y_i)$ in the pattern, $(-x_i, y_i)$ is also in the pattern. The $x$-axis is called the “skew transverse axis” and the $y$-axis is called the “skew-symmetry axis”. In a skew-symmetric pattern, the lines connecting corresponding skew-symmetric points, are parallel and intersect the skew-symmetry axis at a constant angle. If the skew transverse axis and the skew-symmetry axis are orthogonal, the skew-symmetry is reduced to planar mirror-symmetry (see Figure 2.11d). Examples of skew-symmetric shapes are shown in Figure 2.11c-d.

A combination of these two types of symmetries is also possible, thus in [117], a generalization of these two projected symmetries is defined as “extremum symmetry”. Two curves $X_1$ and $X_2$ have extremum symmetry if a (continuous) 1-to-1 mapping exists between the curves such that extreme points of $X_1$ (extremum curvature, intersection points etc) are mapped to extreme points on $X_2$ and vice versa.

Given a 2D projection of a mirror-symmetric 3D object, two approaches can be taken to deal with the symmetries:

- detection of skew and parallel symmetries by finding the axis of skew or the parallel symmetry curves directly in the 2D image.
- evaluating the orientation in space of the surface or object having skewed symmetry, and then obtaining the 3D mirror-symmetry.

### 2.4.1 Direct Detection of Skew and Parallel Symmetries

The direct method of detecting the skew axes in a 2D projection has been approached from several directions.

**Projection Methods and Other Voting Schemes**

The basic method (dealing with skew-symmetry in collections of 2D curves) is the *projection* methods that search for pairs of segment points which are possibly skew-symmetric at a given skew angle. In [83, 71], for each angle $\alpha$, defining the angle of skew, the input curves are segmented into cuts by strips at angle $\alpha$ (see Figure 2.12a). Two cuts are selected per strip and points in these cuts are paired as skewed-symmetric pairs (in Figure 2.12a points $A,B$ in strips $F_A, F_B$ are paired). The midpoint between paired points are stored. A line is fit through the collection of midpoints. Thus, for each angle $\alpha$, a
Figure 2.12: a) The projection method for finding skew-symmetry. For each angle $\alpha$, defining the angle of skew, the curves are divided into cuts by strips at angle $\alpha$. Two cuts are selected per strip and points in these cuts are paired as skewed-symmetric pairs. Thus, points $A, B$ in strips $F_A, F_B$ are paired. The midpoint between paired points are stored as a points on the skew-symmetry axis.

b) The intersection property states that the curve tangent (dotted lines) at two skew-symmetric points having the same coordinate on the skew-symmetry axis, intersect on the skew symmetry axis.

line is found describing the optimal skew-symmetry axis. The line with largest support is selected as the skew-symmetry axis of the image.

In [38], a similar method is described where the symmetry axis is found for projected objects of revolution, by pairing contour points. The matching complexity is reduced in this case by several constraints. In [94] constraints are introduced into the voting scheme to detect projected mirror-symmetry in generalized cones.

An extension to this method is proposed in [90], where curves are approximated by b-splines, thus reducing the number of elements to be matched and reducing the complexity of search for the symmetry axis.

A variant of this approach uses the Hough transform where pairs of points vote for possible symmetry curves or axes and the one with the highest score is selected [73, 82, 114, 117, 72] (see also Section 2.3.2).

Another approach to detection of skew-symmetry is presented in [34, 35]. In this work, skew-symmetry in a planar image is assumed to be generated from a 2D image which has undergone an oblique coordinate transformation followed by a rotation in the image plane (rather than from an orthographic projection of a rotated 3D mirror-symmetric object). A constraint on the parameters of the skew-symmetry axes is found, based on moments of the figure, reducing the parameters to a 1D curve in skew-parameter space.
Voting Schemes - Discussion

The search methods based on voting schemes as described above can be described as local methods for skew-symmetry finding. The methods based on moments can be considered global skew-symmetry approaches (see Section 2.3.3). Global vs. local skew-symmetry finding are discussed in [41]. The global methods are much more efficient in run time, having a linear time complexity whereas local search and voting methods are exponential. However the global methods are encumbered with the drawback that they are highly sensitive to occlusion and noise.

The search and voting are robust, to a certain degree, under noise and occlusion in the input image. This is due to the flexibility and noise tolerance in the line fitting of [83, 117] and in the peak detection in the hough transform methods. However, these same features may also give rise to incorrect or imprecise symmetry axes. The process of line fitting and peak detecting, assume some a-priori knowledge on the existence of symmetry axes and the number of such axes (although thresholding heuristics could provide these variables, the process is image and noise dependent and generally not stable). These studies approach symmetry as a binary feature: either the projection is symmetric or it is not (and accordingly, the original 3D object is symmetric or not). Even though the certainty of line fit or the peak height and density might be used as a quantifier of the certainty and quality of the symmetry axis that was found, there is no assurance that these values correctly measure the quality and “amount” of symmetry in the object (and most probably do not reflect the intuition and perception of continuous symmetry).

2.4.2 Detecting Skew-Symmetry by Recovering 3D Structure

A different approach to dealing with skew-symmetry is to reconstruct surface orientation and object orientation of the original 3D mirror-symmetric object.

![Figure 2.13: Skew Ambiguities.](image)

- a) Many 3D planar objects project onto a given skew-symmetric figure.
- b) Shear ambiguity in a scalene triangle where 3 pairs of skew-symmetry axes can be found.
Deriving the orientation of a planar 3D object from its 2D skew-symmetric projection has no unique solution since there are many 3D planar objects that project and create a given skew-symmetric figure (see Figure 2.13a). This redundancy is described in [34] as being due to shear ambiguity and to gradient ambiguity. Shear ambiguity is a multiplicity of skew-symmetry coordinates and is analogous to multiple axes of mirror symmetry in planar figures (such as a square, equilateral triangle, circle, etc). For example Figure 2.13b shows shear ambiguity in a scalene triangle where 3 pairs of skew-symmetry axes can be found. The gradient ambiguity is defined as the ambiguity arising from the fact that there are an infinite number of rotations in 3D that project a 3D mirror-symmetry axis to a given skew-symmetry axis.

Figure 2.14: Constraints on symmetric surface orientations.
For a given skew-symmetry axis, the possible orientations of the original mirror-symmetric planar figure is constrained to lie on a hyperbola (thick lines) in the \( p - q \) gradient space (b). The asymptotes of the hyperbola are shown to be orthogonal to lines drawn in the \( p - q \) space at angles corresponding to the skew-symmetry axis in the image (a) (angles \( \alpha, \beta \)).

All possible orientations of a surface can be represented as a 2D gradient space \( p - q \) where \( p \) denotes the gradient in the \( x \)-coordinate direction and \( q \) in the \( y \)-coordinate direction. In [53], it is shown that for a given skew-symmetry axis, the possible orientations of the original mirror-symmetric planar figure is constrained to lie on a hyperbola in the \( p - q \) gradient space. The asymptotes of the hyperbola are shown to be orthogonal to lines drawn in the \( p - q \) space at angles corresponding to the skew-symmetry axis in the image (see Figure 2.14). Given this constraint, it is suggested [53] that the orientation to be chosen as the solution is that which is closest to the origin in gradient space (marked as \( m \) in Figure 2.14b), and corresponds to the orientation of minimum slant, where slant is defined as \( \sqrt{p^2 + q^2} \) (\( m' \) in Figure 2.14b corresponds to the interpretation analogous to the necker cube reversal). This choice is generally in accord with human interpretation of the orientation of a surface, given its skew-symmetric projection.
In [97] an extension of skew-symmetry to cylindrical surfaces is described. Specifically, this work deals with cylindrical surfaces where one of the principle curvatures is zero and all lines of zero curvature are parallel to each other (these lines correspond to the skew-transverse axis direction in planar figures). The lines of greatest curvature, denoted symmetry lines, are planar (and correspond to the skew-symmetry axis direction).

c) The lines of zero curvature and the lines of symmetry are assumed to be orthogonal in 3D, thus the lines of zero curvature project as straight lines and each intersects successive symmetry lines at constant angle (Figure 2.15c). Based on this orthogonality assumption, the skew-symmetry constraint defined above [53] is used to evaluate the surface orientation at every intersection of the zero curvature lines and the symmetry lines.

An extension to this work is found in [114] dealing with general bounded surfaces. In this work, the zero curvature lines and the lines of symmetry are not given (as in [97]) but are created on the surface as a net, from information on the boundary curves which are assumed to be lines of curvature. The lines of the created net are assumed to be orthogonal in 3D. In both [114, 97], surface orientation is evaluated at special points on the net where error is small and then propagated to other points on the net.

These two methods [97, 114], however, do not always give the correct result due to errors in propagation. For example, as is shown in [106], these methods do not distinguish between the orientation of the cylindrical surface in Figure 2.16a and that of Figure 2.16b. The problem according to [106] is that every surface is considered independently and no consideration is given to the object as a whole, i.e. to all emerging surfaces. Interpretation of neighboring surface orientation influences the perception of any given surface. For
Figure 2.16: Cylindrical 3D surfaces.

a-b) The cylindrical surface in (a) has different orientation than that in (b).

c-d) The top surface in (c) is perceived as planar in 3D whereas the same contour is interpreted as non-planar in (d).

example, as shown in [106], the top surface in Figure 2.16c is perceived as planar in 3D whereas the same contour is interpreted as non-planar in Figure 2.16d.

The idea of determining orientation or pose, is extended from single surfaces to combinations of surfaces, thus exploiting constraints existing between the surfaces.

In [53], the skew symmetry constraint in the $p - q$ gradient space is combined with intra-surface constraints to determine orientation of surfaces in polyhedral scenes. The intra-surface constraints are derived from the fact that the segment joining two points in gradient space must be orthogonal to the edge in the image separating two planes of orientation equivalent to the two points in gradient space (Figure 2.17).

In [106], the local orientation of surfaces are evaluated for cylindrical objects by combining skew-symmetry and intra-surface constraints. The shared boundary constraint, as described above for two planar surfaces, is extended in [106] to curved surfaces by approximating the surface using polygonal approximations.

Figure 2.17: The intra-surface constraint.

The intra-surface constraints are derived from the fact that the segment joining two points $T_1, T_2$ in gradient space (b) must be orthogonal to the edge in the image (a) separating two planes of orientation equivalent to $T_1, T_2$. 

The image gradient space
The studies described above, reduce the number of possible solutions for orientation of a 2D projection, however they do not determine the orientation absolutely (the necker-cube reversal, for instance, always exists for line drawings). In order to determine absolute orientation, additional assumptions or constraints must be included. For example, for specific classes of objects such as cylindrical objects, more precise results for structure and orientation can be obtained. In [107], cross-section of cylindrical objects are approximated by ellipses such that orientation of the surface can be determined uniquely. Assuming 3D objects are Straight Homogeneous Generalized Cylinders (SHGC), constraints and invariants can be derived to assist in finding symmetry axes and orientation [82, 46]. Further assuming the cross-section of the SHGC is symmetric, the orientation and structure is found in [114], based on skew-symmetry finding in the cross-section and based on the orthogonality of the cylinder axis and the skew-symmetry axes. Another class of shapes, the surfaces of revolution, are studied from their projections [70] and the symmetry axis is reconstructed [70, 38, 33].

![Cross-section and boundary contour](image)

**Figure 2.18: 3d reconstruction.**
a) Twisted 2D manifolds in 3D, project bounding curves which do not necessarily form a continuous projected symmetry.
b) Full 3D objects can be generated from the projected bounding curves and the projection of cross-sections of the object.

Given additional information, such as depth information at certain positions in the image, symmetric objects can be reconstructed. Thus, in [98], 3D objects are generated from 2D line drawings by interpolating orientation (and then depth) parameters from initial values given as input. This work can generate twisted 2D manifolds in 3D, from their projection bounding curves which do not necessarily form a continuous projected symmetry (see Figure 2.18a). Full 3D objects are generated from the projected bounding curves and the projection of cross-sections of the object (see Figure 2.18b).
2D Projections of Rotational Symmetries

A last note on projected symmetries: The study of projections of symmetrical 3D objects onto 2D planes, predominantly deals with mirror-symmetry, most probably, since these symmetries are dominant in our world. Additionally, projections of 3D mirror-symmetry retain some symmetric features, namely, skew and parallel symmetry, as discussed above. Projections of rotationally symmetric 3D objects lose their symmetry features. In [117], an algorithm for detecting projected rotational symmetry is presented, where the projected image is backprojected to 3D according to given gradient parameters. Perfect rotational symmetry is searched for in the backprojection using a voting scheme (see Section 2.3 for detection of 2D rotational symmetry). The projected rotational symmetry is found by following the voting scheme for all possible gradient parameters.

2.5 Additional Approaches to 3D Symmetry Reconstruction from 2D Projections

A class of 3D reconstruction methods from 2D projections is that of computerized tomography. These methods assume the 2D projection represents density of the object along the lines of projection. Classic tomography requires infinite projections to fully reconstruct the object. However, it is shown in [27] that symmetry of the 3D object can be exploited to assist in reconstruction. Specifically, a surface of revolution having axial symmetry, is reconstructed from a single 2D projection in the direction perpendicular to the axis of rotation of the object. Bayesian regularization is incorporated into the reconstruction method inorder to reduce noise effects.

A different approach to 3D symmetry reconstruction is found in [99] for reconstruction of axially symmetric objects from 2D intensity images. The method uses deformable models (also known as “active contours” or “snakes”) which can be viewed as elastic tubes coupled with elastic spines. The model deformation is controlled by internal and external energy functionals. The internal energy determines the elastic properties of the deformable surface, and the external energy enforces constraints on the deformation according to the given 2D intensity image. The deformable models are constrained by the internal energies to be axially symmetric and by the external energies to projects as the given intensity image.

A third approach to reconstruction from 2D projections, follows structure-from-motion schemes. In this class of reconstructions, the 2D projections are assumed to be composed of feature points, and possibly linked by edges. The reconstructed object is a 3D configuration of points (or a graph-like structure). Standard structure from motion techniques
require at least three 2D views of a 3D object inorder to reconstruct the object [50, 104]. At least two views (orthographic) are required for constructing the affine structure which can be used for recognition [105, 80]. When the 3D objects are known to be symmetric, the symmetries can be exploited to reduce the number of views required. Specifically, given a single 2D projection (orthographic) of a mirror-symmetric 3D object which is a non-accidental view (i.e. the projection is not along the mirror-plane of the object), then an additional non-accidental view can be generated by reflecting the projection image about a 2D axis (see Figure 2.19a). Thus, in [80], it is proven that recognition of a mirror-symmetric object can be achieved from a single 2D view and that for higher order symmetries (i.e. more than one mirror-plane) the object can be reconstructed from a single view. In the case of perspective projection - it is shown in [67, 88] that the structure of a 3D mirror-symmetric object can be reconstructed from a single view by calculating the direction vector to the vanishing point (see Figure 2.19b). Exploiting mirror-symmetry for reconstruction has also led to methods involving actual mirrors, thus creating a mirror-symmetric image from a single asymmetric object [74, 51, 1, 54].

2.6 Applications Using Symmetry

2.6.1 2D Applications

The classic use of symmetry in 2D is for pattern and shape description and correspondingly for pattern and shape recognition. Most of the detection methods described in
Section 2.3 can be used as shape descriptors and with some pre or postprocessing can become invariant to rigid transformations of the shape. Given a shape described by its symmetry, it can be matched with symmetry descriptions of models in the database.

Symmetry can also be exploited for recognition of shapes which are not necessarily symmetric. In [65], 2D object recognition is performed by superimposing a reflected 2D model on the test image. If the test and model images match, the superimposed image is a mirror-symmetric image. Using symmetry detection algorithms [64], the superimposed image can be tested for symmetry and the 2D model, verified. In [96] 2D shapes (not necessarily symmetric) are represented by a sequence of symmetry values as follows: radial rays are taken at constant angular intervals from the centroid of the shape. Along these rays, local symmetry and an associated confidence measure is evaluated. For each ray, the symmetry value associated with the highest confidence measure is recorded. The sequence of recorded symmetry values is used as a shape descriptor for both the test pattern and the model. Searching for permutations of the shape descriptor sequence, allows for recognition of rotated patterns.

