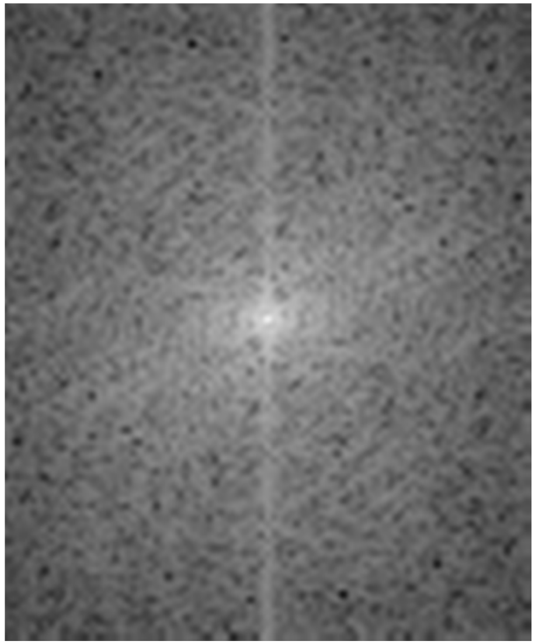
Fourier Transform 1D

The Fourier Transform

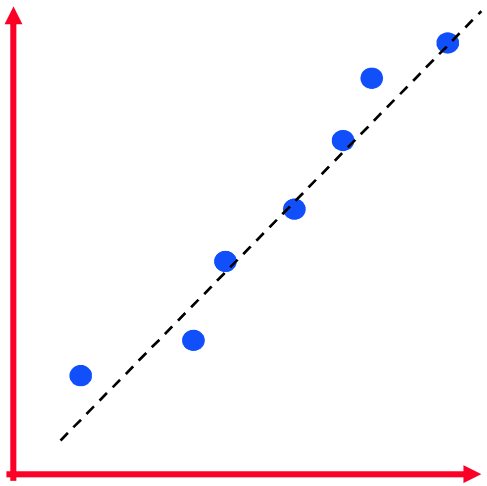


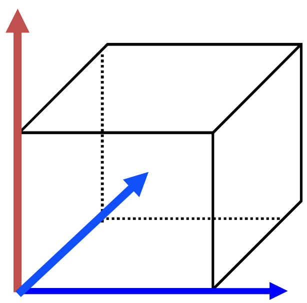
Jean Baptiste Joseph Fourier

1768-1830

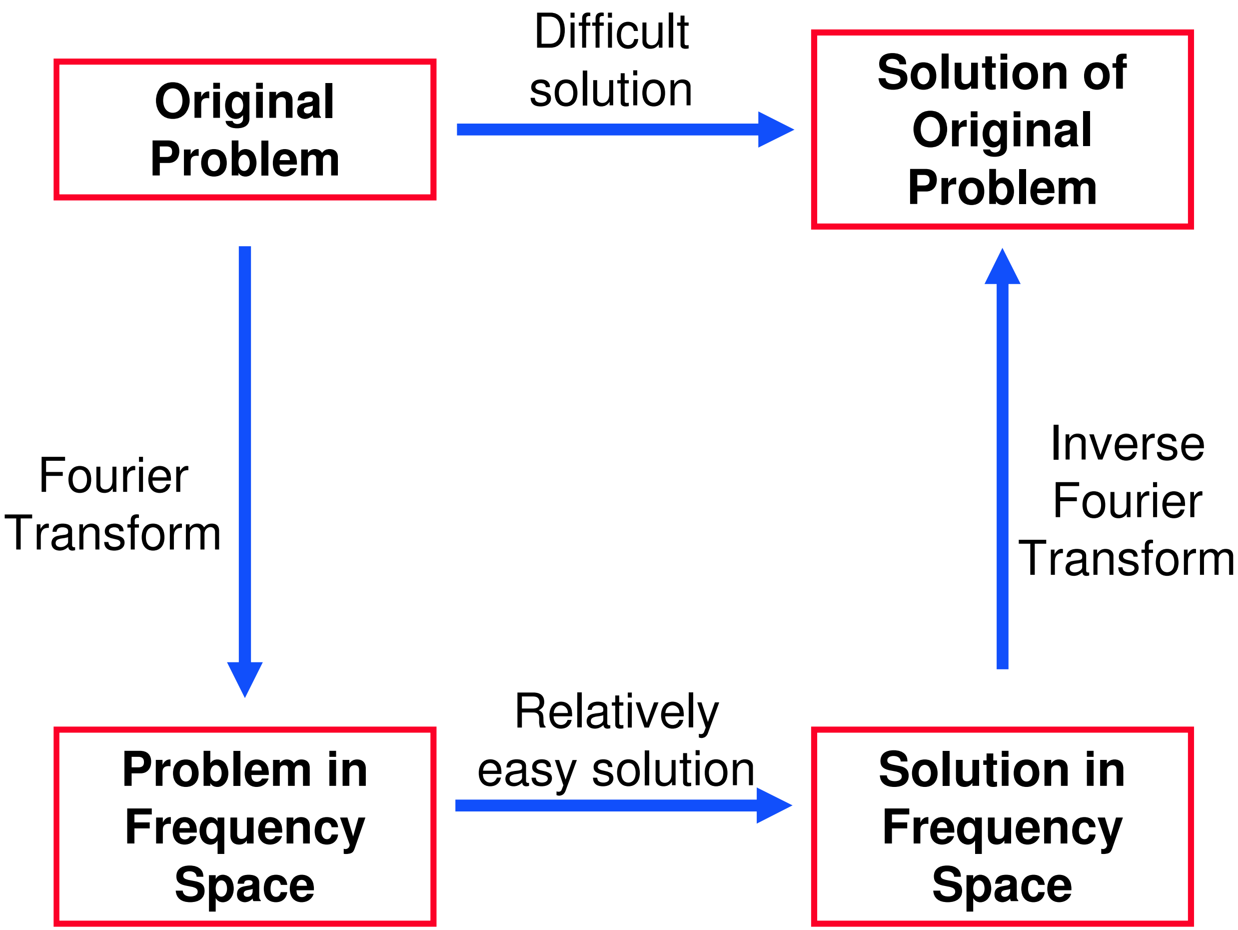
Efficient Data Representation

