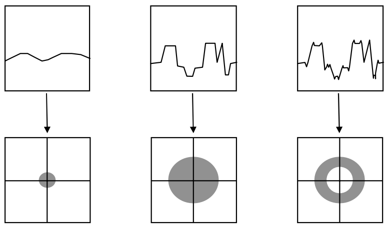
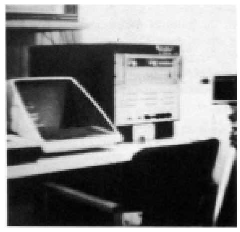
Image Operations in the Frequency Domain

- Low Pass Filter
- High Pass Filter
- Band pass Filter
- Blurring
- Sharpening

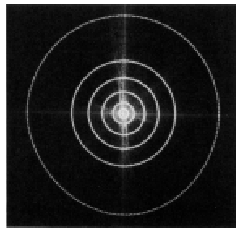


Frequency Bands

Image



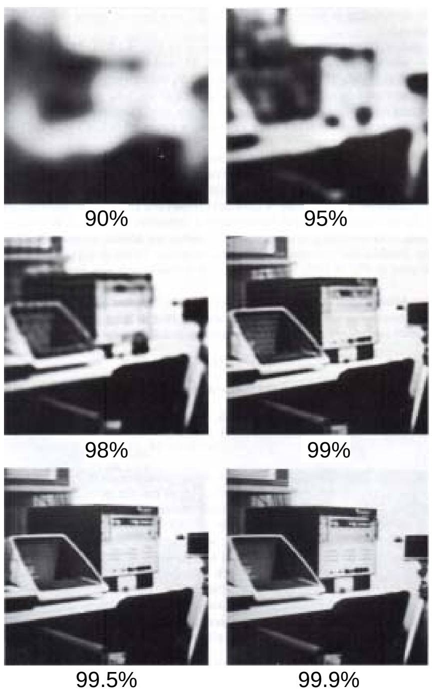
Fourier Spectrum

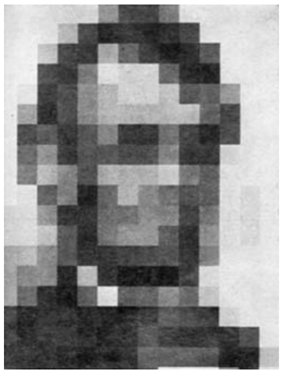


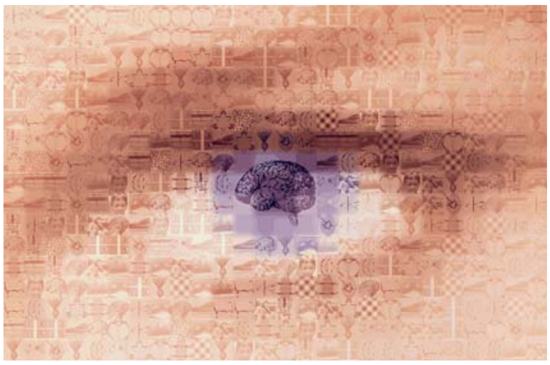
Percentage of image power enclosed in circles (small to large) :

90, 95, 98, 99, 99.5, 99.9

Blurring - Ideal Low pass Filter







The Power Law of Natural Images

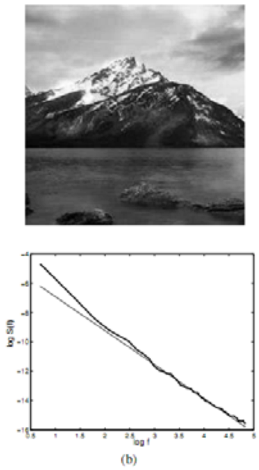


Figure 1: (a) A natural image (256 × 256 pixels), and (b) its circularly averaged power spectrum (thick line) and a linear fit to the high frequency portion (thin line). The slope in (b) is 2.3.