In a different application, local symmetry is used to detect facial features in face images [86]. These features are then used to normalize the face image and creating a standard representation for matching with standard models in a face database.

Symmetry is used in [15] to discriminate textures.

Symmetry takes an important role in perceptual organization - i.e. grouping and segmentation (see [62] for example). Thus heuristic for grouping of elements extracted from images, usually involve symmetry as an important criteria for grouping (see for example [93, 115]).

Specific object recognition applications can be found in [58, 128] for detection and following of vehicles.

### 2.6.2 3D Applications

As described in Section 2.4.2, symmetries are exploited for reconstruction of 3D structure from 2D images.

Symmetry is also exploited in object recognition processes inorder to reduce search complexity. Thus, in [73], model based matching from 2D images is performed by first finding skew-symmetry in the 2D image and selecting hypothesis 3D models having mirror-symmetry axes greater or equal in length, to the skew-symmetry axis (this assumes that the 3D symmetry axis may only decrease in length under projection, due to slant). Verification of the model is performed by reconstructing the projected image from the 3D object, after evaluating the slant from the ratio of model and skew symmetry axes.
In another approach [32], model databases of polyhedral objects are greatly compressed by eliminating redundancies arising from symmetries in the polyhedral objects. Specific 3D application can be found in [13], where symmetry is used in guiding robot grasping.

2.7 Measuring Symmetry

In general, many of the symmetry detection methods described in this chapter, include some form of thresholding in order to overcome the noise and imprecisions in digital images. However the notion of measuring of symmetry is very scarcely discussed. In an early work, Grünbaum [42] reviews methods of geometrically measuring symmetry of convex sets. Yodogawa [116] has presented an evaluation of symmetry (namely “Symmetry”) in single patterns which uses information theory to evaluate the distribution of symmetries in a pattern. Marola [64] presents a coefficient of mirror-symmetry with respect to a given axis. Global mirror-symmetry of an object (image) is found by roughly estimating the axis location and then fine tuning the location by minimizing the symmetry coefficient. Gilat [37], Hel-Or et.al.[44] and Avnir et.al. [10] present the idea of a Measure of Chirality (a measure of deviation from mirror-symmetry). Similar to Marola, Gilat’s chirality measure is based on minimizing the volume difference between the object and its reflection through a varying plane of reflection. Hel-Or et al. present a measure of chirality for 2D objects based on rotational effects of chiral bodies on the surround. Additional discussion on measuring chirality can be found in chemistry research and is reviewed in [120].

These symmetry evaluation methods are each limited to a certain type of symmetry (mirror or circular symmetry) and are generally of high complexity.
Chapter 3

Measuring Symmetry

3.1 A Continuous Symmetry Measure - Definition

We define the Symmetry Distance (SD) as a quantifier of the minimum effort required to turn a given shape into a symmetric shape. This effort is measured by the mean of the square distances each shape point is moved from its location in the original shape to its location in the symmetric shape. Note that no a priori symmetric reference shape is assumed.

Denote by $\Omega$ the space of all shapes of a given dimension, where each shape $P$ is represented by a sequence of $n$ points $\{P_i\}_{i=0}^{n-1}$. We define a metric $d$ on this space as follows:

$$d : \Omega \times \Omega \rightarrow R$$

$$d(P, Q) = d(\{P_i\}, \{Q_i\}) = \frac{1}{n} \sum_{i=0}^{n-1} \|P_i - Q_i\|^2$$

This metric defines a distance function between every two shapes in $\Omega$.

We define the Symmetry Transform of a shape $P$, as the symmetric shape $\hat{P}$ closest to $P$ in terms of the metric $d$.

The Symmetry Distance (SD) of a shape $P$ is now defined as the distance between $P$ and its Symmetry Transform $\hat{P}$:

$$SD = d(P, \hat{P})$$

The SD of a shape $P = \{P_i\}_{i=0}^{n-1}$ is evaluated by finding the symmetry transform $\hat{P} = \{\hat{P}_i\}_{i=0}^{n-1}$ of $P$ (Figure3.1d) and computing:

$$SD = \frac{1}{n} \sum_{i=0}^{n-1} \|P_i - \hat{P}_i\|^2$$

This definition of the Symmetry Distance implicitly implies invariance to rotation and translation. Normalization of the original shape prior to the transformation additionally
Figure 3.1: Calculating the Symmetry Distance of a shape:

a) Original shape \( \{P_0, P_1, P_2\} \).

b) Normalized shape \( \{P'_0, P'_1, P'_2\} \), such that maximum distance to the center of mass is one.

c) Applying the symmetry transform to obtain a symmetric shape \( \{\hat{P}_0, \hat{P}_1, \hat{P}_2\} \).

d) SD = \frac{1}{3}(\|P'_0 - \hat{P}_0\|^2 + \|P'_1 - \hat{P}_1\|^2 + \|P'_2 - \hat{P}_2\|^2)

SD values are multiplied by 100 for convenience of handling.

allows invariance to scale (Figure 3.1). We normalize by scaling the shape so that the maximum distance between points on the contour and the centroid is a given constant (in this paper all examples are given following normalization to 100). The normalization presents an upper bound of on the mean squared distance moved by points of the shape. Thus the SD value is limited in range, where SD=0 for perfectly symmetric shapes (see Appendix B).

The general definition of the Symmetry Distance enables evaluation of a given shape for different types of symmetries (mirror-symmetries, rotational symmetries etc). Moreover, this generalization allows comparisons between the different symmetry types, and allows expressions such as “a shape is more mirror-symmetric than rotationally-symmetric of order two”. An additional feature of the Symmetry Distance is that we obtain the symmetric shape which is ‘closest’ to the given one, enabling visual evaluation of the SD.

An example of a 2D polygon and its symmetry transforms and SD values are shown in Figure 3.2. Note that shape 3.2e is the most similar to the original shape 3.2a and,
Figure 3.2: Symmetry Transforms and Symmetry Distances of a 2D polygon.
a) The 2D polygon.
b) Symmetry Transform of (a) with respect to $C_2$-symmetry (SD = 1.87).
c) Symmetry Transform of (a) with respect to $C_3$-symmetry (SD = 1.64).
d) Symmetry Transform of (a) with respect to $C_6$-symmetry (SD = 2.53).
e) Symmetry Transform of (a) with respect to Mirror-symmetry (SD = 0.66).

indeed, its SD value is the smallest.

In the next Section we describe a geometric algorithm for deriving the Symmetry Transform of a shape. In Section 3.3 we deal with the initial step of representing a shape by a collection of points.

### 3.2 Evaluating the Symmetry Transform

In this Section we describe a geometric algorithm for deriving the Symmetry Transform of a shape represented by a sequence of points $\{P_i\}_{i=0}^{n-1}$. In practice we find the Symmetry Transform of the shape with respect to a given point-symmetry group (see Section 4.1.1 for a review of algebraic definitions). For simplicity and clarity of explanation, we describe the method by using some examples. Mathematical proofs and derivations can be found in Section 4.1.2.

Following is a geometrical algorithm for deriving the symmetry transform of a shape $P$ having $n$ points with respect to rotational symmetry of order $n$ ($C_n$-symmetry). This method transforms $P$ into a regular $n$-gon, keeping the centroid in place.
Figure 3.3: The $C_n$-symmetry Transform of 3 points.

a) original 3 points $\{P_i\}_{i=0}^2$.
b) Fold $\{P_i\}_{i=0}^2$ into $\{\hat{P}_i\}_{i=0}^2$.
c) Average $\{\hat{P}_i\}_{i=0}^2$ obtaining $\hat{P}_0 = \frac{1}{3} \sum_{i=0}^2 \hat{P}_i$.
d) Unfold the average point obtaining $\{\hat{P}_i\}_{i=0}^2$.
The centroid is marked by $\oplus$.

**Algorithm for finding the $C_n$-symmetry transform:**

1. **Fold** the points $\{P_i\}_{i=0}^{n-1}$ by rotating each point $P_i$ counterclockwise about the centroid by $2\pi i/n$ radians obtaining the points $\{\hat{P}_i\}_{i=0}^{n-1}$ (Figure 3.3b).

2. **Average** the points $\{\hat{P}_i\}_{i=0}^{n-1}$ obtaining point $\hat{P}_0$ (Figure 3.3c).

3. **Unfold** the points by duplicating $\hat{P}_0$ and rotating clockwise about the centroid by $2\pi i/n$ radians obtaining the $C_n$-symmetric points $\{\hat{P}_i\}_{i=0}^{n-1}$ (Figure 3.3d).

The set of points $\{\hat{P}_i\}_{i=0}^{n-1}$ is the symmetry transform of the points $\{P_i\}_{i=0}^{n-1}$, i.e., they are the $C_n$-symmetric configuration of points closest to $\{P_i\}_{i=0}^{n-1}$ in terms of the metric $d$ defined in Section 3.1 (in terms of the average distance squared). Proof is given in Section 4.1.2.

The common case, however, is that shapes have more points than the order of the symmetry. For symmetry of order $n$, the folding method can be extended to shapes having a number of points which is a multiple of $n$. A 2D shape $P$ having $qn$ points is represented as $q$ sets $\{S_r\}_{r=0}^{q-1}$ of $n$ interlaced points $S_r = \{P_{rn+i}\}_{i=0}^{n-1}$ (see discussion in Appendix A). The $C_n$-symmetry transform of $P$ (Figure 3.4) is obtained by applying the above algorithm to each set of $n$ points separately, where the folding is performed about the centroid of all the points (See Equation 4.7 in Appendix 4.1.2).

The procedure for evaluating the symmetry transform for mirror-symmetry is similar:
Figure 3.4: Geometric description of the $C_3$-symmetry transform for 6
points.
The centroid of the points is marked by $\oplus$.
a) The original points shown as two sets of 3 points: $S_0 = \{P_0, P_2, P_4\}$ and
$S_1 = \{P_1, P_3, P_5\}$.
b) The obtained $C_3$-symmetric configuration.

Given a shape having $m = 2q$ points we divide the points into $q$ pairs of points (see Appendix A) and given an initial guess of the symmetry axis, we apply the folding/unfolding method as follows (see Figure 3.5):

**Algorithm for finding the mirror-symmetry transform:**

1. for every pair of points $\{P_0, P_1\}$:
   
   (a) fold - by applying the identity to point $P_0$ and reflecting point $P_1$ across the
   mirror symmetry axis obtaining $\{\hat{P}_0, \hat{P}_1\}$ (Figure 3.5a).
   
   (b) average - the folded points obtaining a single averaged point $\hat{P}_0$ (Figure 3.5b).

   (c) unfold - by reflecting $\hat{P}_0$ back across the mirror symmetry axis obtaining point
   $\hat{P}_1$ (Figure 3.5c).

2. minimize over all possible axis of mirror-symmetry.

The minimization performed in step 2 is, in practice, replaced by an analytic solution
(derivation and proof can be found in Section 4.1.3).
This method extends to any finite point-symmetry group $G$ in any dimension, where the
folding and unfolding are performed by applying the group elements about the centroid
(see derivations in Section 4.1.2):
Figure 3.5: The mirror-symmetry transform of a single pair of points for angle $\theta$, where the centroid of the shape is assumed to be at the origin.

a) The two points $\{P_0, P_1\}$ are folded to obtain $\{\tilde{P}_0, \tilde{P}_1\}$.

b) Points $\tilde{P}_0$ and $\tilde{P}_1$ are averaged to obtain $\tilde{P}_0$.

c) $\tilde{P}_1$ is obtained by reflecting $\tilde{P}_0$ about the symmetry axis.

Given a symmetry group $G$ (having $n$ elements) and given a shape $P$ represented by $m = qn$ points, the symmetry transform of the shape with respect to $G$-symmetry is obtained as follows:

**Algorithm for finding the $G$-symmetry transform:**

- The points are divided into $q$ sets of $n$ points.

- For every set of $n$ points:
  - The points are folded by applying the elements of the $G$-symmetry group.
  - The folded points are averaged, obtaining a single averaged point.
  - The averaged point is unfolded by applying the inverse of the elements of the $G$-symmetry group. A $G$-symmetric set of $n$ points is obtained.

- The above procedure is performed over all possible orientations of the symmetry axis and planes of $G$. Select that orientation which minimizes the Symmetry Distance value. As previously noted, this minimization is analytic in 2D (derivation is given in Section 4.1.3) but requires an iterative minimization process in 3D (except for the 3D mirror-symmetry group where a closed form solution has been derived - see Chapter 5).

Figure 3.6 displays the Symmetry Distance values obtained using the folding method described above with respect to mirror-symmetry, for the shapes shown in Figure 1.2.
Figure 3.6: The Symmetry Distance values with respect to mirror symmetry of the shapes of Figure 1.2 as a function of the percentage of arm shortening. The points a, b, c, and d are the symmetry values obtained for the corresponding shapes in Figure 1.2.

### 3.3 Point Selection for Shape Representation

As symmetry has been defined on a sequence of points, representing a given shape by points must precede the application of the symmetry transform. Selection of points influences the value of SD and depends on the type of object to be measured. If a shape is inherently created from points (such as a graph structure or cyclically connected points creating a polygon) we can represent a shape by these points (Figure 3.7). This is the case when analyzing symmetry of molecules ([125, 124, 118]).

There are several ways to select a sequence of points to represent continuous 2D shapes.

Figure 3.7: When measuring symmetry of shapes inherently created from points we represent the shape by these points.
Chapter 3: Measuring Symmetry

Figure 3.8: Point selection by equal distance: points are selected along the contour such that each point is equidistant to the next in terms of curve length. In this example six points are distributed along the contour spaced by \( \frac{1}{6} \) of the full contour length.

One such method is selection at **equal distances** - the points are selected along the shape's contour such that the curve length between every pair of adjacent points is equal (Figure 3.8).

Figure 3.9: Selection by smoothing

a) Original continuous contour.
b) Points are selected at equal distances along the continuous contour.
c-f) The smoothed shape is obtained by replacing each boundary point with the average of its neighborhood. The amount of smoothing depends on the size of the neighborhood. The smoothed shapes (c-f) are obtained when neighborhood includes 5,10,15 and 20 percent of the boundary points, respectively.
g-j) The resampling of points on the original shape using the smoothed shapes (c-f) respectively. Notice that fewer points are selected on the “noisy” part of the contour.
Chapter 3: Measuring Symmetry

Figure 3.10: Selection by smoothing (in practice)

a) Continuous contour sampled at equal distances. Points \{M_j\} are obtained.

b) Each sampled point \(M_j\) is replaced by the centroid \(M'_j\) of a finite number of its neighboring sampled points (in this example the neighborhood includes 15 points). The “smoothed” shape is obtained by connecting the centroid points \{M'_j\}.

c) A second sampling of points is performed at equal distances on the smoothed contour obtaining points \{P'_i\} (white points in Figure). The sample point \(P'_i\) is considered as an interpolation between the two points \(M'_{j1}\) and \(M'_{j2}\).

d) The sample point \(P'_i\) is backprojected onto the original contour at the point \(P_i\) which is the interpolated point between the corresponding sampled points \(M_{j1}\) and \(M_{j2}\) on the original contour.