- Data can be represented in many ways.
- Advantage using an appropriate representation.
- Examples:
  - Noisy points along a line
  - Color space red/green/blue v.s. Hue/Brightness





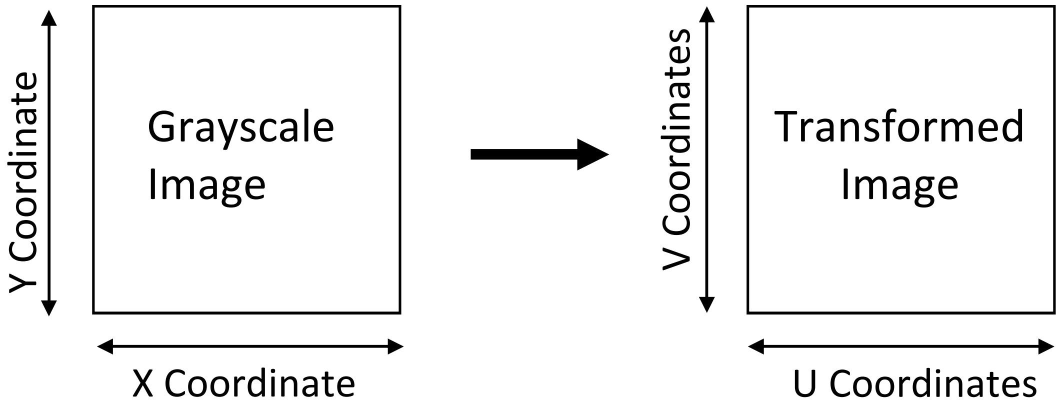
Why do we need representation in the frequency domain?