- The power in a disk of radii $r = \mathrm{sqrt}(u^2+v^2)$ follows: $P(r)=Ar^{-\alpha}$ where $\alpha \approx 2$

Images from: Millane, Alzaidi & Hsiao - 2003

Recall: The Convolution Theorem

$g = f * h$	$g = f \cdot h$
implies	implies
$G = F \cdot H$	$G = F * H$

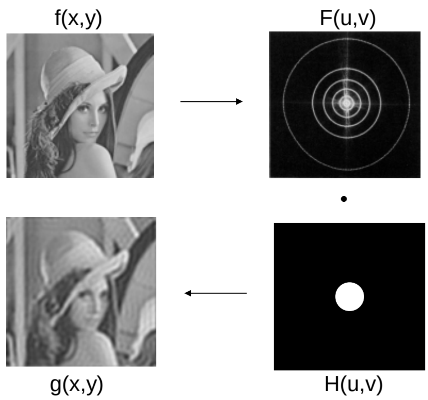
Low pass Filter

spatial domain — frequency domain

f(x,y) → F(u,v)

G(u,v) = F(u,v) • H(u,v) (filter: H(u,v))

g(x,y) ← G(u,v)

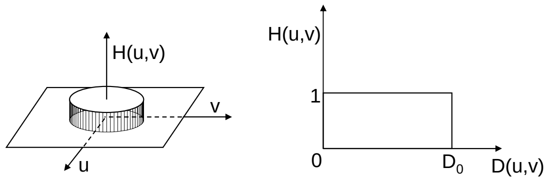


H(u,v) - Ideal Low Pass Filter

$$H(u,v) = \begin{cases} 1 & D(u,v) \le D_0 \\ 0 & D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \sqrt{u^2 + v^2}$$