In many cases, however, contour length is not a meaningful measure, as in noisy or occluded shapes. In such cases we propose to select points on a smoothed version of the contour and then project them back onto the original contour. The smoothing of the continuous contour is performed by moving each point on the continuous contour to the centroid of its contour neighborhood. The greater the size of the neighborhood, the greater is the smoothing (see Figure 3.9). The level of smoothing can vary and for a high level of smoothing, the resulting shape becomes almost a circle about the centroid ([40, 68]). We use the following procedure for selection by smoothing:

The original contour is first sampled at very high density and at equal distances along the contour obtaining the sampled points \{M_j\} (Figure 3.10a). Following, each sampled point \(M_j\) is replaced by the centroid \(M'_j\) of a finite number of its neighboring sampled points. The “smoothed” shape is obtained by connecting the centroid points \{M'_j\} (Figure 3.10b). A second sampling of points is performed on the smoothed contour, where the points \{P'_i\} are, again, selected at equal distances along the contour. The backprojection is performed by considering each sampled point \(P'_i\) as an interpolated point between two points of the smoothed shape \(M'_{j1}\) and \(M'_{j2}\) (Figure 3.10c). The sample point \(P'_i\) is backprojected onto the original contour at the point \(P_i\) which is the interpolated point between the corresponding sampled points \(M_{j1}\) and \(M_{j2}\) on the original contour (Figure 3.10d). i.e.

\[
P_i = (P'_i - M_{j1}) \frac{(M_{j1} - M_{j2})}{(M'_j_{j1} - M'_j_{j2})} + M_{j1}.
\]

The ultimate smoothing is when the shape is smoothed into a circle. In this case, equal
distances on the circular contour is equivalent to equal angles about the center. For maximum smoothing we, therefore, use selection at equal angles (Figure 3.11) where points are selected on the original contour at equal angular intervals around the centroid.

Figure 3.11: Selection at equal angles: points are distributed along the contour at regular angular intervals around the centroid.

3.4 Symmetry of Occluded Shapes - Center of Symmetry

When a symmetric object is partially occluded, we use the symmetry distance to evaluate the symmetry of the occluded shapes, locate the center of symmetry and reconstruct the symmetric shape most similar to the unoccluded original.

Figure 3.12: An occluded shape with sampled points selected at equal angles about the center of symmetry (marked by ⊕). The symmetry distance obtained using these points is greater than the symmetry distance obtained using points selected at equal angles about the centroid (marked by +).

As described in Section 3.3, a shape can be represented by points selected at regular angular intervals about the centroid. Angular selection of points about a point other than the centroid will give a different symmetry distance value. We define the center of symmetry of a shape as that point about which selection at equal angles gives the minimum symmetry distance value. When a symmetric shape is complete the center of symmetry coincides with the centroid of the shape. However, the center of symmetry
of truncated or occluded objects does not align with its centroid but is closer to the centroid of the complete shape. Thus the center of symmetry of a shape is robust under truncation and occlusion.

![Figure 3.13: Reconstruction of an occluded shape.](image)

a) Original occluded shape, its centroid (+) and its center of symmetry (⊕). b,c) The closest $C_5$-symmetric shapes following angular selection about the centroid (b) and about the center of symmetry (c). Selection about the centroid gives an almost featureless shape, while selection about the center of symmetry yields a more meaningful shape.

To locate the center of symmetry, we use an iterative procedure of gradient descent that converges from the centroid of an occluded shape to the center of symmetry. Denote by center of selection, that point about which points are selected using selection at equal angles. We initialize the iterative process by setting the centroid as the center of selection. At each step we compare the symmetry value obtained from points selected at equal angles about the center of selection with the symmetry value obtained by selection about points in the center of selection’s immediate neighborhood. That point about which selection at equal angles gives minimum symmetry value, is set to be the new center of selection. If center of selection does not change, the neighborhood size is decreased. The process is terminated when neighborhood size reaches a predefined minimum size. The center of selection at the end of the process is taken as the center of symmetry.

In the case of occlusions (Figure 3.13), the closest symmetric shape obtained by angular selection about the center of symmetry is visually more similar to the original than that obtained by angular selection about the centroid. We can reconstruct the symmetric shape closest to the unoccluded shape by obtaining the symmetry transform of the occluded shape following angular selection about the center of symmetry (see Figure 3.13c). In Figure 3.14 the center of symmetry and the closest symmetric shapes were found for several occluded flowers.
Figure 3.14: Application example.

a) A collection of occluded asymmetric flowers.
b) Contours of the occluded flowers were extracted manually.
c) The closest symmetric shapes and their center of symmetry.
d) The center of symmetry of the occluded flowers are marked by ‘+’.

The process of reconstructing the occluded shape can be improved by altering the method of evaluating the symmetry of a set of points. As described in Section 3.2, the symmetry of a set of points is evaluated by folding, averaging and unfolding about the centroid of the points. We alter the method as follows:

1. The folding and unfolding (steps 1 and 3) will be performed about the center of selection rather than about the centroid of the points.

2. Rather than averaging the folded points (step 2), we apply other robust clustering methods. In practice we average over the folded points, drop the points farthest from the average and then reaverage (see Figure 3.15).

The improvement in reconstruction of an occluded shape is shown in Figure 3.16. This method improves both the shape and the localization of the reconstruction. Under the assumption that the original shape was symmetric, this method can reconstruct an occluded shape very accurately.
Figure 3.15: Improving the averaging of folded points
a) An occluded shape with points selected using angular selection about the center of symmetry (marked as ∗).
b) A single set (orbit) of the selected points of a) is shown.
c) Folding the points about the centroid of the shape (marked as +), points are clustered sparsely.
d) Folding the points about the center of symmetry of the shape, points are clustered tightly. Eliminating the extremes (the two farthest points) and averaging will result in smaller averaging error and better reconstruction.

Figure 3.16: Reconstruction of an occluded almost symmetric shape.
The original shape is shown as a dashed line and the reconstructed shape is shown as a solid line.
a) The closest symmetric shape following angular selection about the centroid.
b) The closest symmetric shape following angular selection about the center of symmetry.
c) The closest symmetric shape following angular selection about the center of symmetry and altered symmetry evaluation (see text). The reconstructed shape is more similar to the existing shape as well as more accurately located.
3.5 Symmetry of Points with Uncertain Locations

In most cases, sensors do not have absolute accuracy and the location of each point in a sensed pattern can be given only as a probability distribution. Given sensed points with such uncertain locations, the following properties are of interest:

- The most probable symmetric configuration represented by the sensed points.
- The probability distribution of symmetry distance values for the sensed points.

3.5.1 The Most Probable Symmetric Shape

![Diagram](image)

Figure 3.17: Folding measured points.

a) A configuration of 6 measurement points $Q_0\ldots Q_5$. The dot represents the expected location of the point and the rectangle has width and length proportional to the standard deviation.

b) Each measurement $Q_i$ was rotated by $2\pi i/6$ radians about the centroid of the expected point locations (marked as '+'

Figure 3.17a shows a configuration of points whose locations are given by a normal distribution function. The dot represents the expected location of the point and the rectangle represents the standard deviation marked as rectangles having width and length proportional to the standard deviation. In this section we briefly describe a method of evaluating the most probable symmetric shape under the Maximum Likelihood criterion ([26]) given the sensed points. Detailed derivations and proofs are given in Section 4.2.1. For simplicity we describe the method with respect to rotational symmetry of order $n$ ($C_n$-symmetry). The solution for mirror symmetry or any other symmetry is similar (see Section 4.2.2).
Chapter 3: Measuring Symmetry

Given \( n \) ordered points in 2D whose locations are given as normal probability distributions with expected location \( P_i \) and covariance matrix \( \Lambda_i \):

\[
Q_i \sim \mathcal{N}(P_i, \Lambda_i), \quad i = 0 \ldots n - 1,
\]

we find the \( C_n \)-symmetric configuration of points at locations \( \{\hat{P}_i\}_{0}^{n-1} \) which is optimal under the Maximum Likelihood criterion ([26]).

Denote by \( \omega \) the centroid of the most probable \( C_n \)-symmetric set of locations \( \hat{P}_i \):

\[
\omega = \frac{1}{n} \sum_{i=0}^{n-1} \hat{P}_i.
\]

The point \( \omega \) is dependent on the location of the measurements \( \{P_i\} \) and on the probability distribution associated with them \( \{\Lambda_i\} \). Intuitively, \( \omega \) is positioned at that point about which the folding (described below) gives the tightest cluster of points with small uncertainty (small s.t.d.). We assume for the moment that the centroid \( \omega \) is given. A method for finding \( \omega \) is derived in Section 4.2.1. We use a variant of the folding method which was described in Section 3.2 for evaluating \( C_n \)-symmetry of a set of points:

1. The \( n \) measurements \( Q_i \sim \mathcal{N}(P_i, \Lambda_i) \) are folded by rotating each measurement \( Q_i \) by \( 2\pi i/n \) radians about the centroid \( \omega \). A new set of measurements \( \tilde{Q}_i \sim \mathcal{N}(\tilde{P}_i, \tilde{\Lambda}_i) \) is obtained (see Figure 3.17b).

2. The folded measurements are averaged using a weighted average, obtaining a single point \( \tilde{P}_0 \). Averaging is done by considering the \( n \) folded measurements \( \tilde{Q}_i \) as \( n \) measurements of a single point and \( \tilde{P}_0 \) represents the most probable location of that point under the Maximum Likelihood criterion.

\[
\tilde{P}_0 - \omega = \left( \sum_{j=0}^{n-1} \tilde{\Lambda}_j^{-1} \right)^{-1} \sum_{i=0}^{n-1} \tilde{\Lambda}_i^{-1}(\tilde{P}_i - \omega)
\]

3. The “average” point \( \tilde{P}_0 \) is unfolded as described in Section 3.2 obtaining points \( \{\tilde{P}_i\}_{0}^{n-1} \) which are perfectly \( C_n \)-symmetric.

When we are given \( m = qn \) measurements, we find the most probable \( C_n \)-symmetric configuration of points, similar to the folding method of Section 3.2. The \( m \) measurements \( \{Q_i\}_{i=0}^{n-1} \), are divided into \( q \) interlaced sets of \( n \) points each, and the folding method as described above is applied separately to each set of measurements. Derivations and proof of this case is also found in Section 4.2.1.

Several examples are shown in Figure 3.18, where for a given set of measurements, the most probable symmetric shapes were found.

Figure 3.19 shows an example of varying the probability distribution of the measurements on the resulting symmetric shape.
Chapter 3: Measuring Symmetry

Figure 3.18: The most probable symmetric shape.
a) A configuration of 6 measured points.
b) The most probable symmetric shapes with respect to $C_2$-symmetry.
c) The most probable symmetric shapes with respect to $C_3$-symmetry.
d) The most probable symmetric shapes with respect to $C_4$-symmetry.
e) The most probable symmetric shapes with respect to mirror-symmetry.

Figure 3.19: The most probable $C_2$-symmetric shape for a set of measurements after varying the probability distribution and expected locations of the measurements.
a-c) Changing the uncertainty (s.t.d.) of the measurements.
d-e) Changing both the uncertainty and the expected location of the measurements.
3.5.2 The Probability Distribution of Symmetry Values

Figure 3.20a displays a Laue photograph ([1, 9]) which is an interference pattern created by projecting X-ray beams onto crystals. Crystal quality is determined by evaluating the symmetry of the pattern. In this case the interesting feature is not the closest symmetric configuration, but the probability distribution of the symmetry distance values.

Consider the configuration of 2D measurements given in Figure 3.17a. Each measurement $Q_i$ is a normal probability distribution $Q_i \sim N(P_i, \Lambda_i)$. We assume the centroid of the expectation of the measurements is at the origin. The probability distribution of the symmetry distance values of the original measurements is equivalent to the probability distribution of the location of the “average” point ($\bar{P}_0$) given the folded measurements as obtained in Step 1 and Step 2 of the algorithm in Section 3.5.1. It is shown in Section 4.2.3 that this probability distribution is a $\chi^2$ distribution of order $n - 1$. However, we can approximate the distribution by a gaussian distribution. Details of the derivation are given in Section 4.2.3.

In Figure 3.20 we display distributions of the symmetry distance value as obtained for the Laue photograph given in Figure 3.20a. In this example we considered every dark patch as a measured point with variance proportional to the size of the patch. Thus in Figure 3.20b the rectangles which are proportional in size to the corresponding dark patches of Figure 3.20a, represent the standard deviation of the locations of point measurements. Note that a different analysis could be used in which the variance of the measurement location is taken as inversely proportional to the size of the dark patch.

In Figure 3.21 we display distributions of the symmetry distance value for various measurements. As expected, the distribution of symmetry distance values becomes broader as the uncertainties (the variance of the distribution) of the measurements increase.
Figure 3.20: Probability distribution of symmetry values
a) Interference pattern of crystals.
b) Probability distribution of point locations corresponding to a.
c) Probability distribution of symmetry distance values with respect to $C_{10}$-symmetry was evaluated as described in Sections 3.5.2 and 4.2.3. Expectation value $= 0.003663$. 
Figure 3.21: Probability distribution of the symmetry distance value as a function of the variance of the measured points.

a-d) Some examples of configurations of measured points.
e) Probability distribution of symmetry distance values with respect to $C_6$-symmetry for the configurations a-d.
3.6 Application to Images

3.6.1 Finding Orientation of Symmetric 3D Objects

When dealing with images, we let pixel values denote elevation, and consider an image as a 3D object (see Figure 3.22) on which we can measure 3D symmetries.

![Image as a 3D object](image)

Figure 3.22: An image as a 3D object. The pixel values denote elevation.

We applied the SD to find orientation of symmetric 3D objects by finding their 3D mirror-symmetry. The 3D shape is represented by a set of 3D points: for a possible reflection plane, the plane perpendicular to it is sampled. Each sampled points is projected onto the 3D object and its elevation is recomputed relative to the sampling plane (Figure 3.23). The symmetry value for 3D mirror-symmetry is evaluated using the projected sampling points. The final reflection plane of the 3D object is determined by minimizing the symmetry value over all possible reflection planes. In practice only feasible symmetry planes were tested (i.e. planes which intersect the 3D image) and a gradient descent algorithm was used to increase efficiency of convergence to the minimum symmetry value (the SD).

![Diagram of reflection plane](diagram)

Figure 3.23: Selecting points on the 3D object (a 2D analog is shown). For a possible reflection plane, the plane perpendicular to it is sampled. Elevations are recomputed on the object relative to the sampling plane.
Examples are shown in Figure 3.24, where a symmetric 3D object is rotated into a frontal vertical view after the reflection plane was found.

![Figure 3.24: Applying the SD with respect to 3D mirror-symmetry in order to find orientation of a 3D object.](image1)

- a,c) Original depth maps.
- b,d) The symmetry reflection plane has been found and the image rotated to a frontal vertical view.

### 3.6.2 Using a Multiresolution Scheme

In many images the process of finding the reflection plane did not converge to the correct solution, i.e. the process converged to a local minima due to the sensitivity of the symmetry value to noise and digitization errors. To overcome this problem we introduced a multiresolution scheme, which is based on the multiresolution features of symmetry mentioned in Section 1.1. In this multiresolution scheme, an initial estimation of the symmetry transform is obtained at low resolution (see Figure 3.25) and is fine tuned using high resolution images. The solution obtained for the low resolution image is used as an initial guess of the solution for the high resolution image. The low resolution images were obtained by creating gaussian pyramids [21, 20], i.e. convolving the original image with a gaussian.
Chapter 3: Measuring Symmetry

Figure 3.25: Using multiresolution to find symmetry.
The grey-level image is treated as a depth map and 3D mirror-symmetry is computed. The computed symmetry plane is used to bring the image into a frontal vertical view.

a) Original image.
b) Applying the mirror-symmetry transform on (a) does not find correct reflection plane.
c) A low resolution image obtained by convolving (a) with a gaussian.
d) Applying the mirror-symmetry transform on the low resolution image (c) gives a good estimation of the reflection plane and direction of gaze.
e) Using the reflection plane found in (d) as an initial guess, the process now converges to the correct symmetry plane.
3.6.3 Finding Locally symmetric Regions

Most images cannot be assumed to have a single global symmetry, but contain a collection of symmetric and almost symmetric patterns (be they objects or background). We aim to locate these local symmetries and segment the symmetrical objects from the background. The following three staged process is used to extract locally symmetric regions in images.

1. The first stage locates symmetry foci - those points about which the image is locally symmetric. Several methods can be used to find the symmetry foci (see [85, 86]). We used a variant of the multiresolution method presented in [119] for sampling and transmitting an image, which uses a simple model of the human visual attention mechanism.