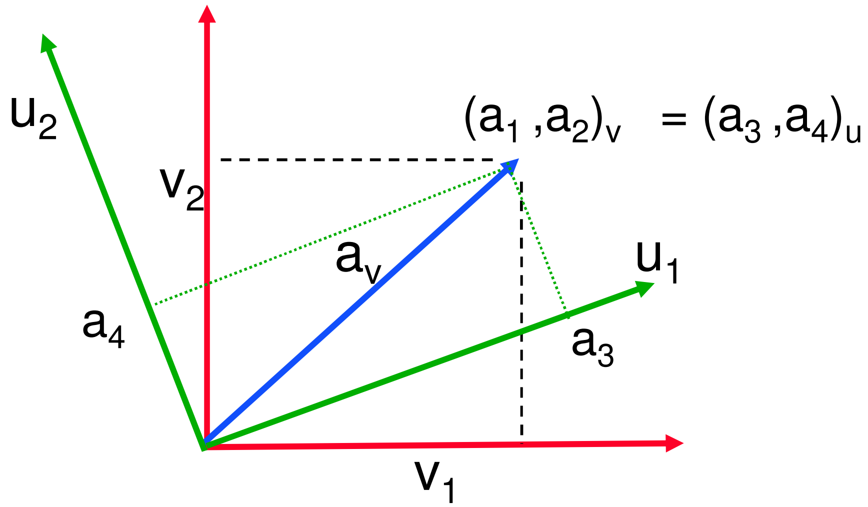
How can we enhance such an image?

Transforms

1. Basis Functions.
2. Method for finding the image given the transform coefficients.
3. Method for finding the transform coefficients given the image.



Change of Basis



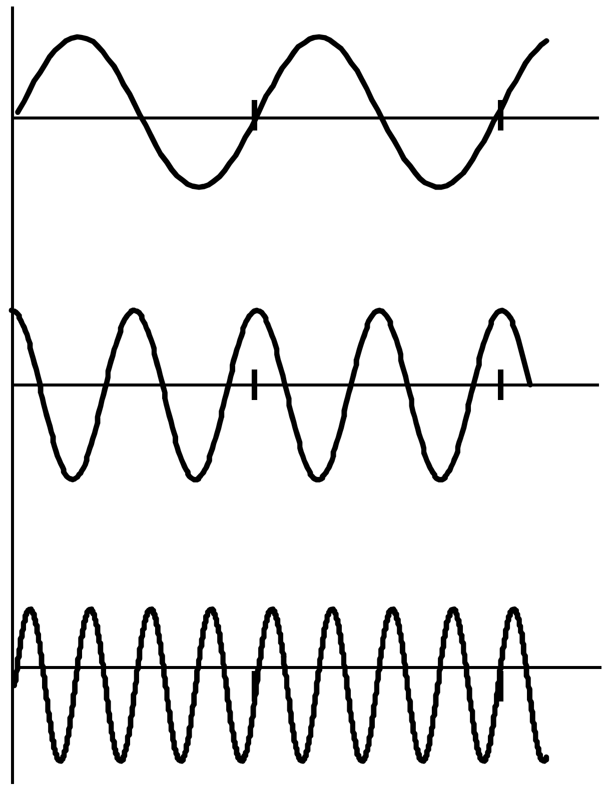
$$a_{iu} = \langle a_v , u_i \rangle$$

$$a_v = \sum_i a_{iu} \, u_i$$

where $\langle c, b \rangle = c^{T*} \, \mathbf{b} = \sum_i c^*(i)\mathbf{b}(i)$

The Fourier basis functions

Basis Functions are sines and cosines



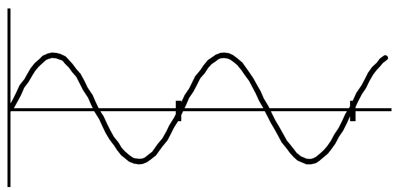
sin(x)

cos(2x)