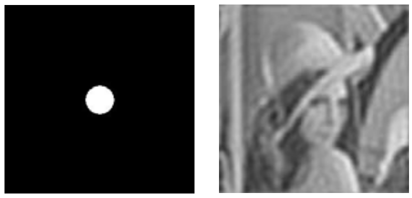
D_0 = cut off frequency



Blurring - Ideal Low pass Filter

99.7%

99.37%



98.65%

Blurring - Ideal Low pass Filter

99.7%

99.6%

99.4%

99.0%

98.0%

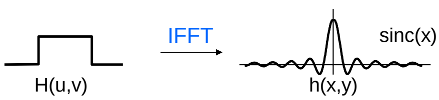
96.6%

The Ringing Problem

$G(u,v) = F(u,v) \bullet H(u,v)$

↓ Convolution Theorem

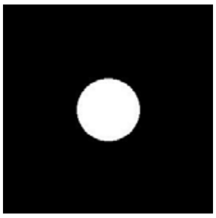
$g(x,y) = f(x,y) * h(x,y)$



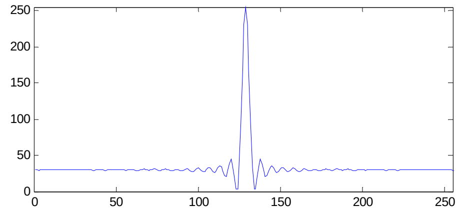
↑ D_0 ⟶ ↓ Ringing radius + ↓ blur

The Ringing Problem

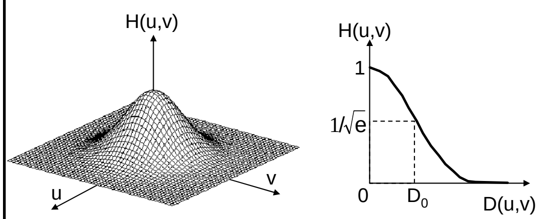
Freq. domain



Spatial domain



H(u,v) - Gaussian Filter

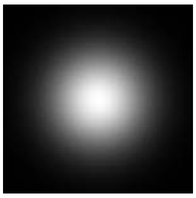


$$H(u,v) = e^{-D^2(u,v)/(2D^2_0)}$$

$$D(u,v) = \sqrt{u_2 + v_2}$$

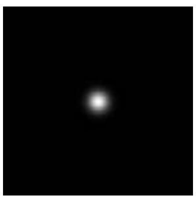
Softer Blurring + no Ringing

Blurring - Gaussain Lowpass Filter



99.11%

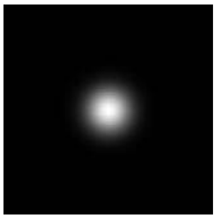
98.74%



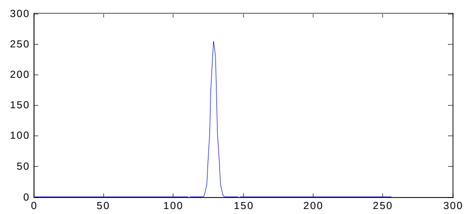
96.44%

The Gaussian Lowpass Filter

Freq. domain



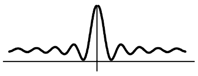
Spatial domain



Blurring in the Spatial Domain:

Averaging = convolution with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

= point multiplication of the transform with sinc:



Gaussian Averaging = convolution with $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

= point multiplication of the transform with a gaussian.

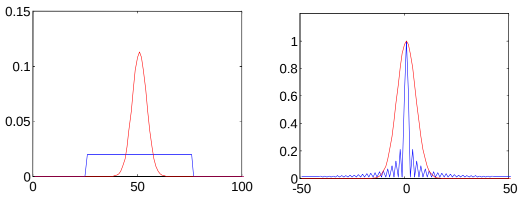


Image Domain   Frequency Domain

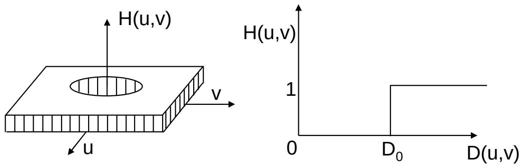
Image Sharpening - High Pass Filter

H(u,v) - Ideal Filter

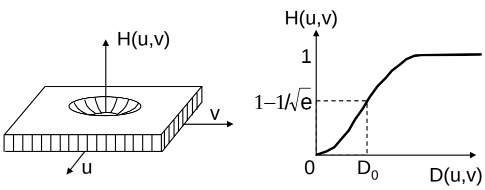
$$H(u,v) = \begin{cases} 0 & D(u,v) \le D_0 \\ 1 & D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \sqrt{u^2 + v^2}$$

D_0 = cut off frequency



High Pass Gaussian Filter



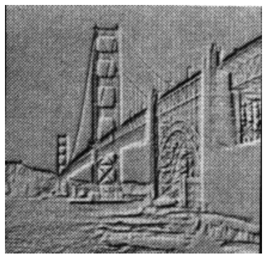
$$H(u,v) = 1 - e^{-D^2(u,v)/(2D^2_0)}$$

$$D(u,v) = \sqrt{u_2 + v_2}$$

High Pass Filtering

Original

High Pass Filtered

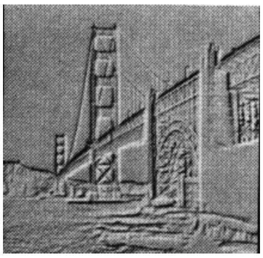


High Frequency Emphasis

Original

+

High Pass Filtered



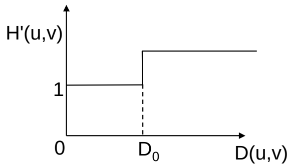
=

High Frequency Emphasis

Emphasize High Frequency.
Maintain Low frequencies and Mean.

$$H'(u,v) = K_0 + H(u,v)$$

(Typically $K_0 = 1$)