We used the Quad Tree [92, 91] structure which is a hierarchical representation of an image, based on recursive subdivisions of the image array into quadrants as is shown in Figure 3.26. Every node in the quad tree represents a square region in the image obtained by recursive quadration as follows: the root node of the tree corresponds to the entire image; the four nodes at the next level (if they exist) correspond to the four quadrants of the image and so on; the leaf nodes of the full quad tree correspond to single pixels in the image.

The process of locating symmetry focials builds a sequence of quad trees and a sequence of corresponding divisions of the image into quadrants. The process is initialized by setting the current quad tree to a single root node (corresponding to the whole image). At each step all quadrants of the image corresponding to the leaves of the quad tree are tested for a given “interest” function. That node of the quad tree that corresponds to the quadrant with highest “interest” value is selected. The current quad tree is expanded by creating the son nodes of the selected node and accordingly, subdividing the image quadrant associated with the selected node. Several steps of this procedure are shown in Figure 3.27. In our case the “interest” function was chosen as to be a function of the Symmetry Distance value obtained for the image quadrant. Thus the procedure described focuses onto regions of high symmetry values and finds symmetry focials. In practice, the “interest” function also took into account the busyness of the image quadrant, thus regions that had low busyness values (i.e. the grey-scale values were almost constant) gave low “interest” values although they were highly symmetric (and gave low Symmetry Distance values). In Figure 3.28b, a mirror-symmetry focal and an associated reflection plane (passing through the focal) were found.
Chapter 3: Measuring Symmetry

Figure 3.26: Examples of quad trees. The tessellation of the image is displayed on the left, and the tree representation is on the right.

2. Given a symmetry focal and a reflection plane, a symmetry map of the image is created as follows: the original image is sampled at points which are pairwise symmetric with respect to the given reflection plane. The symmetry value obtained using the folding method described in Section 3.2 for each pair of sampled points is recorded and marked in the symmetry map at the location corresponding to the coordinates of the sampled points. Thus the symmetry map displays the “amount” of mirror-symmetry at every point (with respect to the given reflection plane) where low grey values denote high symmetry content and high grey values denote low symmetry content (Figure 3.28c).

3. Starting from the symmetry focal regions, regions are expanded using “active contours” [55] to include compact symmetric regions. The expansion is guided by the symmetry map and continues as long as the pixels included in the locally symmetric region do not degrade the symmetry of the region more than a predefined threshold (Figure 3.28d).

The process can be continued to extract several locally symmetric regions, as shown in the example of Figure 3.29.
Figure 3.27: Several steps in the process of finding symmetry focal.
A sequence of quad trees (bottom row) and the corresponding recursive
division of the image (top row) is created. The process is initialized by
creating a quad tree with a single root node corresponding to the whole
image (left). At each step all quadrants of the image corresponding to the
leaves of the quad tree are tested for the “interest” function. The leaf of the
quad tree that corresponds to the quadrant with highest value (marked in
grey) is expanded to create the quad tree of the next step. Three additional
steps are shown.
Figure 3.28: Applying the Multiresolution scheme to detect a mirror-symmetric region
a) Original image.
b) A mirror-symmetry focal was found.
c) Symmetry map of the image for the mirror-symmetry found in b).
d) Extracted locally symmetric region.
Figure 3.29: Applying the Multiresolution scheme to detect multiple mirror-symmetric regions.
a) Original image.
b) The mirror-symmetry foci found.
c) Symmetry maps of for each mirror-symmetry are merged into a single image.
d) Extracted locally symmetric regions.
Chapter 3: Measuring Symmetry
Chapter 4

Mathematical Proofs

4.1 Mathematical Proof of the Folding Method

In this section we derive mathematically the folding method described in Section 3.2 for finding the symmetry transform of a shape represented by points.

4.1.1 Background

We first review some basic definitions required for our proofs and derivations.

A symmetry group \( G \) is a group whose elements \( g \) are matrices representing isometries (distance preserving transformations) in a \( d \) dimensional Euclidian space:
\[
g \in G \Rightarrow g : \mathcal{R}^d \rightarrow \mathcal{R}^d
\]
For example, the set of all rotations about the origin in 2D, \( \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \), is a symmetry group.

A symmetry group having a finite number of elements is a finite symmetry group. For example, the \( C_3 \)-Symmetry Group is a finite point symmetry group having 3 elements:
rotation by \( 0^\circ \) (I - identity), rotation by \( 120^\circ \) and rotation by \( 240^\circ \) about the origin.

A subset of points \( S \subset \mathcal{R}^d \) is G-invariant if every element of \( G \) transforms \( S \) onto itself, i.e.: \( \{gs \mid s \in S\} = S \) for all \( g \in G \).
For example, the set of 2D points \( \{(0,1), (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2)\} \) is \( C_3 \)-invariant.
The orbit of a point $x \in \mathcal{R}^d$ under the symmetry group $G$ is the set of points into which $x$ is transformed using elements of $G$, i.e.:

$$\{gx \mid g \in G\}.$$ 

Given a finite group $G$ and given an ordering of its $n$ elements: $g_0, g_1, \ldots, g_{n-1}$, the ordered orbit under $G$ of a point $x$ in Euclidean space is the finite set of points $x_0, \ldots, x_{n-1}$ such that $x_i = g_i x$ for $i = 0 \ldots n - 1$. If $g_0 = I$ (the identity element of $G$) then $x_i = g_i x_0$ $i = 0 \ldots n - 1$.

If $S \subset \mathcal{R}^d$ is $G$-invariant then $S$ is a union of separate orbits. Specifically, an orbit of symmetry group $G$ is itself $G$-invariant.

A finite point symmetry group is a finite symmetry group that has at least one orbit containing a single point. For example the origin is a single point orbit of the $C_3^*$-symmetry group described above. Thus the $C_3^*$-symmetry group is a finite point symmetry group.

A point $x \in X$ is a general point (or is in general position) with respect to $G$ if for all $g \in G$, $g \neq I$ (where $I$ is the identity in $G$) we have $gx \neq x$.

**Lemma 1** If $G$ is a finite point symmetry group having $n$ elements and if $x$ is a general point with respect to $G$, then there are $n$ different points in the orbit of $x$ and all of them are general points.

**Lemma 2** The centroid of a $G$-invariant set of points $S$ is a fixed point under $G$ (i.e. the centroid of $S$ is a single point orbit under $G$).

**Lemma 3** If $S$ is $G$-invariant and points $s_1, s_2 \in S$ are fixed points under $G$, then the translation of the set by the vector $s_2 - s_1$ leaves the set $G$-invariant (see example in Figure 4.1).

A subset of points $S \subset \mathcal{R}^n$ is $G$-symmetric if there is a rotation $R$ and a translation vector $\omega$ such that the set $S'$ obtained by translating $S$ by $t$ and rotating by $R$, is $G$-invariant. $S' = \{R(s - \omega) \mid s \in S\}$.

For example, the set $\{(5 + 0.3 + 1), (5 + \sqrt{3}/2, 3 - 1/2), (5 - \sqrt{3}/2, 3 - 1/2)\}$ in 2D is $C_3^*$-symmetric.

For a more detailed review and proofs see [66].
\[ \sum_{i=0}^{n-1} \| P_i - \hat{P}_i \|^2 \]  

Since G has a fixed point at the origin and the centroid of the orbit formed by the rotated and translated \( \hat{P}_i \) is a fixed point under \( G \) (see Lemma 2), we can assume without loss of generality (from Lemma 3) that the translation vector \( w \) is the centroid of points \( \hat{P}_i \):  

\[ w = \frac{1}{n} \sum_{i=0}^{n-1} \hat{P}_i \]  

The points \( \hat{P}_0, \ldots, \hat{P}_{n-1} \) translated by \( \omega \) and rotated by \( R \), form an orbit of \( G \), thus the following must be satisfied:  

\[ \hat{P}_i = R^i g_i R(\hat{P}_0 - w) + w \quad i = 0 \ldots n - 1 \]
Using Lagrange multipliers with Equations 4.1-4.3 we minimize the following:

$$
\sum_{i=0}^{n-1} \| P_i - \hat{P}_i \|^2 + \sum_{i=0}^{n-1} \lambda_i (\hat{P}_i - R^t g_i R (\hat{P}_0 - w) - w) + \varepsilon (w - \frac{1}{n} \sum_{i=0}^{n-1} \hat{P}_i) \tag{4.4}
$$

where \( \varepsilon, \lambda_i \) for \( i = 0 \ldots n - 1 \) are the Lagrange multipliers.

Setting the derivatives equal to zero we obtain:

$$
\sum_{i=0}^{n-1} (P_i - \hat{P}_i) = 0
$$

and using the last constraint (Eq. 4.2) we obtain:

\[
\begin{align*}
w &= \frac{1}{n} \sum_{i=0}^{n-1} P_i \\
\end{align*}
\]  

(4.5)

i.e. the centroid of \( P_0, \ldots, P_{n-1} \) coincides with the centroid of \( \hat{P}_0, \ldots, \hat{P}_{n-1} \) (in terms of the symmetry distance defined in Section 1.1, the centroid of a configuration and the centroid of the closest symmetric configuration is the same for any point symmetry group \( G \)).

Noting that \( R^t g_i R \) for \( i = 0 \ldots n - 1 \) are isometries and distance preserving, we have from the derivatives:

$$
\sum_{i=0}^{n-1} R^t g_i^t R (P_i - \hat{P}_i) = 0
$$

Expanding using the constraints we obtain:

$$
n \hat{P}_0 - n \omega = \sum_{i=0}^{n-1} R^t g_i^t R P_i - \sum_{i=0}^{n-1} R^t g_i^t R w
$$

or

\[
\begin{align*}
\hat{P}_0 - w &= \frac{1}{n} \sum_{i=0}^{n-1} R^t g_i^t R (P_i - w) \\
\end{align*}
\]  

(4.6)

The derivation of the rotation matrix \( R \) is given in the next section, however, given \( R \), the geometric interpretation of Equation 4.6 is the folding method, as described in Section 3.2, thus proving that the folding method results in the \( G \)-symmetric set of points
closest to the given set.

The common case, however, is that shapes have more points than the order of the point group for symmetry. For symmetry of order \( n \), the folding method can be extended to shapes having a number of points which is a multiple of \( n \). Given \( m = qn \) points (i.e. \( q \) sets of \( n \) points) \( \{P_j^1, \ldots, P_j^{n-1}\} \) \( j = 0 \ldots q - 1 \) we follow the above derivation and obtain a result similar to that given in Equation 4.6. For each set of \( n \) points, i.e. for \( j = 0 \ldots q - 1 \):

\[
\hat{P}_0^j - w = \frac{1}{m} \sum_{i=0}^{n-1} R_i^j g_i R(P_i^j - w)
\]

where \( w = \frac{1}{m} \sum_{j=0}^{q-1} \sum_{i=0}^{n-1} P_i^j \) is the centroid of all \( m \) points.

The geometric interpretation of Equation 4.7 is the folding method, as described in Section 3.2 for the case of a shape represented by \( m = qn \) points.

The folding method for the cases where the number of points is not a multiple of the number of elements in \( G \) is not derived here. Details of this case can be found in [124].

### 4.1.3 Finding the Optimal Orientation in 2D

Following the derivation in Appendix 4.1.2 we derive here an analytic solution for finding the rotation matrix \( R \) which minimizes Equation 4.4, i.e. finding the orientation of the reflection axes of the \( G \)-symmetry group. In [125] (Appendix A.2) we gave the derivation for the specific case of the mirror-symmetry group \( (D_1) \) having the two elements: \( \{E, \sigma\} \). In 2D there are 2 classes of point symmetry groups: the class \( C_n \) having rotational symmetry of order \( n \) and the class \( D_n \) having rotational symmetry of order \( n \) and \( n \) reflection axes (see Section 1.2). The problem of finding the minimizing orientation is irrelevant for the \( C_n \) symmetry groups since every element \( g \) of these groups is a rotation and \( R^t g R = g \). In the case of \( C_n \)-symmetry groups, \( R \) is usually taken as \( I \) (the identity matrix). We derive here a solution for the orientation in the case where \( G \) is a \( D_n \) symmetry group.

In 2D there are 2 classes of point symmetry groups: the class \( C_n \) having rotational symmetry of order \( n \) and the class \( D_n \) having rotational symmetry of order \( n \) and \( n \) reflection axes (see Section 1.2). The problem of finding the minimizing orientation is irrelevant for the \( C_n \) symmetry groups since every element \( g \) of these groups is a rotation and \( R^t g R = g \). In the case of \( C_n \)-symmetry groups, \( R \) is usually taken as \( I \) (the identity matrix). We derive here a solution for the orientation in the case where \( G \) is a \( D_n \) symmetry group.

The 2n elements of the \( D_n \)-symmetry group \( \{g_0, \ldots, g_{2n-1}\} \) can be described as the \( n \) elements \( g_0 = I, g_1 = R_1^n, g_2 = R_2^n, \ldots, g_{n-1} = R_{n-1}^n \) (where \( R_i^n \) is the rotation of \( 2\pi/n \) radians about the origin) and the \( n \) elements obtained by applying a reflection \( R_f \) on each of these elements: \( g_n = R_f, g_{n+1} = R_f R_1^n, g_{n+2} = R_f R_2^n, \ldots, g_{2n-1} = R_f R_{n-1}^n \). We denote the orientation of the symmetry group as the angle \( \theta \) between the reflection axis
and the y axis. Thus
\[ R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]
and the reflection operation \( R_f \) is given by:
\[ R_f = R^T g_{n+1} R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \]
Without loss of generality, we assume the centroid \( w \) is at the origin (\( \omega = 0 \)). Following Appendix 4.1.2, we minimize Equation 4.1 over \( \theta \). Using Equation 4.3 and noting that \( R^T g_i R \) for \( i = 0 \ldots 2n - 1 \) are distance preserving, we minimize the following over \( \theta \):
\[ \sum_{i=0}^{2n-1} \| P_i - \hat{P}_i \|^2 = \sum_{i=0}^{2n-1} \| R^T g^j_i R P_i - \hat{P}_i \|^2 \]
Substituting Equation 4.6 we minimize:
\[ \sum_{i=0}^{2n-1} \| R^T g^j_i R P_i \| = \sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1} R^T g^j_i R P_i \|
Rearranging and noting that \( R^T g^j_i R = g^j_i \) for \( i = 0 \ldots n - 1 \), we minimize:
\[ \sum_{i=0}^{2n-1} \| 2n R^T g^j_i R P_i - \sum_{j=0}^{n-1} g^j_i P_j - \sum_{j=n}^{2n-1} R_j g^j_{n-j} P_j \|^2 \] (4.8)
Denoting by \( x_i, y_i \) the coordinates of the point \( P_i \) and taking the derivative of Equation 4.8 with respect to \( \theta \) we obtain:
\[ \tan 2\theta = \frac{\sum_{i=0}^{n-1} \sum_{j=m}^{2n-1} (x_i y_j + x_j y_i)}{\sum_{i=0}^{n-1} \sum_{j=m}^{2n-1} (x_i x_j - y_i y_j)} \] (4.9)
which is an analytic solution for the case of optimal orientation in 2D. In higher dimensions, however, no analytic solution was found and a minimization procedure is used (except for the mirror symmetry group in 3D where a closed form solution is given - see [120]).

### 4.2 Uncertain Point Locations

#### 4.2.1 The Most Probable \( C_n \)-Symmetric Shape

In Section 3.5.1 we described a method of evaluating the most probable symmetric shape given a set of measurements. In this Section we derive mathematically and prove the
method. For simplicity we derive the method with respect to rotational symmetry of order \( n \) (\( C_n \)-symmetry). The solution for mirror symmetry is similar (see Section 4.2.2).

Given \( n \) points in 2D whose positions are given as normal probability distributions: \( Q_i \sim \mathcal{N}(P_i, \Lambda_i) \), \( i = 0 \ldots n-1 \), we find the \( C_n \)-symmetric configuration of points \( \{ \hat{P}_i \}_{i=0}^{n-1} \) which is most optimal under the Maximum Likelihood criterion (\([26]\)).