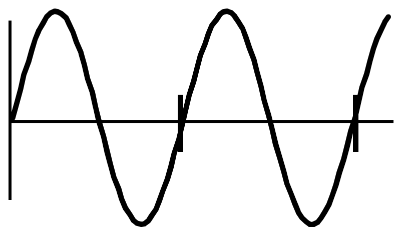
sin(4x)

The Fourier basis functions

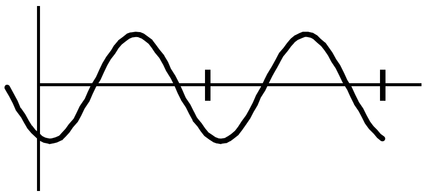
The transform coefficients determine the amplitude and phase:



$a \sin(2x)$

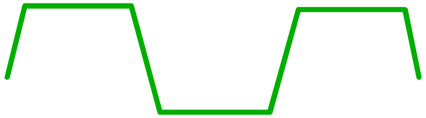


$2a \sin(2x)$



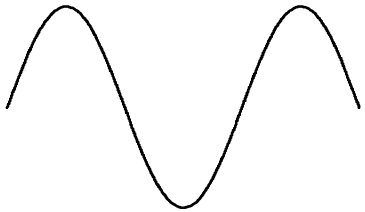
$-a \sin(2x+\phi)$

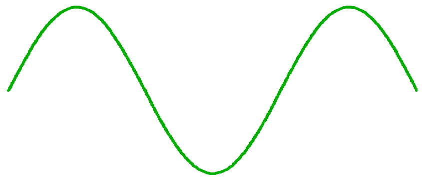
Every function equals a sum of sines and cosines



=

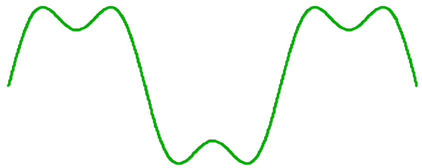
3 sin(x)

A 



+ 1 sin(3x)

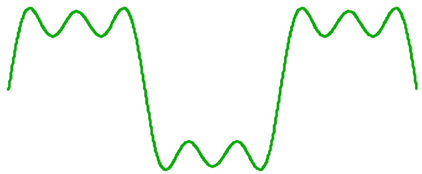
B 



A+B

+ 0.8 sin(5x)

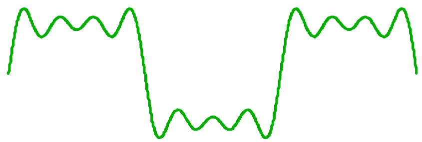
C 



A+B+C

+ 0.4 sin(7x)