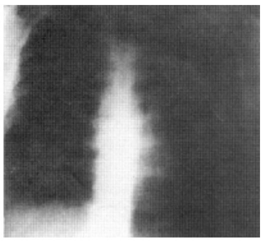
High Frequency Emphasis - Example

Original | High Frequency Emphasis

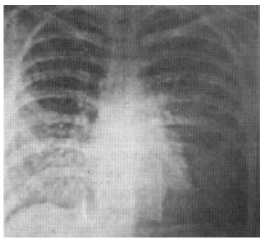
Original | High Frequency Emphasis

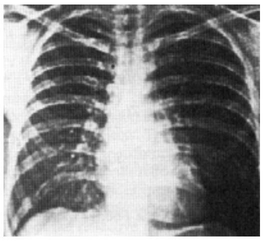
High Pass Filtering - Examples

Original



High pass Emphasis





High Frequency Emphasis
+
Histogram Equalization

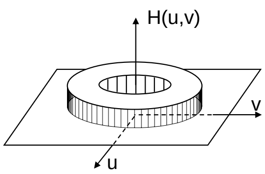
Band Pass Filtering

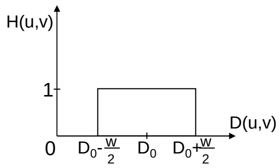
$$H(u,v) = \begin{cases} 0 & D(u,v) \le D_0 - \frac{w}{2} \\ 1 & D_0 - \frac{w}{2} \le D(u,v) \le D_0 + \frac{w}{2} \\ 0 & D(u,v) > D_0 + \frac{w}{2} \end{cases}$$

$D(u,v) = \sqrt{u^2 + v^2}$

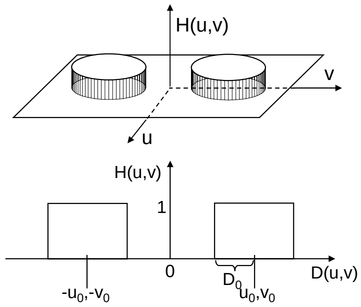
D_0 = cut off frequency

w = band width





Local Frequency Filtering



$$H(u,v) = \begin{cases} 1 & D_1(u,v) \le D_0 \textbf{ or } D_2(u,v) \le D_0 \\ 0 & \textbf{otherwise} \end{cases}$$

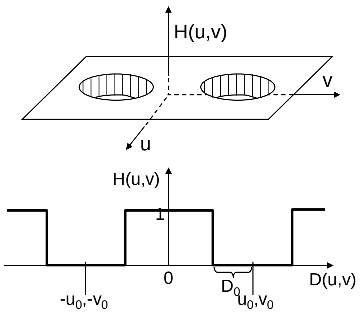
$D_1(u,v) = \sqrt{(u-u_0)^2 + (v-v_0)^2}$

$D_2(u,v) = \sqrt{(u+u_0)^2 + (v+v_0)^2}$

D_0 = local frequency radius

u_0,v_0 = local frequency coordinates

Band Rejection Filtering



$$H(u,v) = \begin{cases} 0 & D_1(u,v) \le D_0 \textbf{ or } D_2(u,v) \le D_0 \\ 1 & \textbf{otherwise} \end{cases}$$

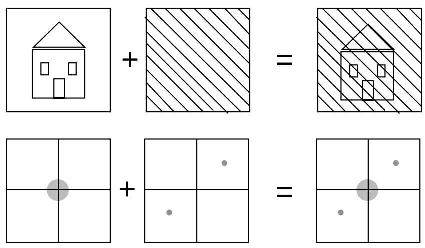
$D_1(u,v) = \sqrt{(u-u_0)^2 + (v-v_0)^2}$

$D_2(u,v) = \sqrt{(u+u_0)^2 + (v+v_0)^2}$

D_0 = local frequency radius

u_0,v_0 = local frequency coordinates

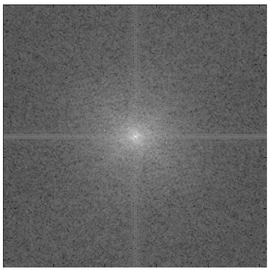
Additive Noise Filtering



Local Reject Filter - Example

Original Noisy image

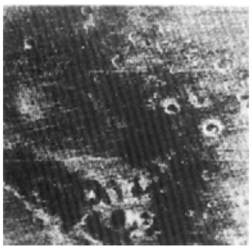
Fourier Spectrum



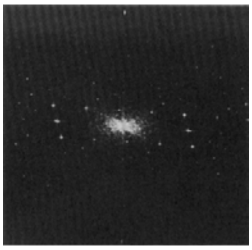
Band Reject Filter

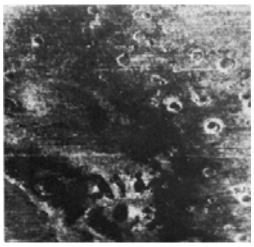
Local Reject Filter - Example

Original Noisy image



Fourier Spectrum





Local Reject Filter

Homomorphic Filtering (multiplicative Noise Filtering)