Denote by \( \omega \) the center of mass of \( \hat{P}_i \): \( \omega = \frac{1}{n} \sum_{i=0}^{n-1} \hat{P}_i \).

Having that \( \{ \hat{P}_i \}_{i=0}^{n-1} \) are \( C_n \)-symmetric, the following is satisfied:

\[
\hat{P}_i = R_i(\hat{P}_0 - \omega) + \omega
\]  

(4.10)

for \( i = 0 \ldots n-1 \) where \( R_i \) is a matrix representing a rotation of \( 2\pi i/n \) radians.

Thus, given the measurements \( Q_0, \ldots, Q_{n-1} \) we find the most probable \( \hat{P}_0 \) and \( \omega \). We find \( \hat{P}_0 \) and \( \omega \) that maximize \( \text{Prob}(\{ \hat{P}_i \}_{i=0}^{n-1} \mid \omega, \hat{P}_0) \) under the symmetry constraints of Equation 4.10.

Considering the normal distribution we have:

\[
\prod_{i=0}^{n-1} k_i \exp\left( -\frac{1}{2}(\hat{P}_i - P_i)^\top \Lambda_i^{-1}(\hat{P}_i - P_i) \right)
\]

where \( k_i = \frac{1}{2\pi} | \Lambda_i |^{1/2} \). Having log being a monotonic function, we maximize:

\[
\log \prod_{i=0}^{n-1} k_i \exp\left( -\frac{1}{2}(\hat{P}_i - P_i)^\top \Lambda_i^{-1}(\hat{P}_i - P_i) \right)
\]

Thus we find those parameters which maximize:

\[
-\frac{1}{2} \sum_{i=0}^{n-1} (\hat{P}_i - P_i)^\top \Lambda_i^{-1}(\hat{P}_i - P_i)
\]

under the symmetry constraint of Equation 4.10.

Substituting Equation 4.10, taking the derivative with respect to \( \hat{P}_0 \) and equating to zero we obtain:

\[
\left( \sum_{i=0}^{n-1} R_i^\top \Lambda_i^{-1} R_i \right) \hat{P}_0 + \sum_{i=0}^{n-1} R_i^\top \Lambda_i^{-1} (I - R_i) \omega = \sum_{i=0}^{n-1} R_i^\top \Lambda_i^{-1} P_i
\]  

(4.11)
Note that $R_0 = I$ where $I$ is the identity matrix.
When the derivative with respect to $\omega$ is zero:

$$
\left( \sum_{i=0}^{n-1} (I - R_i) \Lambda_i^{-1} R_i \right) \hat{P}_0 + \sum_{i=0}^{n-1} (I - R_i) \Lambda_i^{-1} (I - R_i) \omega = \sum_{i=0}^{n-1} (I - R_i) \Lambda_i^{-1} P_i
$$

(4.12)

Notice that when all $\Lambda_i$ are equal (i.e., all points have the same uncertainty, which is equivalent to the cases in the previous sections where point location is known with no uncertainty), Eqs. 4.11-4.12 reduce to Eqs. 4.5-4.6 in Section 4.1.2.

From Equation 4.11 we obtain

$$
\hat{P}_0 - \omega = \left( \sum_{j=0}^{n-1} R_j^i \Lambda_j^{-1} R_j \right)^{-1} \sum_{i=0}^{n-1} (R_j^i \Lambda_j^{-1} R_i) P_i
$$

Which gives the folding method described in Section 3.5.1, where $R_i^i P_i$ is the location of the folded measurement (denoted $\tilde{P}_i$ in the text) and $R_j^i \Lambda_j^{-1} R_i$ is its probability distribution (denoted $\Lambda_i$ in the text). The factor $(\sum_{i=0}^{n-1} R_j^i \Lambda_j^{-1} R_j)$ is the normalization factor.

Reformulating Eqs. 4.11 and 4.12 in matrix formation we obtain:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
\hat{P}_0 \\
\omega
\end{pmatrix}
= 
\begin{pmatrix}
E \\
F
\end{pmatrix}
$$

Noting that $U$ is symmetric we solve by inversion $V = U^{-1} Z$ and obtain the parameters $\omega$ and $\hat{P}_0$, and obtain the most probable $C_n$-symmetric configuration, given the measurements $\{Q_i\}_{i=0}^{n-1}$.

Similar to the representation in Section 3.2, given $m = qn$ measurements $\{Q_i\}_{i=0}^{m-1}$, we consider them as $q$ sets of $n$ interlaced measurements: $\{Q_{i+q+j}\}_{i=0}^{n-1}$ for $j = 0 \ldots q - 1$. The derivations given above are applied to each set of $n$ measurements separately, in order to obtain the most probable $C_n$-symmetric set of points $\{P_i\}_{i=0}^{m-1}$. Thus the symmetry constraints that must be satisfied are:

$$
\hat{P}_{i+q+j} = R_i (\hat{P}_j - \omega) + \omega
$$

for $j = 0 \ldots q - 1$ and $i = 0 \ldots n - 1$ where, again, $R_i$ is a matrix representing a rotation of $2\pi i/n$ radians and $\omega$ is the centroid of all points $\{\hat{P}_i\}_{i=0}^{m-1}$. 

---

Chapter 4: Mathematical Proofs

Note that $R_0 = I$ where $I$ is the identity matrix. When the derivative with respect to $\omega$ is zero:

$$
\left( \sum_{i=0}^{n-1} (I - R_i) \Lambda_i^{-1} R_i \right) \hat{P}_0 + \sum_{i=0}^{n-1} (I - R_i) \Lambda_i^{-1} (I - R_i) \omega = \sum_{i=0}^{n-1} (I - R_i) \Lambda_i^{-1} P_i
$$

(4.12)

Notice that when all $\Lambda_i$ are equal (i.e., all points have the same uncertainty, which is equivalent to the cases in the previous sections where point location is known with no uncertainty), Eqs. 4.11-4.12 reduce to Eqs. 4.5-4.6 in Section 4.1.2.

From Equation 4.11 we obtain

$$
\hat{P}_0 - \omega = \left( \sum_{j=0}^{n-1} R_j^i \Lambda_j^{-1} R_j \right)^{-1} \sum_{i=0}^{n-1} (R_j^i \Lambda_j^{-1} R_i) P_i
$$

Which gives the folding method described in Section 3.5.1, where $R_i^i P_i$ is the location of the folded measurement (denoted $\tilde{P}_i$ in the text) and $R_j^i \Lambda_j^{-1} R_i$ is its probability distribution (denoted $\Lambda_i$ in the text). The factor $(\sum_{i=0}^{n-1} R_j^i \Lambda_j^{-1} R_j)$ is the normalization factor.

Reformulating Eqs. 4.11 and 4.12 in matrix formation we obtain:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
\hat{P}_0 \\
\omega
\end{pmatrix}
= 
\begin{pmatrix}
E \\
F
\end{pmatrix}
$$

Noting that $U$ is symmetric we solve by inversion $V = U^{-1} Z$ and obtain the parameters $\omega$ and $\hat{P}_0$, and obtain the most probable $C_n$-symmetric configuration, given the measurements $\{Q_i\}_{i=0}^{n-1}$.

Similar to the representation in Section 3.2, given $m = qn$ measurements $\{Q_i\}_{i=0}^{m-1}$, we consider them as $q$ sets of $n$ interlaced measurements: $\{Q_{i+q+j}\}_{i=0}^{n-1}$ for $j = 0 \ldots q - 1$. The derivations given above are applied to each set of $n$ measurements separately, in order to obtain the most probable $C_n$-symmetric set of points $\{P_i\}_{i=0}^{m-1}$. Thus the symmetry constraints that must be satisfied are:

$$
\hat{P}_{i+q+j} = R_i (\hat{P}_j - \omega) + \omega
$$

for $j = 0 \ldots q - 1$ and $i = 0 \ldots n - 1$ where, again, $R_i$ is a matrix representing a rotation of $2\pi i/n$ radians and $\omega$ is the centroid of all points $\{\hat{P}_i\}_{i=0}^{m-1}$.
As derived in Equation 4.11, we obtain for \( j = 0 \ldots q - 1 \):

\[
\begin{align*}
\sum_{i=0}^{n-1} R_i^j \Lambda_{ij+1}^{-1} R_i^j \hat{P}_j + \sum_{i=0}^{n-1} R_i^j \Lambda_{iq+j}^{-1} (I - R_i) \omega = \sum_{i=0}^{n-1} R_i^j \Lambda_{iq+j}^{-1} P_{iq+j} & \\
A_j & B_j & E_j
\end{align*}
\tag{4.13}
\]

and equating to zero, the derivative with respect to \( \omega \), we obtain, similar to Equation 4.12:

\[
\sum_{j=0}^{q-1} \sum_{i=0}^{n-1} (I - R_i)^j \Lambda_{ij+1}^{-1} R_i \hat{P}_j + \sum_{j=0}^{q-1} \sum_{i=0}^{n-1} (I - R_i)^j \Lambda_{iq+j}^{-1} (I - R_i) \omega = \sum_{j=0}^{q-1} \sum_{i=0}^{n-1} (I - R_i)^j \Lambda_{iq+j}^{-1} P_{iq+j} & \\
C_j & D & F
\tag{4.14}
\]

Reformulating Eqs. 4.13 and 4.14 in matrix formation we obtain:

\[
\begin{pmatrix}
A_0 & A_1 & \ldots & A_{q-1} \\
B_0 & B_1 & \ldots & B_{q-1} \\
C_0 & C_1 & \ldots & C_{q-1} \\
D & & & &
\end{pmatrix}
\begin{pmatrix}
P_0 \\
P_1 \\
\vdots \\
P_{q-1} \\
\omega
\end{pmatrix}
= 
\begin{pmatrix}
E_0 \\
E_1 \\
\vdots \\
E_{q-1} \\
F
\end{pmatrix}
\]

Noting that \( U \) is symmetric we solve by inversion \( V = U^{-1} Z \) and obtain the parameters \( \omega \) and \( \{P_j\}_{j=0}^{q-1} \), and obtain the most probable \( C_n \)-symmetric configuration, \( \{P_j\}_{j=0}^{m-1} \) given the measurements \( \{Q_i\}_{i=0}^{m-1} \).

### 4.2.2 The Most Probable Mirror Symmetric Shape

In Section 3.5.1 we described a method for finding the most probable rotationally symmetric shape given measurements of point location. The solution for mirror symmetry is similar. In this case, given \( m \) measurements (where \( m = 2q \)), the unknown parameters are \( \{P_j\}_{j=0}^{q-1}, \omega \) and \( \theta \) where \( \theta \) is the angle of the reflection axis. However these parameters are redundant and we reduce the dimensionality of the problem by replacing the 2 dimensional \( \omega \) with the one dimensional \( x_0 \) representing the x-coordinate at which the reflection axis intersects the x-axis. Additionally we replace \( R_i \) the rotation matrix with \( R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \) the reflection about an axis at an angle \( \theta \) to the x-axis. The angle \( \theta \) is found analytically (see [125]) thus the dimensionality of the problem is \( 2q + 1 \) (rather than \( 2q + 2 \)) and elimination of the last row and column of matrix \( U \) (see Section 4.2.1) allows an inverse solution as in the rotational symmetry case.
4.2.3 Probability Distribution of Symmetry Values

In this section we derive mathematically the probability distribution of symmetry distance values obtained from a set of \( n \) measurements in 2D: \( Q_i \sim \mathcal{N}(P_i, \Lambda_i) \quad i = 0 \ldots n - 1 \) with respect to \( C_n \)-symmetry (see Section 3.5.2).

Denote by \( X_i \) the 2-dimensional random variable having a normal distribution equal to that of measurement \( \hat{Q}_i \) i.e.

\[
E(X_i) = R_i P_i \\
\text{Cov}(X_i) = R_i \Lambda_i R_i^t
\]

where \( R_i \) denotes (as in Section 3.2) the rotation matrix of \( 2\pi i/n \) radians.

Denote by \( Y_i \) the 2-dimensional random variable:

\[
Y_i = X_i - \frac{1}{n} \sum_{j=0}^{n-1} X_j
\]

in matrix notation:

\[
\begin{pmatrix}
Y_0 \\
\vdots \\
Y_{n-1}
\end{pmatrix} = A
\begin{pmatrix}
X_0 \\
\vdots \\
X_{n-1}
\end{pmatrix}
\]

or \( Y = AX \) where \( Y \) and \( X \) are of dimension \( 2n \) and \( A \) is the \( 2n \times 2n \) matrix:

\[
A = \frac{1}{n}
\begin{pmatrix}
n - 1 & 0 & -1 & 0 & -1 & \cdots \\
0 & n - 1 & 0 & -1 & 0 & \cdots \\
-1 & 0 & \ddots & 0 & -1 & \cdots \\
& & & & & \ddots \\
& & & \cdots & n - 1
\end{pmatrix}
\]

And we have

\[
E(X) = \begin{pmatrix}
E(X_0) \\
\vdots \\
E(X_{n-1})
\end{pmatrix} \\
\text{Cov}(X) = \begin{pmatrix}
\text{Cov}(X_0) \\
\vdots \\
\text{Cov}(X_{n-1})
\end{pmatrix}
\]

\[
E(Y) = AE(X) \\
\text{Cov}(Y) = ACov(X)A^t
\]

The matrix \( ACov(X)A^t \), being symmetric and positive definite, we find the \( 2n \times 2n \) matrix \( S \) diagonalizing \( \text{Cov}(Y) \) i.e.

\[
SACov(X)A^tS^t = D
\]
where $D$ is a diagonal matrix (of rank $2(n - 1)$).

Denote by $Z = (Z_0, \ldots, Z_{n-1})^t$ the $2n$-dimensional random variable $SAX$.

$$E(Z) = SA E(X)$$

$$\text{Cov}(Z) = SA \text{Cov}(X) A^t S^t = D$$

Thus the random variables $Z_i$ that compose $Z$ are independent and, being linear combinations of $X_i$, they are of normal distribution.

The symmetry distance, as defined in Section 3.2, is equivalent, in the current notations, to $s = Y^t Y$. Having $S$ orthonormal we have

$$s = (SAX)^t SAX = Z^t Z$$

If $Z$ were a random variable of standard normal distribution, we would have $s$ being of a $\chi^2$ distribution of order $2(n - 1)$. In the general case $Z_i$ are normally distributed but not standard and $Z$ cannot be standardized globally. We approximate the distribution of $s$ as a normal distribution with

$$E(s) = E(Z)^t E(Z) + \text{trace}D^t D$$

$$\text{Cov}(s) = 2 \text{trace}(D^t D)(D^t D) + 4E(Z)^t D^t D E(Z)$$
Chapter 5

3D Mirror Symmetry

The most common symmetry in our environment is three dimensional mirror symmetry (corresponding to the 3D mirror-symmetry group having two elements: the identity element and a reflection element - see Section 4.1.1). It is thus not surprising that the human visual system is most sensitive to reflectional symmetry [89, 75]. Due to its abundance and its importance in our world, we devote a separate chapter to deal with 3D mirror-symmetry.

5.1 Background

In Sections 3.1-3.2 the Symmetry Distance (SD) was defined and a method was described to evaluate this measure for any configuration of points. The measure can be evaluated with respect to any point symmetry group in any dimension. The method for evaluating the symmetry distance in 2D was shown to be analytic but in 3D only numerical methods are available in most cases. In this chapter we describe a closed form solution for the specific case of evaluating the symmetry distance with respect to 3D mirror symmetry. We focus on connected configurations of points which can be considered as graphs. A graph $\mathcal{G}$ is represented as a pair $\mathcal{G} = \{V, E\}$, where $V$ is the set of vertices (points) and $E$ is the set of edges (i.e. a set of pairs of vertices $(i, j)$ representing an edge between the vertex $i$ and the vertex $j$ of $V$). Connected configurations of points can model 3D structures, several examples are shown in Figure 5.1. The 3D graph structure, specifically the connectivity of the graph, enforces constraints on the symmetries of the structure which will be exploited in order to facilitate measuring of symmetry (Section 5.2.1).
Figure 5.1: Several examples of 3D structures modeled as connected configurations of points (graphs). The points are represented as dots and the edges as segments connecting the dots.

