Image Representation

- Gaussian pyramids
- Laplacian Pyramids
- Wavelet Pyramids
- Applications

Image Pyramids

Image features at different resolutions require filters at different scales.

Edges (derivatives):

\[ f(x) \]

\[ f'(x) \]

Image Pyramid = Hierarchical representation of an image

Low Resolution

No details in image - (blurred image)

low frequencies

High Resolution

Details in image -

low+high frequencies

A collection of images at different resolutions.

Image pyramids

- Gaussian Pyramids
- Laplacian Pyramids
- Wavelet/QMF
Image Pyramid

High resolution

Low resolution

Image Pyramid

Frequency Domain

High resolution

Low resolution

Image Blurring = low pass filtering

\[
\text{High resolution} \xrightarrow{\text{Low pass filtering}} \text{Low resolution}
\]
Image Pyramid

Low resolution

High resolution

Gaussian Pyramid

Burt & Adelson (1981)

Normalized: $\sum w_i = 1$

Symmetry: $w_i = w_{-i}$

Unimodal: $w_i \geq w_j$ for $0 < i < j$

Equal Contribution: for all $j$ \[ \sum w_{i,j} = \text{constant} \]
**Gaussian Pyramid**

Burt & Adelson (1981)

- Normalized: $\sum w_i = 1$
- Symmetry: $w_i = w_{-i}$
- Unimodal: $w_i \geq w_j$ for $0 < i < j$
- Equal Contribution: for all $j$, $\sum w_{j+2i} = \text{constant}$

\[
\begin{array}{cccccc}
  w_0 & w_{-1} & w_{-1} & w_{-2} & w_{-2} \\
  w_0 & w_{-1} & w_{-1} & w_{-2} & w_{-2}
\end{array}
\]

For $a = 0.4$ most similar to a Gaussian filter

\[
g = [0.05 \ 0.25 \ 0.4 \ 0.25 \ 0.05]
\]

\[
\text{low_pass_filter} = g' * g =
\]

\[
\begin{bmatrix}
  0.0025 & 0.0125 & 0.0200 & 0.0125 & 0.0025 \\
  0.0125 & 0.0625 & 0.1000 & 0.0625 & 0.0125 \\
  0.0200 & 0.1000 & 0.1600 & 0.1000 & 0.0200 \\
  0.0125 & 0.0625 & 0.1000 & 0.0625 & 0.0125 \\
  0.0025 & 0.0125 & 0.0200 & 0.0125 & 0.0025
\end{bmatrix}
\]
Gaussian Pyramid - Computational Aspects

Memory:
\[ 2^N \times 2^N (1 + 1/4 + 1/16 + \ldots) = 2^N \times 2^N \times 4/3 \]

Computation:
Level \( i \) can be computed with a single convolution with filter: \( h_i = g \ast g \ast g \ast \ldots \), \( i \) times

Example:
\[ h_2 = g \ast g \]

MultiScale Pattern Matching

Option 1:
Scale target and search for each in image.

Option 2:
Search for original target in image pyramid.

Hierarchical Pattern Matching

Pattern matching using Pyramids - Example
Image pyramids

- Gaussian Pyramids
- Laplacian Pyramids
- Wavelet/QMF

Laplacean Pyramid

Motivation = Compression, redundancy removal. Compression rates are higher for predictable values. 
E.g. values around 0.

$G_0, G_1, \ldots = \text{the levels of a Gaussian Pyramid.}$

Predict level $G_i$ from level $G_{i+1}$ by Expanding $G_{i+1}$ to $G'_i$

Denote by $L_i$ the error in prediction:

\[
L_i = G_i - G'_i
\]

$L_0, L_1, \ldots = \text{the levels of a Laplacian Pyramid.}$

What does blurring take away?

original

What does blurring take away?

smoothed (5x5 Gaussian)
What does blurring take away?

Gaussian Pyramid

Frequency Domain

Laplacian Pyramid

No scaling
Laplacian Pyramid - Computational Aspects

Memory:

\[ 2^N \times 2^N \left( 1 + \frac{1}{4} + \frac{1}{16} + \cdots \right) = 2^N \times 2^N \times \frac{4}{3} \]

However, coefficients are highly compressible.