Noise Model:

Image	i(x,y)
Noise	n(x,y)
Brightness	f(x,y) = i(x,y) • n(x,y)

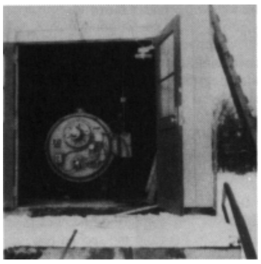
Assumption: noise ≈ low frequencies.

Goal: Clean multiplicative noise
(suppress low frequencies associated with n(x,y))

However:

$$\tilde{F}(i(x,y)\cdot n(x,y)) \neq \tilde{F}(i(x,y))\cdot \tilde{F}(n(x,y))$$

Homomorphic Filtering



Original

Homomorphic Filtering - Example

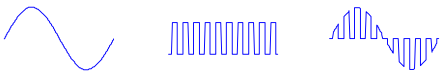
Reflectance Model:

Surface Reflectance	i(x,y)
Illumination	n(x,y)
Brightness	f(x,y) = i(x,y) • n(x,y)

Assumptions:

Illumination changes "slowly" across scene
⟹ Illumination ≈ low frequencies.

Surface reflections change "sharply" across scene
⟹ reflectance ≈ high frequencies.



Illumination   Reflectance   Brightness

Goal: Determine i(x,y)

Perform:

$z(x,y) = \mathbf{log}(f(x,y)) = \mathbf{log}(i(x,y) \cdot n(x,y)) = \mathbf{log}(i(x,y)) + \mathbf{log}(n(x,y))$

↓ $\mathcal{F}$

$Z(u,v) = I(u,v) + N(u,v)$

↓ Apply low attenuating filter H(u,v)

$S(u,v) = H(u,v) \cdot Z(u,v) = H(u,v) \cdot I(u,v) + H(u,v) \cdot N(u,v)$

↓ $\mathcal{F}^{-1}$

$s(x,y) = i'(x,y) + n'(x,y)$

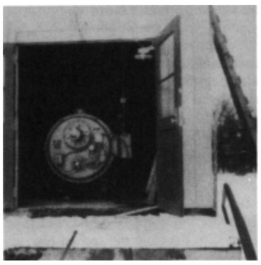
↓

$g(x,y) = \mathbf{exp}(s(x,y)) = \mathbf{exp}(i'(x,y)) \cdot \mathbf{exp}(n'(x,y))$

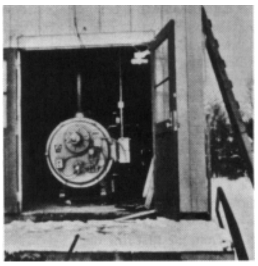
Homomorphic Filtering:

image → log → FFT → H(u,v) → FFT^{-1} → exp → image

Homomorphic Filtering



Original



Filtered

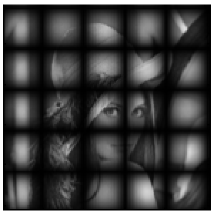
Homomorphic Filtering

Original

Histogram Equalized

Filtered

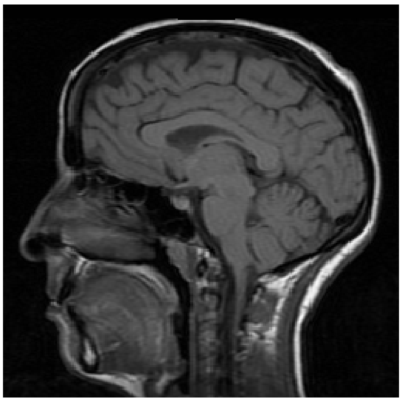
Homomorphic Filtering



Original

Filtered

Computer Tomography using FFT



CT Scanners

- In 1901 W.C. Roentgen won the Nobel Prize (1st in physics) for his discovery of X-rays.