5.2 Closed Form Method for 3D Mirror Symmetry

In order to evaluate the SD of a configuration of 3D points \( \{P_i\}_{i=1}^{n} \), we find the mirror-symmetric configuration \( \{\hat{P}_i\}_{i=1}^{n} \) which is closest to the original configuration in terms of the mean squared distances (Section 3.1). A method for finding the mirror symmetric configuration of 3D points close to a given configuration was described in Section 3.2 (and is detailed in [127]). This method requires minimization over all possible reflection planes and all possible divisions of the given points into sets (Step 2 in algorithm of Section 3.2). In this section, a closed form solution is described for finding the optimal reflection plane, where minimization is performed over all possible divisions of the points into sets. Assuming the input to be a connected configuration of points (as described above), minimization over all possible divisions into sets is greatly simplified.

We first note, that every mirror symmetric 3D configuration of points \( \{\hat{P}_i\}_{i=0}^{n-1} \), implicitly implies a pairing (matching) of the points: for every point \( \hat{P}_i \) there exists a point \( \hat{P}_j \) which is its counterpart under reflection. Note that \( \hat{P}_i = \hat{P}_j \) when the point is on the mirror plane.

Thus in the case of 3D mirror symmetry, the sets into which the points are divided, are pairs representing a point and its counterpart (under reflection). Since some points may lie on the reflection plane and have no counterpart, some pairs may be degenerate pairs composed of a single point. For example, given the configuration of points in Figure 5.2, a possible division into pairs is: \( \{P_1, P_3\} \) and the degenerated pairs \( \{P_0\} \) and \( \{P_2\} \).

The process of finding the closest mirror-symmetric configuration and evaluating the SD value is divided into two stages:

1. The **Topological stage** -
   This stage corresponds to the process of dividing the points in the given config-
Figure 5.2: Pairing points in a connected configuration.

a) The original configuration of points $P_0, \ldots, P_3$. The points are divided into pairs: the proper pair $\{P_1, P_3\}$ and the degenerated pairs $\{P_0\}$ and $\{P_2\}$.

b) The points of the degenerated pairs are projected in the closest mirror-symmetric configuration onto the mirror plane (obtaining points $\hat{P}_0, \hat{P}_2$). The pair $\{P_1, P_3\}$ is transformed to be a mirror-symmetric pair $\{\hat{P}_1, \hat{P}_3\}$.

In this stage all possible divisions into pairs are found. As previously mentioned, the division into pairs is constrained by the connectivity (or topology) of the configuration and no consideration is given to the geometry of the configuration (i.e., to the actual 3D location of the points).

2. The Geometric stage -

For every possible division of the configuration points into pairs (obtained in the Topological stage), we find the closest symmetric configuration having the same connectivity and evaluate the symmetry distance value.

### 5.2.1 Topological Stage

Finding the closest mirror-symmetric configuration and evaluating the SD of a given configuration requires the division of the points into pairs. Each such pair is transformed in the symmetric configuration into a mirror symmetric pair. Although the coordinates of the points in the pair change in order to accommodate symmetry, we assume that any order defined on the points and any feature or characteristic associated with the points remain invariant under the transformation. When dealing with connected configurations
of points such as graph structures (Section 5.1), the invariant features associated with the points are implicitly derived from the connectivity or the topology of the graph (which is defined as the configuration of edges in the graph). For example, the valency of a point (the number of edges converging at a point) is a feature associated with points in a graph structure.

Thus, the connectivity of the points in the original configuration (the topology of the configuration) determines the division of points into pairs. As an example, consider the connected configuration shown in Figure 5.3a. Points $P_0, \ldots, P_6$ are leaf nodes and can be paired between them. Points $P_7, \ldots, P_9$ have the same valency of three and can be paired. Points $P_{10}$ and $P_{11}$ stand alone in their valency of 4 and 2 respectively, and will form single-point pairs (degenerate pairs). Thus a possible division of the points into pairs for measuring mirror-symmetry and for transforming the configuration into a mirror-symmetric configuration, is: $\{P_1, P_4\}, \{P_2, P_3\}, \{P_7, P_8\}, \{P_0\}, \{P_{10}\}, \{P_9\}, \{P_5, P_6\}, \{P_{11}\}$.

\begin{figure}[h]
    \centering
    \includegraphics[width=\textwidth]{figure5.3.png}
    \caption{Connected configurations of points. The graph shown in a) is isomorphic to the graph shown in b) (see text).}
\end{figure}

However the valency of a point (the number of edges converging at a point) is insufficient in determining the division into sets. Consider the example in Figure 5.3a. Points $P_7$ and $P_9$ have the same valency (3) but obviously cannot be geometrically moved to be mirror-symmetric. This is due to the fact that they are not equivalent in their second order connectivity (i.e. in the valency of their neighboring points), thus point $P_7$ has two neighbors of valency 1 and one neighbor of valency 4, whereas point $P_9$ has two neighbors of valency 1 and one neighbor of valency 2. This reasoning does not stop at the second order connectivity (in Figure 5.3a points $P_3$ and $P_5$ do not agree in their third order connectivity) but must be taken to the maximal connectivity of the configuration (which is equal to the width of the graph).

Thus, the topological stage of evaluating the SD of a connected configuration of points, lists all possible division of the points into pairs, by taking into account only the topology (the connectivity of the points). When considering a configuration of points as a graph, the problem of dividing the points into pairs (proper and degenerate pairs) reduces to
the classical question of listing all graph isomorphisms of order 2. A graph isomorphism is a permutation $\Pi$ of the graph vertices which leaves the graph topologically equivalent; i.e. given a graph $G = \{V, E\}$, replacing each vertex $i \in V$ with its permuted vertex $\Pi(i)$ results in a graph $G' = \{V', E'\}$ such that the set of edges $E'$ equals $E$. Note that if $\Pi$ is an isomorphism of $G$ then if $(i, j) \in E$ also $(\Pi(i), \Pi(j)) \in E$. A graph isomorphism of order 2 is an isomorphism where $\Pi(\Pi(i)) = i$ (i.e. either $\Pi(i) = i$, or, $\Pi(i) = j$ and $\Pi(j) = i$). For example, graph $G$ shown in Figure 5.3a has the following 12 isomorphisms of order two:

<table>
<thead>
<tr>
<th>$\Pi(P_0)$</th>
<th>$\Pi(P_1)$</th>
<th>$\Pi(P_2)$</th>
<th>$\Pi(P_3)$</th>
<th>$\Pi(P_4)$</th>
<th>$\Pi(P_5)$</th>
<th>$\Pi(P_6)$</th>
<th>$\Pi(P_7)$</th>
<th>$\Pi(P_8)$</th>
<th>$\Pi(P_9)$</th>
<th>$\Pi(P_{10})$</th>
<th>$\Pi(P_{11})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$P_0$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
<td>$P_6$</td>
<td>$P_7$</td>
<td>$P_8$</td>
<td>$P_9$</td>
<td>$P_{10}$</td>
</tr>
<tr>
<td>b</td>
<td>$P_0$</td>
<td>$P_1$</td>
<td>$P_2$</td>
<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
<td>$P_6$</td>
<td>$P_7$</td>
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<td>$P_9$</td>
<td>$P_{10}$</td>
</tr>
<tr>
<td>c</td>
<td>$P_0$</td>
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<td>$P_4$</td>
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<tr>
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<tr>
<td>k</td>
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<tr>
<td>l</td>
<td>$P_0$</td>
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<td>$P_3$</td>
<td>$P_4$</td>
<td>$P_5$</td>
<td>$P_6$</td>
<td>$P_7$</td>
<td>$P_8$</td>
<td>$P_9$</td>
<td>$P_{10}$</td>
</tr>
</tbody>
</table>

Thus replacing every point with its permuted point of isomorphism (l), for example, we obtain the graph $G'$ of Figure 5.3b which is topologically equivalent to graph $G$.

To compute all possible pairings of points in a graph structure, we use a recursive algorithm to find graph isomorphisms of order two (see Appendix of [120] for details).

### 5.2.2 Geometrical Stage

The topological stage of the process, as described above, finds all possible divisions into pairs for the given configuration of points represented by a graph. In the second stage of the process, the symmetry distance is evaluated for every possible pairing. The final symmetry distance is obtained by taking the minimum of all these values.

Since the division into pairs has been determined by the topology, we may disregard the topology in the second stage and deal only with the geometrical 3D locations by considering the input as a collection of (disconnected) point locations in 3D. Note that in the topological stage, only the topology was considered and the geometrical locations were disregarded. In the geometrical stage, the topology is disregarded and only the geometry is considered. Thus we can reformulate the problem: given a collection of points in
3D and given a permutation of these points, find the closest set of points (in the mean squared distance sense) which are mirror symmetric under this permutation (i.e. the mirror-symmetry group permutes the points by the given permutation).

Figure 5.4: Finding the closest mirror-symmetric configuration of points.

a) The original set of points $P_0, P_1, P_2, P_3$.
b) The original set of points reflected about an arbitrary axis (the y axis) obtaining points $\hat{P}_0, \ldots, \hat{P}_3$.
c) The transformation $R$ (rotation and translation) which minimizes $\sum_{i=0}^{3} \| P_i - R\hat{P}_i \|^2$ is found. The optimally transformed set of reflected points is shown superimposed on the original set.
d) The points $\hat{P}_i$ of the closest mirror-symmetric configuration are obtained by averaging: $\hat{P}_i = \frac{1}{2}(P_i + R\hat{P}_i)$.

According to the SD definition (Section 3.1), the squared distance to be minimized is the sum of squared distances between points in the original configuration and points in the symmetric configuration. The minimization is over all possible reflection planes which determine the symmetric configuration. However, reflection $R_{f_x}$ about a given reflection plane is equivalent to a reflection about an arbitrary reflection plane (say w.l.g. reflection $R_{f_z}$ about the z-y plane) followed by a rotation and a translation. This induces an alternative algorithm to the folding method: Given a set of points and given a pairing (permutation of order two) of these points, we find the closest mirror-symmetric configuration of points by reflecting all the points about an arbitrary reflection plane and finding the rotation and translation of the reflected set which minimizes the sum of squared distances between every reflected and rotated point and its corresponding original point defined by the pairing. For example, given the set of points $(P_0, P_1, P_2, P_3)$ in Figure 5.4a and given the permutation: $\Pi(P_0) = P_0, \Pi(P_1) = P_3, \Pi(P_2) = P_3, \Pi(P_3) = P_2$. We reflect the points about an arbitrarily chosen plane obtaining $\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3$ in Figure 5.4b and then find the transformation $R$ (rotation and translation) of the reflected set which minimizes $\| P_0 - R\hat{P}_0 \|^2 + \| P_1 - R\hat{P}_3 \|^2 + \| P_2 - R\hat{P}_1 \|^2 + \| P_3 - R\hat{P}_2 \|^2$. The optimally rotated set of reflected points is shown in Figure 5.4c superimposed on the original set. The points $\hat{P}_i$ of the closest mirror-symmetric configuration are obtained by averaging:
\[ \hat{P}_i = \frac{1}{2}(P_i + R\Pi(P_i)) \]. Thus \[ \hat{P}_0 = \frac{1}{2}(P_0 + R\hat{P}_0) \], \[ \hat{P}_1 = \frac{1}{2}(P_1 + R\hat{P}_3) \], \[ \hat{P}_2 = \frac{1}{2}(P_2 + R\hat{P}_2) \] and \[ \hat{P}_3 = \frac{1}{2}(P_3 + R\hat{P}_1) \] (Figure 5.4d).

Finding the optimal rotation and translation which minimizes the sum of squared distances between corresponding points is a well known classic problem of pose estimation. Several methods have been suggested to solve this problem analytically [5, 47]. We follow the method of Arun et. al. [5];

Given 2 sets of points \( \{P_i\}_{i=0}^{n} \) and \( \{\hat{P}_i\}_{i=0}^{n} \) and given a correspondence between them (without loss of generality, we assume point \( P_i \) corresponds to point \( \hat{P}_i \)):

1. Calculate the centroids \( P \) and \( \hat{P} \) of the two sets.
2. Translate each set so that its centroid aligns with the origin, i.e. \( Q_i = P_i - P \) and \( \hat{Q}_i = \hat{P}_i - \hat{P} \) for \( i = 0 \ldots n \)
3. calculate the 3x3 matrix \( H \):
   \[
   H = \sum_{i=0}^{n} Q_i \hat{Q}_i^T
   \]
4. Find the Singular Value Decomposition (SVD) of \( H \) [84, 39] i.e. find 2 orthonormal 3x3 matrices \( U \) and \( V \) and find a diagonal 3x3 matrix \( W \) such that \( H = UVW^T \). (Computational algorithms are readily available - see [84] for example).
5. Calculate the rotation matrix \( R \):\(^1\)
   \[
   R = VU^T
   \]

The translation \( T = P - \hat{P} \) and the rotation matrix \( R \) optimally transform points \( \{\hat{P}_i\} \) such that the sum of squared distances between these points and their corresponding point \( P_i \) is minimal.

This method replaces the folding method and finds the optimal mirror plane (Step 2 in the folding algorithm of Section 3.2) using SVD calculations rather than iterative minimization procedures. The equivalence of this method to the folding algorithm is detailed in [127].

### 5.3 Examples

Several examples of connected configurations of points and the closest 3D mirror-symmetric configurations are shown in Figures 5.5-5.8. The closest symmetric configurations were found using the algorithm described in this chapter.

\(^1\)The determinant of \( R \) should be one, in some extreme cases this is not true and additional steps are required, see [5] for further details.
Figure 5.5: Example of a connected configuration and its closest symmetric configuration with respect to 3D mirror-symmetry. 

a-c) The original connected configuration from various view points. 

d-f) The closest mirror-symmetric configuration from the same view points as a-c respectively.

Figure 5.6: Example of a connected configuration and its closest symmetric configuration with respect to 3D mirror-symmetry. 

a-c) The original connected configuration from various view points. 

d-f) The closest mirror-symmetric configuration from the same view points as a-c respectively.
5.4 Extensions - Chirality

Mirror-symmetry, as a characteristic of shapes and objects extends to the notion of chirality (handedness) [43]. A shape or object is chiral if it can not be superimposed on its reflection (and it is achiral otherwise). In 2D, a shape is achiral if and only if it is mirror-symmetric. In 3D, all mirror-symmetric objects are achiral, however there are achiral objects which are not mirror symmetric (several examples are shown in Figure 5.7). In 3D an object is a-chiral if and only if it contains an improper rotation axis of even order (i.e. a rotation by $\pi/n$ radians followed by a reflection perpendicular to the rotation axis). The structure in Figure 5.7a has an improper rotation of order 2 and that in Figure 5.7b has an improper rotation of order 6.

![Example of achiral 3D structures](image)

Figure 5.7: Example of achiral 3D structures (which can be superimposed on their reflection) and which are not mirror-symmetric.

Due to the influence of this characteristic on chemical reactions and processes, the notion of chirality is widely studied in the natural and physical sciences, and particularly in chemistry [43]. Although measuring of chirality has been discussed in the chemical literature (see [37, 44, 10] also see [120] for a review of measures of chirality) these measures are usually specific to a given chemical structure or assume some reference structure. Additionally no simple and efficient algorithms are suggested for computing these measures for a given structure.

The Symmetry Distance (SD), as defined in Section 3.1, is easily extended to measure chirality. A measure of chirality is defined as the mean squared distance required to move points of an object or shape in order for it to become achiral. This is equivalent to the minimum SD value obtained with respect to any of the symmetry groups which define achiral configurations, i.e. with respect to any symmetry group having a mirror plane or an element of improper rotation of even order (the latter groups are denoted $S_{2n}$ for $n \geq 1$).
Given the definition of chirality, above, a variant of the folding/unfolding method described in Section 5.2 for 3D mirror-symmetry is used to find the closest achiral configuration and to evaluate the chirality distance. The geometrical stage remains the same however the topological stage is varied so that all isomorphisms of even order are listed. Further details are given in [120].