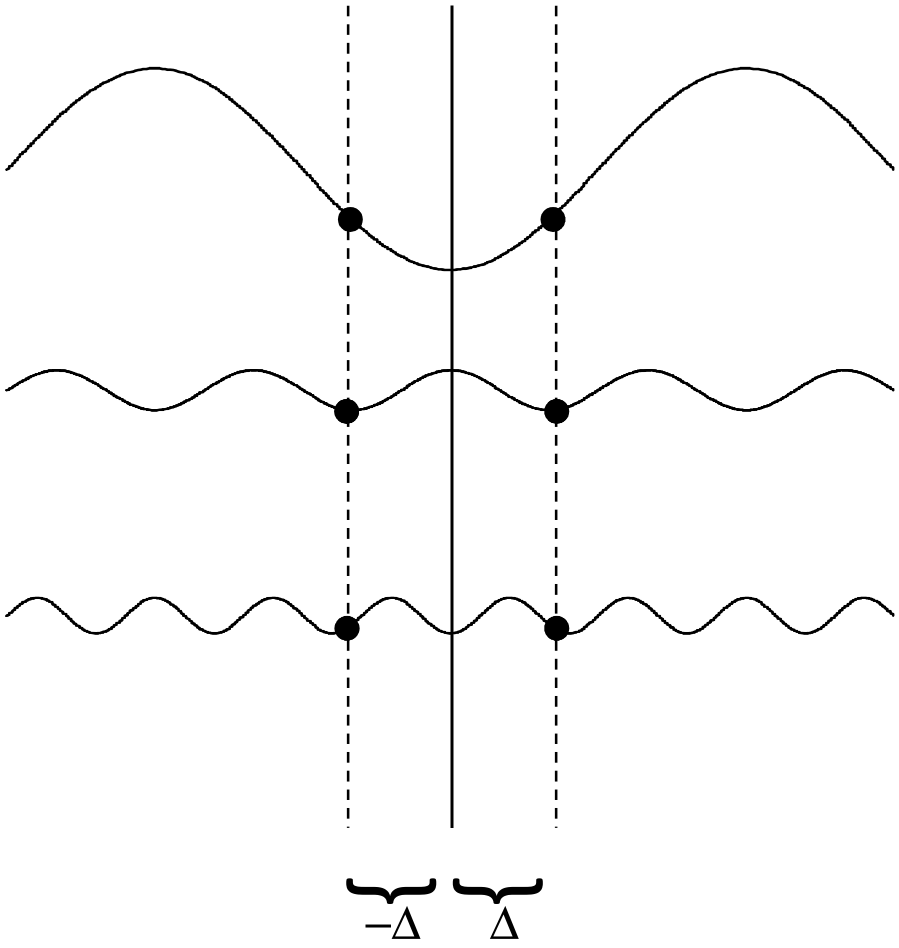
D 



A+B+C+D

Sum of cosines only → symmetric functions

Sum of sines only → antisymmetric functions



Fourier Coefficients

$$f(x) = C_0 + \underbrace{C_1\cos(x) + S_1\sin(x)} + \ldots + \underbrace{C_k\cos(kx) + S_k\sin(kx)} + \ldots$$

Terms are considered in pairs:

$$C_k\cos(kx) + S_k\sin(kx) = R_k \sin(kx + \theta_k)$$

$$\text{where} \quad R_k = \sqrt{C_k^{\,2} + S_k^{\,2}} \quad \text{and} \quad \theta_k = \tan^{-1}\left(\frac{S_k}{C_k}\right)$$

Using Complex Numbers:

$$\cos(kx), \sin(kx) \longrightarrow e^{ikx}$$

$$C_k\cos(kx) + S_k\sin(kx) \longrightarrow \underbrace{R\, e^{i\theta}}_{\text{Amplitude+phase}} e^{ikx}$$

The 1D Continuous Fourier Transform

The Continuous Fourier Transform finds F(ω) given the (cont.) signal f(x):

$$F(\omega) = \int_x f(x)\, e^{-i2\pi\omega x}\, dx$$

$B_\omega(x) = e^{-i2\pi\omega}$ is a complex wave function for each w .

The Inverse Continuous Fourier Transform composes a signal f(x) given F(ω):

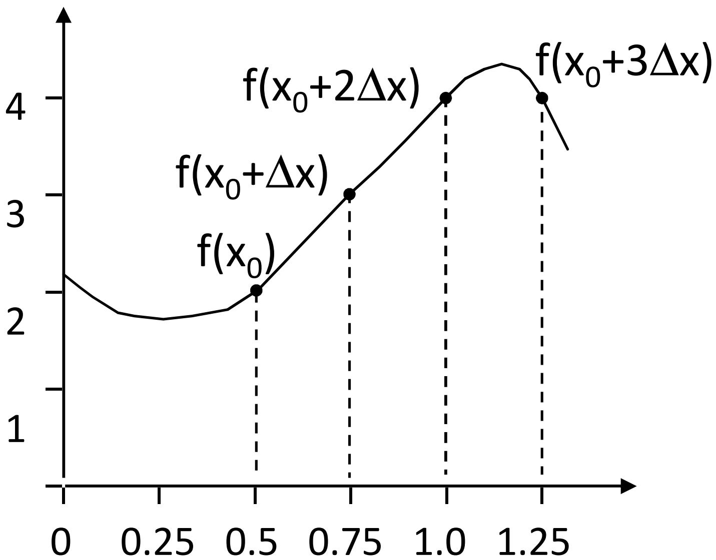
$$f(x) = \int_\omega F(\omega)\, e^{i2\pi\omega x}\, d\omega$$