Computation:

\( L_i \) can be computed from \( G_0 \) with a single convolution with filter:  \( k_i = h_{i-1} - h_{i} \)

\[ h_{i-1} \quad h_i \quad k_i \]

\[ k_1 \quad k_2 \quad k_3 \]

Reconstruction of the original image from the Laplacian Pyramid

\[ G_i = L_i + G_i \]

Original Image

Image Mosaicing

Registration
When splining two images, transition from one image to the other should behave:

- **High Frequencies**
- **Middle Frequencies**
- **Low Frequencies**
Multiresolution Spline - Example

Left Image  Right Image

Left + Right  Narrow Transition

Wide Transition

(Burt & Adelson)

Multiresolution Spline - Using Laplacian Pyramid

(Burt & Adelson)
Multiresolution Spline

Multi-Res. Blending
What is a good representation for image analysis?

- Pixel domain representation tells you “where” (pixel location), but not “what”.
  - In space, this representation is too localized
- Fourier transform domain tells you “what” (textural properties), but not “where”.
  - In space, this representation is too spread out.
- Want an image representation that gives you a local description of image events—what is happening where.
  - That representation might be “just right”.

Image pyramids

- Gaussian Pyramids
- Laplacian Pyramids
- Wavelet/QMF

Space-Frequency Tiling

<table>
<thead>
<tr>
<th>Freq.</th>
<th>Spatial</th>
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<tbody>
<tr>
<td></td>
<td>Standard basis</td>
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</table>
Various Wavelet basis

Wavelet - Frequency domain

Wavelet bands are split recursively

Wavelet bands are split recursively

Wavelet decomposition - 2D

Apply the wavelet transform separably in both dimensions
• Splitting can be applied recursively:

Pyramids in Frequency Domain

Gaussian Pyramid  Laplacian Pyramid

Wavelet Decomposition

Wavelet Transform - Step By Step Example

Fourier Space
Wavelet Transform - Example

Application: Wavelet Shrinkage Denoising

Noisy image

Clean image

From: B. Freeman
Get top histogram but want to get bottom histogram.

Wavelet Shrinkage Denoising

For every Wavelet Band define Shrinkage function:

Wavelet Shrinkage Pipe-line

More results
Image Pyramids - Comparison

Image pyramid levels = Filter then sample.

Filters:
- Gaussian Pyramid
- Laplacian Pyramid
- Wavelet Pyramid

Image Linear Transforms

<table>
<thead>
<tr>
<th>Transform</th>
<th>Basis</th>
<th>Characteristics</th>
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<tbody>
<tr>
<td>Delta</td>
<td>Standard</td>
<td>Localized in space</td>
</tr>
<tr>
<td>Fourier</td>
<td>Sines+Cosines</td>
<td>Not localized in Frequency</td>
</tr>
<tr>
<td>Wavelet Pyramid</td>
<td>Wavelet Filters</td>
<td>Localized in space</td>
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</table>

\[ \vec{F} = U \vec{f} \]

Convolution and Transforms in matrix notation (1D case)

transformed image
Basis vectors (Fourier, Wavelet, etc)

Vectorized image
From: B. Freeman

Fourier Transform

Fourier bases are global: each transform coefficient depends on all pixel locations.

Inverse Fourier Transform

Every image pixel is a linear combination of the Fourier basis weighted by the coefficient.

Note that if $U$ is orthonormal basis then $U^{-1} = U^T$

Transform in matrix notation (1D case)

Forward Transform:

$$\tilde{F} = U\tilde{f}$$

Inverse Transform:

$$U^{-1}\tilde{F} = \tilde{f}$$

Convolution

Convolution Result

Circular Matrix of Filter Kernels

From: B. Freeman
Pyramid = Convolution + Sampling

Pyramid Level 1

Pyramid Level 2

Pyramid Level 2

Pyramid Level 2

Pyramid = Convolution + Sampling

Pyramid Level 1

Pyramid Level 2

Pyramid Level 2

Pyramid Level 2

Pyramid as Matrix Computation - Example

From: B. Freeman
Gaussian Pyramid

\[
\text{pixel image} = \ast \text{Gaussian pyramid}
\]
Overcomplete representation. Low-pass filters, sampled appropriately for their blur.

From: B. Freeman

Laplacian Pyramid

\[
\text{pixel image} = \ast \text{Laplacian pyramid}
\]
Overcomplete representation. Transformed pixels represent bandpassed image information.

From: B. Freeman

Wavelet Transform

\[
\text{pixel image} = \ast \text{Wavelet pyramid}
\]
Ortho-normal transform (like Fourier transform), but with localized basis functions.

From: B. Freeman

The End