Figure 5.8 shows a chemical structure - a Fullerene molecule ([25]) having 28 carbon atoms, described as a connected graph. Due to the chirality in the connectivity, the closest a-chiral structure is mirror-symmetric and is equivalent to the original Fullerene collapsed to a plane.

Figure 5.8: A chiral Fullerene having 28 Carbon atoms and its closest a-chiral configuration.

a-d) The original Fullerene as a connected configuration from various view points.

e-h) The closest chiral configuration from the same view points as a-d respectively. Notice that due to the chirality in the connectivity, the closest a-chiral structure is mirror-symmetric and is equivalent to the original Fullerene collapsed to a plane.
Chapter 6

3D Symmetry from 2D Data

In the previous chapters we discussed two and three dimensional symmetry distance when the input data was given in 2D and 3D respectively. However a common case in computer vision is that only 2D (projective) data is given on a 3D object. Several studies can be found in the literature that deal with inferring 3D symmetry from 2D data. These studies deal with perfect non-noisy data. In this chapter, we deal with noisy 2D data by extending the notion of Symmetry Distance to 2D projections of 3D objects which are not necessarily perfectly symmetric. As a test case, we describe in this chapter, the reconstruction of 3D mirror symmetric connected configurations from their noisy 2D projections.

Reconstruction of general 3D structures from 2D projections, or the problem of structure from motion, is widely studied in computer vision [102, 95, 101, 110]. However, this topic is outside the scope of this dissertation and we will describe here only the enhancement in performance that can be obtained, using existing structure-from-2D methods, when the reconstructed object is known to be mirror-symmetric.

6.1 Previous Work

When dealing with 3D symmetries, several studies concentrate on finding the projected or skewed symmetries in 2D images [34, 70, 81]. Other studies reconstruct 3D objects from 2D images using symmetry as a constraint [99, 27, 106]. Additionally, symmetry has been used in guiding robot grasping [13]. Recently, symmetry has been exploited for reducing complexity and reducing number of frames in structure from motion problems [79, 67]. See also Section 2.5. However none of these studies deal with exploiting symmetry for improving the input data for structure from motion algorithms.
In this chapter we combine the invariant reconstruction algorithm [110] (reviewed in Section 6.6.1), with the method dealing with inexact symmetries described in previous chapters, in order to pre-process the input and post-process the results of the structure reconstruction from several views.

6.2 Reconstruction of 3D Mirror-Symmetric Structures from 2D Projections

We assume in this chapter that the examined 3D structure is a mirror-symmetric connected configuration (a 3D graph structure composed of one or more connected components - as described in Section 5.1). Further, we assume that noisy 2D projections of this object are given, where the projection is assumed to be weak perspective (i.e. orthographic projection with a possible scaling factor). Several examples of such projections are shown in Figure 6.1.

![Figure 6.1: Weak perspective projections of 3D mirror-symmetric connected configurations.](image)

Using existing methods for structure-from-2D data, we employ two approaches to exploit the fact that the 3D structure to be reconstructed is mirror-symmetric:

- correct for symmetry following reconstruction
- correct for symmetry prior to reconstruction

Correction for symmetry following reconstruction is performed by applying any existing method of structure-from-2D with no a-priori symmetry assumption on the reconstructed
object. Following the reconstruction, the symmetry assumption is exploited and the mirror-symmetric structure closest to the reconstruction is found. This last stage is performed using the closed form method described in Chapter 5 for finding the closest mirror-symmetric configuration to a given 3D connected configuration. In fact, following the reconstruction of a general 3D structure from 2D data, we have a 3D connected configuration which can be measured and can be transformed into the closest symmetric structure with respect to any 3D symmetry group as described in Section 3.2.

Correction for symmetry prior to reconstruction requires application of some symmetrizing procedure on the 2D data with respect to 3D symmetry. Following the symmetrization procedure, any existing method of reconstruction of general 3D structure from 2D data is applied. Notice that this procedure does not ensure that the final reconstructed 3D structure will be mirror-symmetric, however, as will be shown in Section 6.6.2 the error in reconstruction is greatly reduced. In the following section we describe a symmetrization procedure of 2D data for projected 3D mirror-symmetry.

In Section 6.6 we give examples and comparisons between correction for symmetry prior and following 3D reconstruction, using real and simulated data.

### 6.3 2D Symmetrization Procedure

Dealing with mirror-symmetry and assuming weak perspective projection, a 3D mirror-symmetric object has the property that if the projection of the mirror-symmetric pairs of 3D points are connected by segments in the 2D plane, then all these segments are parallel i.e. have the same orientation. We will denote this property as the "projected mirror-symmetry constraint". For example, Figure 6.2a shows a weak perspective projection of a 3D mirror-symmetric connected configuration. Points \( P_i \) and \( P'_i \) \( i = 0, \ldots \) are projections of corresponding mirror-symmetric pairs of points in the 3D structure. Connecting points \( P_i \) with the corresponding \( P'_i \), we obtain the collection of segments of Figure 6.2b which are parallel. Thus the projected mirror-symmetry constraint is satisfied. Notice that if perspective projection were used, these line segments would not be parallel, rather they would be oriented such that the rays extending and including these segments all meet at a single point (the epipole) [67].

We use the projected mirror-symmetry constraint of the 2D weak perspective projection of a 3D mirror-symmetric object in order to correct for symmetry in the 2D data, prior to the reconstruction of the 3D structure. Following the basic idea of the Symmetry Distance and the closest symmetric configuration, as defined in Chapters 3.1, we define a projected mirror-symmetric distance and the closest projected mirror-symmetric configuration, as follows:
Figure 6.2: The projected mirror-symmetry constraint.

a) A weak perspective projection of a 3D mirror-symmetric configuration. Points $P_i$ and $P'_i \ i = 0 \ldots 4$ are projections of corresponding mirror-symmetric pairs of points in the 3D structure.

b) Connecting points $P_i$ with the corresponding $P'_i$, we obtain the collection of parallel segments (i.e. having the same orientation).

Given a configuration of points in 2D $P_0, \ldots, P_{n-1}$, the closest projected mirror-symmetric configuration is the configuration of 2D points $\hat{P}_0, \ldots, \hat{P}_{n-1}$ which satisfies the projected mirror-symmetry constraint and which minimizes

$$\sum_{i=0}^{n-1} \| P_i - \hat{P}_i \|^2$$

(6.1)

The projected mirror-symmetry distance is defined as the minimal value obtained in Equation 6.1. In the following we describe a closed form method for finding the closest projected mirror-symmetric configuration.

### 6.4 Closed Form Method for finding the Closest Projected Mirror-Symmetric Configuration

Given a 2D configuration of connected points $P_i, i = 0 \ldots n - 1$ (as defined in Section 5.1 for 3D) and given a pairing of the points of the configuration (where a degenerate pair of a single point is also permissible), we find a connected configuration of points $\hat{P}_i, i = 0 \ldots n - 1$ which satisfy:
Chapter 6: 3D Symmetry from 2D Data

1. The configuration of points \( \hat{P}_i \) has the same topology as the configuration of points \( P_i \), i.e. points \( \hat{P}_i \) and \( P_j \) are connected if and only if points \( P_i \) and \( P_j \) are connected.

2. Points \( \hat{P}_i, i = 0 \ldots n - 1 \) satisfy the projected mirror-symmetry constraint, i.e. all segments connecting paired points are parallel.

3. The following sum is minimized:

\[
\sum_{i=0}^{n-1} \| P_i - \hat{P}_i \|^2
\]

We first consider a simple case where we are given two points \( P_0 \) and \( P_1 \) in 2D and we are given an orientation \( \theta \) (depicting the orientation of the parallel segments). Without loss of generality, \( \theta \) is the angle to the positive x-axis. We find 2 points \( \hat{P}_0 \) and \( \hat{P}_1 \) such that the segment connecting them is at orientation \( \theta \) and the following sum is minimized:

\[
\| P_0 - \hat{P}_0 \|^2 + \| P_1 - \hat{P}_1 \|^2
\]

Claim 1: Given a line \( y = \tan(\theta)x + c \) \( (c \in \mathcal{R}) \), points \( \hat{P}_0 \) and \( \hat{P}_1 \) which minimize Equation 6.3 are obtained by projecting \( P_0 \) and \( P_1 \) respectively, onto the line.

Furthermore, the line of orientation \( \theta = 0^\circ \), on which positioning points \( \hat{P}_0 \) and \( \hat{P}_1 \) minimizes Equation 6.3, passes through the centroid (or mid-section point) of \( P_0 \) and \( P_1 \) (Figure 6.3).

![Figure 6.3: Finding the closest projected mirror-symmetric configuration.](image)

A simple case of two points:

Given two points \( P_0 \) and \( P_1 \), the points located on a line of orientation \( \theta \) which are closest to \( P_0 \) and \( P_1 \) are obtained by projecting \( P_0 \) and \( P_1 \) onto a line of orientation \( \theta \) passing through the midpoint between \( P_0 \) and \( P_1 \).
Thus, given 2 points in 2D, \( P_0 \) and \( P_1 \), and given an orientation \( \theta \), the value of Equation 6.3 is:

\[
\| P_0 - \hat{P}_0 \|^2 + \| P_1 - \hat{P}_1 \|^2 = \frac{1}{2} \left[ (x_1 - x_0) \sin(\theta) - (y_1 - y_0) \cos(\theta) \right]^2
\]

Consider, now, the case in which \( n \) points in 2D are given \( P_0 \ldots P_{n-1} \), and a pairing \( \Pi \) (permutation of order two) of these points are given (i.e., point \( P_i \) is paired with point \( \Pi(P_i) \) for \( i = 0 \ldots n - 1 \)). In order to find the points \( \hat{P}_0 \ldots \hat{P}_{n-1} \) that minimize Equation 6.2 and that satisfy the projected mirror-symmetry constraint, we must find the orientation \( \theta \) which minimizes Equation 6.2. For a given orientation \( \theta \), the value of Equation 6.2 is

\[
\sum_{i=0}^{n-1} \| P_i - \hat{P}_i \|^2 = \sum_{i=0}^{n-1} \left[ (x_i - \Pi(x_i)) \sin(\theta) - (y_i - \Pi(y_i)) \cos(\theta) \right]^2
\]

where \( x_i, y_i \) and \( \Pi(x_i), \Pi(y_i) \) are the 2D coordinates of the points \( P_i \) and \( \Pi(P_i) \) respectively.

Taking the derivative with respect to \( \theta \) and equating to zero we obtain for the minimizing \( \theta \):

\[
\tan 2\theta = \frac{2 \sum_{i=0}^{n-1} (x_i - \Pi(x_i))(y_i - \Pi(y_i))}{\sum_{i=0}^{n-1} (x_i - \Pi(x_i))^2 - (y_i - \Pi(y_i))^2} \quad (6.4)
\]

Notice the similarity between this equation and Equation 4.9 in Section 4.1.3. As in Equation 4.9, 2 possible solutions exist for Equation 6.4, and the minimizing solution is achieved when \( \sin \theta \cos \theta \) is of opposite sign to the numerator of Equation 6.4.

Thus we have a closed form solution for finding the closest projected mirror-symmetric set of points: Given the 2D points \( P_0 \ldots P_{n-1} \) and a pairing \( \Pi \):

1. calculate the optimal orientation \( \theta \) according to Equation 6.4.
2. calculate the coordinates of each point \( \hat{P}_i \) by projecting the points \( P_i \) onto a line at orientation \( \theta \) passing through the midpoint between \( P_i \) and \( \Pi(P_i) \).

Several examples of noisy 2D projections of mirror-symmetric configurations of points are shown in Figure 6.4 with the closest projected mirror-symmetric configuration, which was obtained using the above algorithm. The pairing is shown by the connecting segments.
Figure 6.4: Finding the closest projected mirror-symmetric configuration. a-c) Several examples of noisy 2D projections of mirror-symmetric configurations of points (left) and the closest projected mirror-symmetric configuration (right).

6.5 The Pairing

As described above, the closest projected mirror-symmetric configuration of a set of connected points in 2D can be found if a pairing between the points is given. In the case when the pairing is not given, the closest projected mirror-symmetric configuration is found by minimizing the projected symmetry distance (Equation 6.2) over all possible pairings of the 2D data points. Given a connected configuration of points, finding all possible matchings is not as formidable as it seems. In fact, this procedure was discussed in Chapter 5, where a recursive algorithm was described for finding all possible pairings (isomorphisms of order 2) in a 3D graph structure. In our case, we are given a 2D connected configuration, i.e. a 2D graph structure. However, the algorithm described in Section 5.2.1 for finding all matchings in a 3D graph, is independent of the point location and dependent only on the topology of the graph. Thus the discussion and algorithm given in Section 5.2.1 is also applicable to finding all possible pairings in a 2D projection of a 3D connected configuration of points.

6.6 Experiments

In this section we describe simulations in which 3D mirror-symmetric connected configurations are reconstructed from noisy 2D weak perspective projections. We use the two
approaches of correction for symmetry which were described in Section 6.2.

The reconstruction method used in the simulations is the invariant reconstruction method [110]. This method is briefly described in the following subsection.

The correction procedures were the following:

1. The reconstruction method reviewed in Section 6.6.1 was applied directly to the 2D data with no symmetry assumption. Following the reconstruction, correction for symmetry was applied to the 3D reconstruction by finding the closest 3D mirror-symmetric configuration using the method described in Section 5.2.

2. Correction for symmetry was applied to the 2D projected data by finding, for every projection, the closest projected mirror-symmetric configuration, using the method described in Section 6.4. Following the correction for symmetry, the invariant reconstruction method was applied on the modified projections.

3. Correction for symmetry was performed both prior and following the reconstruction of the 3D configuration from 2D data.

The reconstruction obtained from these two procedures were compared with the original mirror-symmetric 3D configuration. The difference was measured as the mean squared-distance between the reconstructed and the original configurations of 3D points.

### 6.6.1 Reconstruction from Invariants

This linear method was described in [110]. It computes an invariant description of the Euclidean structure of points from a sequence of images assuming weak perspective.

Let \(\{\mathbf{p}_i\}_{i=0}^{n-1}\), \(\mathbf{p}_i \in \mathbb{R}^3\), denote the 3D coordinates of an object composed of \(n\) features in some Cartesian coordinate system. For simplicity and clarity, we start with the case \(n = 4\) and \(\mathbf{p}_0 = (0, 0, 0)\). Let \(P\) denote the \(3 \times 3\) matrix whose columns are the vectors \(\{\mathbf{p}_i\}_{i=0}^{n-1}\), namely, \(P = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3]\).

A representation of the object shape, which is invariant to rigid transformations of the camera, is the Gramian matrix\(^1\) \(G = P^T P\):

\[
G = \begin{pmatrix}
\mathbf{p}_1^T \mathbf{p}_1 & \mathbf{p}_1^T \mathbf{p}_2 & \mathbf{p}_1^T \mathbf{p}_3 \\
\mathbf{p}_2^T \mathbf{p}_1 & \mathbf{p}_2^T \mathbf{p}_2 & \mathbf{p}_2^T \mathbf{p}_3 \\
\mathbf{p}_3^T \mathbf{p}_1 & \mathbf{p}_3^T \mathbf{p}_2 & \mathbf{p}_3^T \mathbf{p}_3
\end{pmatrix}
\]

\(^1\)computing depth (matrix \(P\)) from \(G\) is straightforward, and computationally very fast (a decomposition known as Choleski factorization).
Using the weak perspective approximation, it can be shown [109] that:

\[
\begin{align*}
x^T G^{-1} x &= y^T G^{-1} y \\
x^T G^{-1} y &= 0
\end{align*}
\] (6.5)

where the vectors \( \mathbf{x} = (x_1, x_2, x_3) \) and \( \mathbf{y} = (y_1, y_2, y_3) \) are obtained from the image data points \( \{P_i = (x_i, y_i)\}_{i=1}^3 \). We compute the Gramian of the 4 points by solving the linear system of equations given in Eq (6.5) (note that Eq (6.5) is linear in the elements of the inverse Gramian). The Gramian gives the complete Euclidean-invariant (metric) structure of the 4 points [109].