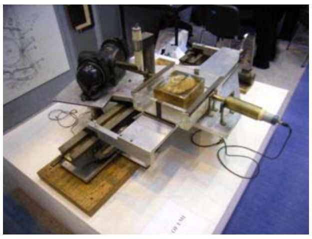
Wilhelm Conrad Röntgen

CT Scanners

- In 1979 G. Hounsfield & A. Cormack, won the Nobel Prize for developing the computer tomography.
- The invention revolutionized medical imaging.

Allan M. Cormack

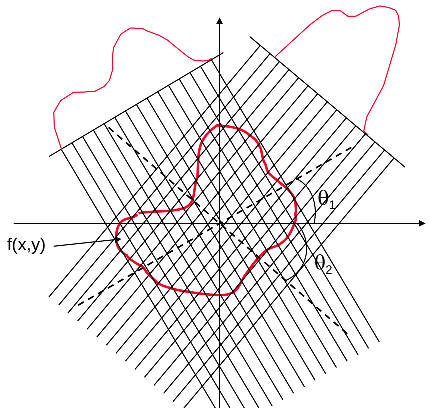
Godfrey N. Hounsfield



1st prototype of CT scanner

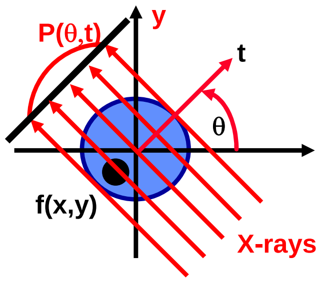
Computerized Tomography

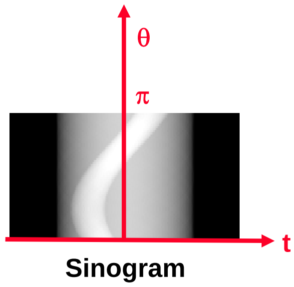
Reconstruction from projections



Projection & Sinogram

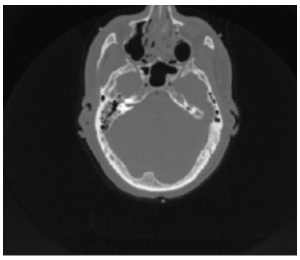
Projection: All ray-sums in a direction

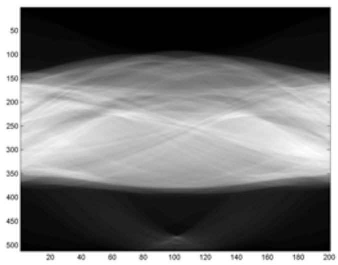




Sinogram:
All projections

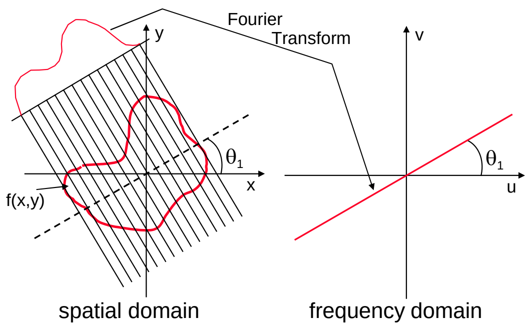
CT Image & Its Sinogram





K. Thomenius & B. Roysam

The Slice Theorem



f(x,y) = object

g(x) = projection of f(x,y) parallel to the y-axis.

$$g(x) = \int f(x,y)\,dy$$

Fourier Transform of f(x,y):

$$F(u,v) = \int\int f(x,y)\, e^{-2\pi i(ux+vy)}\, dxdy$$

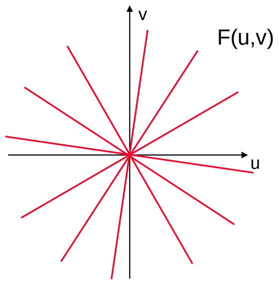
Fourier Transform at v=0 :