Given more than 4 points, the algorithm proceeds as follows:

- select 4 basis points from the data (using QR factorization to maximize the independence of the selected points);
- compute the affine structure of all the points by solving a linear system of equations;
- compute the Euclidean structure of the 4 basis points by solving a linear system (given in Eq (6.5));
- obtain the Euclidean structure of all the points if necessary (this can be done by multiplying a vector of affine coordinates by the root of the Gramian \( G \) of the basis points).

### 6.6.2 Simulation Results

Two examples of the simulation are shown in Figures 6.5-6.6. Two randomly chosen 3D mirror-symmetric connected configuration of 10 points are shown in Figures 6.5a and 6.6a. Points were selected randomly in the box \([0,1]^3\). Eight noisy 2D projections were created for each of the 3D configurations. Perspective projections were used with a focal length of 5. The projections are from randomly chosen viewpoints and the noise was added to the 2D projections and was set at a predefined level of \( \sigma = 0.005 \) for the simulation of Figure 6.5 and of \( \sigma = 0.05 \) for the simulation of Figure 6.6. Reconstruction of the connected configuration directly from the 2D projections, with no symmetry assumption is shown in Figures 6.5b and 6.6b. When correcting for symmetry prior to reconstruction by finding the closest projected mirror-symmetric configuration, the 3D reconstruction obtained is shown in Figures 6.5c and 6.6c. When correcting for symmetry following the reconstruction by finding the closest 3D mirror-symmetric configuration, the 3D reconstruction obtained is shown in Figures 6.5d and 6.6d. Finally, Figures 6.5e and 6.6e show the 3D reconstructed configuration following correction for symmetry prior and
following the reconstruction. The differences between the reconstructions and the original 3D mirror-symmetric configurations were measured as the least squared-distance between the reconstructed configuration and the original configuration [47]. The percentage of improvement was calculated using the following notations:

\[ D_1 = \text{the distance between the original and the reconstruction with no symmetry assumption.} \]

\[ D_2 = \text{the distance between the original and the reconstruction with symmetrization prior to reconstruction.} \]

\[ D_3 = \text{the distance between the original and the reconstruction with symmetrization following the reconstruction.} \]

\[ D_4 = \text{the distance between the original and the reconstruction with symmetrization both prior and following the reconstruction.} \]

The three values for percentage of improvement are \( (D_1 - D_i)/D_1 \) for \( i = 2, 3, 4 \).

The differences and percentage of improvement are summarized in Table 6.1.

<table>
<thead>
<tr>
<th>Sigma</th>
<th>No Symmetrization</th>
<th>Symmetrization prior to reconstruction</th>
<th>Symmetrization following reconstruction</th>
<th>Symmetrization prior &amp; following reconstruction</th>
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<td></td>
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<td>% improvement</td>
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<td>sim 1</td>
<td>0.005</td>
<td>0.081967</td>
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<td>0.057879</td>
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<td>0.046645</td>
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<td>50.48%</td>
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<td>sim 2</td>
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<td>0.058274</td>
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<td>50.48%</td>
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</table>

Table 6.1: Quality of the reconstructions of 3D mirror-symmetric configurations from noisy 2D projections.

The differences between the reconstructions and the original 3D mirror-symmetric configurations are measured by the mean squared-distance required to move points of the reconstructed configuration in order to obtain the original configuration.

As seen in the examples, reconstruction of 3D mirror-symmetric configurations from noisy 2D projected data, can be greatly improved by correcting for symmetry either prior and/or following reconstruction. Although, correcting for symmetry prior to reconstruction improves the result, correcting for symmetry following reconstruction generally gives a greater improvement. Not surprisingly, the greatest improvement in reconstruction is obtained when correction for symmetry is performed both prior and following reconstruction.
Figure 6.5: Reconstruction of 3D mirror-symmetric configurations from noisy 2D projections - simulation
a) the original 3D mirror-symmetric configuration having 10 points (a noise-free projection is shown). Eight noisy 2D projections were created from randomly chosen viewpoints and the noise was set at a level of $\sigma = 0.005$.
b) Reconstruction of the connected configuration directly from the noisy 2D projections, with no symmetry assumption. The mean-squared distance between this reconstruction and the original configuration is 0.084967.
c) Reconstruction of the connected configuration when correction for symmetry was performed prior to reconstruction by finding the closest projected mirror-symmetric configuration. The mean-squared distance between this reconstruction and the original configuration is 0.072156 (15.08% improvement).
d) Reconstruction of the connected configuration when correction for symmetry was performed following the reconstruction by finding the closest 3D mirror-symmetric configuration. The mean-squared distance between this reconstruction and the original configuration is 0.057879 (31.88% improvement).
e) Reconstruction of the connected configuration when correction for symmetry was performed both prior and following the reconstruction. The mean-squared distance between this reconstruction and the original configuration is 0.048645 (42.75% improvement).
Figure 6.6: Reconstruction of 3D mirror-symmetric configurations from noisy 2D projections - simulation
a) the original 3D mirror-symmetric configuration having 10 points (a noise-free projection is shown). Eight noisy 2D projections were created from randomly chosen viewpoints and the noise was set at a level of $\sigma = 0.05$.
b) Reconstruction of the connected configuration directly from the noisy 2D projections, with no symmetry assumption. The mean-squared distance between this reconstruction and the original configuration is 0.094200.
c) Reconstruction of the connected configuration when correction for symmetry was performed prior to reconstruction by finding the closest projected mirror-symmetric configuration. The mean-squared distance between this reconstruction and the original configuration is 0.086757 (7.90% improvement).
d) Reconstruction of the connected configuration when correction for symmetry was performed following the reconstruction by finding the closest 3D mirror-symmetric configuration. The mean-squared distance between this reconstruction and the original configuration is 0.058274 (38.14% improvement).
e) Reconstruction of the connected configuration when correction for symmetry was performed both prior and following the reconstruction. The mean-squared distance between this reconstruction and the original configuration is 0.046645 (50.48% improvement).
In order to obtain some statistical appraisal of the improvement obtained by correcting for symmetry, we applied the simulation many times while varying the number of points in the configuration, the number of 2D projections created from random views and the amount of noise in the 2D projections. Points were, again, selected randomly in the box $[0, 1]^3$, however, in this simulation, perspective projections were used with a focal length of 5. The number of points was varied between 8 and 24, the number of views was varied between 8 and 24, and the noise level was taken as $\sigma = 0.001, 0.005, 0.01, 0.05$ and 0.1. Every combination of parameters was simulated 300 times. The differences between the reconstructions and the original configurations were measured as in the above two examples. The percentage of improvement between the reconstruction with no symmetry assumption and the reconstructions with correction for symmetry, were calculated and averaged over the simulations. Thus for every $\sigma$ value the percentage of improvement was averaged over 7500 trials. The percentage of improvement was calculated as above. The results are given in Table 6.6.2. Using $\sigma$ greater than 0.1 the percentage of improvement breaks down, although when using orthographic projections the improvement is significant up to $\sigma = 0.3$.

<table>
<thead>
<tr>
<th>$\sigma$ (noise)</th>
<th>Symmetry prior to reconstruction vs. No Symmetry % improvement</th>
<th>Symmetry following reconstruction vs. No Symmetry % improvement</th>
<th>Symmetry prior &amp; following reconstruction vs. No Symmetry % improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>11.4</td>
<td>37.7</td>
<td>42.0</td>
</tr>
<tr>
<td>0.005</td>
<td>12.6</td>
<td>38.4</td>
<td>43.3</td>
</tr>
<tr>
<td>0.01</td>
<td>11.3</td>
<td>38.3</td>
<td>43.2</td>
</tr>
<tr>
<td>0.05</td>
<td>4.0</td>
<td>28.9</td>
<td>29.3</td>
</tr>
<tr>
<td>0.1</td>
<td>4.8</td>
<td>23.1</td>
<td>22.2</td>
</tr>
<tr>
<td>All</td>
<td>8.8</td>
<td>33.3</td>
<td>36.0</td>
</tr>
</tbody>
</table>

Table 6.2: Improvement in reconstruction of 3D mirror-symmetric configurations from noisy 2D perspective projections. Reconstructions with correction for symmetry are compared with the reconstruction with no symmetry assumption. The improvement in the reconstruction is measured as the percentage of decrease in the difference between the reconstruction and the original mirror-symmetric configuration when correction for symmetry is performed compared to the case where no symmetry assumption is used. The results are given as a function of the noise ($\sigma$ value) and are the average over 7500 trials per each value of $\sigma$. 
6.7 Real data

Our algorithm was applied to measurements taken from 2D real images of an object.

![Figure 6.7: Three 2D images of a 3D mirror-symmetric object from different points of view.](image)

In the following example we took images of the object at three different positions (Figure 6.7). 16 feature points were manually extracted from each of the three images. Using the 16 points and the three views, the 3D object was reconstructed using the invariant reconstruction method with symmetrization performed prior, following or both prior and following the reconstruction, as discussed above. The reconstructions were compared to the real (measured) 3D coordinates of the object. The results are given in Table 6.3.

<table>
<thead>
<tr>
<th></th>
<th>No Symmetrization</th>
<th>Symmetrization prior to reconstruction</th>
<th>Symmetrization following reconstruction</th>
<th>Symmetrization prior &amp; following reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>1.619283</td>
<td>1.388134</td>
<td>1.339260</td>
<td>1.329660</td>
</tr>
<tr>
<td>% improvement</td>
<td>14.3</td>
<td>17.3</td>
<td>17.9</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.3: Improvement in reconstruction of a real 3D mirror-symmetric object from three 2D images.

Reconstructions with correction for symmetry are compared with the reconstruction with no symmetry assumption. The differences between the reconstructions and the original 3D mirror-symmetric object were measured as the least squared-distance between the reconstructed object and the original object. The improvement in the reconstruction is measured as the percentage of decrease in the difference between the reconstruction and the original mirror-symmetric object when correction for symmetry is performed compared to the case where no symmetry assumption is used.
Chapter 7
Conclusion

In this thesis symmetry is viewed as an inherently continuous property rather than a binary “yes/no” feature.

A “Symmetry Distance” (Symmetry Distance) was introduced, that can measure and quantify all types of symmetries of objects. The Symmetry Distance of an object is defined as the minimum distortion required to transform the object into a symmetric one. The Symmetry Distance is presented in a mathematical framework and forms a metric in the space of all shapes of a given dimension. It is also in accord with human perception of symmetry as a continuous feature.

A geometrical algorithm is presented and proven, to evaluate the Symmetry Distance. This method first finds the Symmetry Transform of the object, which is the symmetric object closest to the original in terms of minimal distortion. The distance between the object and its symmetry transform is evaluated as the Symmetry Distance.

The Symmetry Distance is a continuous measure which reflects the intuitive perception of symmetry. It is invariant to rigid and similarity transformations and, due to its generality, allows comparisons between the “amount” of symmetry of different shapes and between “amounts” of different symmetries of a single shape with respect to any point symmetry group in any dimension.

Associated with the value of Symmetry Distance, is the Symmetry Transform of the shape which is the symmetric shape “closest” to the test shape. The Symmetry Transform enables visual evaluation of the Symmetry Distance, since similarity of a shape and its closest symmetric shape, correlates with low Symmetry Distance values.

The features of the SD are used to deal with occluded and noisy shapes. Thus, occluded almost symmetric shapes are reconstructed by locating the center of symmetry of the shape using the Symmetry Distance. Additionally, the Symmetry Distance is extended
to evaluate imperfect symmetries of fuzzy shapes, i.e., shapes with uncertain point localization. For every such fuzzy shape, the most probable symmetric shape is found. The probability distribution of symmetry values for a given fuzzy shape, is also evaluated.

Several applications of the Symmetry Distance are described in this work: Considering grayscale images, the Symmetry Distance is used to find the orientation of symmetric objects (specifically faces) from range data and intensity images. The Symmetry Distance is combined with a multiresolution scheme to find locally symmetric regions in images. Finally, symmetry, as a characteristic of objects, is exploited in reconstruction of 3D structure from 2D data by extending the Symmetry Distance techniques to deal with projected symmetry.
Bibliography


Bibliography


Bibliography


Bibliography


Appendix A

Dividing Points of a Shape into Sets

Figure A.1: Dividing $m$ selected points into interlaced sets:

a) $C_n$-symmetry - 1 possibility.
b) $D_n$-symmetry - $m/2n$ possibilities

As described in Section 3.2, when measuring $C_n$-symmetry (rotational symmetry of order $n$) of a shape represented by a multiple of $n$ points, the points must be divided into sets of $n$ points. In general, this problem is exponential, however when the points are ordered along a contour, as in our case, the possible divisions into sets are more restricted since the ordering is preserved under the symmetry transform of a shape. For example, points in 2D along the contour of a $C_n$-symmetric shape form orbits which are interlaced. An example is shown in Figure A.1a for $C_3$-symmetry, where 3 interlaced orbits are shown marked as $\bullet$, $\circ$ and $\square$. Thus, given a set of $m = nq$ ordered points there is only one possible division of the points into $q$ sets of $n$ points such that the ordering is preserved in the symmetric shape - the $q$ sets must be interlaced (as was shown in Figure 3.4). In the case of $D_n$-symmetry (rotational and reflective symmetry of order $n$) the $m = 2nq$ ordered points, form $q$ orbits which are interlaced and partially inverted to account for the reflection symmetry. An example is shown in Figure A.1b for $D_4$-symmetry where instead of 3 interlaced orbits $\bullet \circ \square \bullet \circ \square \ldots \bullet \circ \square \bullet$, every other run is inverted: $\bullet \circ \square \circ \bullet \ldots \circ \square \circ \bullet$. Thus, given a set of $m = 2nq$ ordered points there are $m/2n = q$ possible division of the points.
Appendix B

The Bounds of S Values

Following the definition of the SD in Section 3.1, the SD values are limited to the range [0, 1] (where 1 is the normalization scale). The lower bound of the SD is obvious from the fact that the average of the square of the distances moved by the object points, is necessarily non-negative. The upper bound of the average is limited to 1 since the object is previously normalized to maximum distance of 1 and by translation of all vertex points to the center of mass, a symmetric shape is obtained.

The upper bound on the SD can be tightened for specific cases. For instance in 2D one can show that the maximum SD value for a triangle, with respect to $C_3$ is 1/3: Consider the 3 vertices of a normalized triangle $P_1, P_2, P_3$ in 2D (the centroid is at the origin). W.l.g. assume $P_1 = (0, 1)$ and that $P_2$ has a positive x-coordinate and denote by $(x, y)$ the coordinates of $P_2$. Given the constraint that the centroid is at the origin, one has $P_3 = (-x, -1 - y)$. In fact $P_2$ is limited to a circle sector due to the centroid constraint and the normalization constraint (limiting all $P_i$’s to be in the unit circle). Given these notations, we have that the SD value of the triangle with respect to $C_3$-symmetry, is given by:

$$\frac{1}{3}(1 + y^2 + y - \sqrt{3}x + x^2)$$

Considering the limited range of the $P_2$ coordinates, the maximum value is obtained when $P_2 = (0, 0)$ or $P_2 = (0, -1)$ (which are equivalent cases) and the maximum SD value is 1/3.

The maximum SD value is actually obtained for extreme cases such as a polygon of $m$ vertices ($m = qn$) whose contour outlines a regular $q$-gon (i.e., every $q$-th vertex of the $m$-gon coincides with a vertex of a regular $q$-gon). For details, see Appendix in [125].
Appendix C

Applications in Chemistry

The Symmetry Distance has been applied to structural and organic chemistry. Following are studies previously reported in [125, 121].