$$F(u,0) = \int\int f(x,y)\, e^{-2\pi iux} dxdy$$

$$= \int \left[\int f(x,y)dy \right] e^{-2\pi iux} dx$$

$$= \int g(x)\, e^{-2\pi iux} dx \;=\; G(u)$$

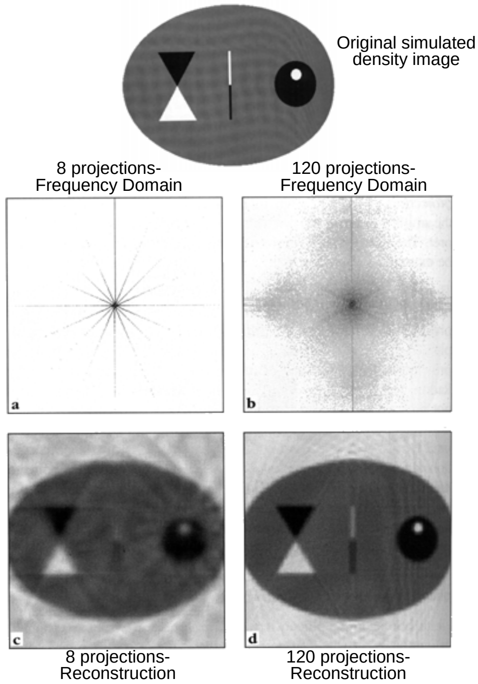
The 1D Fourier Transform of g(x)



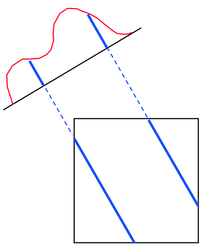
Interpolations Method:

Interpolate (linear, quadratic etc) in the frequency space to obtain all values in F(u,v).
Perform **Inverse Fourier Transform** to obtain the image f(x,y).

Reconstruction from Projections - Example



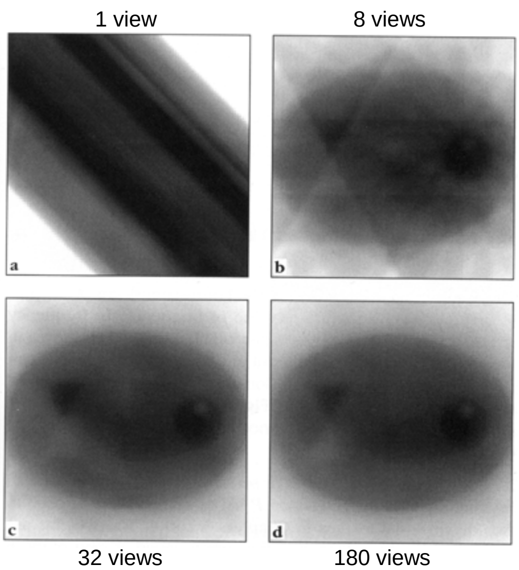
Back Projection Reconstruction



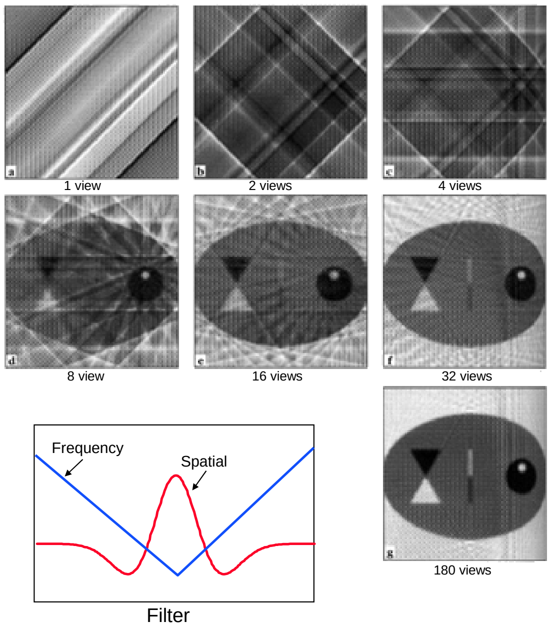
g(x) is Back Projected along the line of projection. The value of g(x) is added to the existing values at each point which were obtained from other back projections.

Note: a blurred version of the original is obtained. (for example consider a single point object is back projected into a blurred delta).

Back Projection Reconstruction - Example



Filtered Back Projection - Example



THE END