Continuous vs sampled Signals

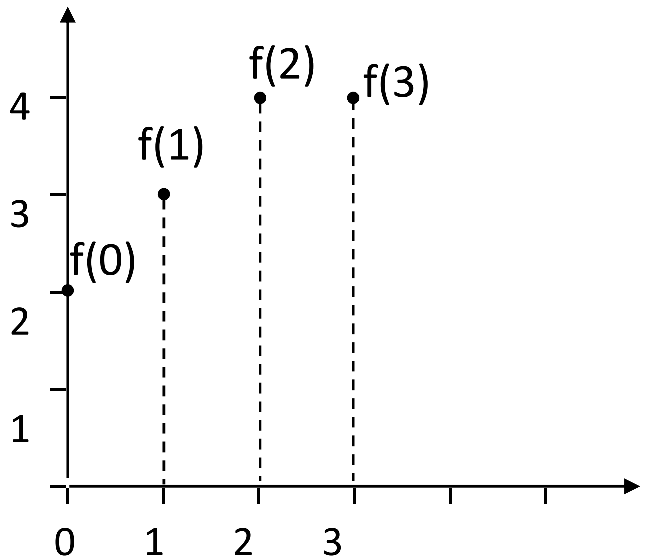
Move from $f(x)$ $(x \in R)$ to $f(x_j)$ $(j \in Z)$ by sampling at equal intervals.

$$f(j) = f(x_0 + j\Delta x) \qquad j = 0, 1, 2, \dots, N-1$$

f(x)



$f(j) = f(x_0 + j\Delta x)$



The discrete Fourier basis functions are

$$b_k(x) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i k x}{N}} \qquad \begin{array}{l} k = 0..N-1 \\ x = 0..N-1 \end{array}$$

For frequency k the Fourier coefficient is:

$$C_k \cos\left(\frac{2\pi kx}{N}\right) + S_k \sin\left(\frac{2\pi kx}{N}\right) \rightarrow \left(R_k e^{i\theta_k}\right) e^{i2\pi kx/N}$$

$$F(k) = R_k e^{i\theta_k}$$

The 1D Discrete Fourier Transform (DFT)

$$F(k) = \langle f(x), b_k(x) \rangle = \sum_{x=0}^{N-1} f(x) e^{\frac{-2\pi i k x}{N}}$$

k = 0, 1, 2, ..., N-1

Matlab: F=fft(f);

The Inverse Discrete Fourier Transform (IDFT) is defined as:

$$f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{\frac{2\pi i k x}{N}}$$

x = 0, 1, 2, ..., N-1

Matlab: F=ifft(f);

Discrete Fourier Transform - Example

$$f(x) = [2 \;\; 3 \;\; 4 \;\; 4]$$

$$F(0) = \sum_{x=0}^{3} f(x) e^{\frac{-2\pi 0 x}{4}} = \sum_{x=0}^{3} f(x) \cdot 1 =$$

$$= (f(0) + f(1) + f(2) + f(3)) = (2+3+4+4) = 13$$

$$F(1) = \sum_{x=0}^{3} f(x) e^{\frac{-2\pi i x}{4}} = \; = [2e^{0}+3e^{-i\pi/2}+4e^{-\pi i}+4e^{-i3\pi/2}] = [-2+i]$$

$$F(2) = \sum_{x=0}^{3} f(x) e^{\frac{-4\pi i x}{4}} = [2e^{0}+3e^{-i\pi}+4e^{-2\pi i}+4e^{-3\pi i}] = [-1-0i] = -1$$

$$F(3) = \sum_{x=0}^{3} f(x) e^{\frac{-6\pi i x}{4}} = [2e^{0}+3e^{-i3\pi/2}+4e^{-3\pi i}+4e^{-i9\pi/2}] = [-2-i]$$

DFT of [2 3 4 4] is [13 (-2+i) -1 (-2-i)]

The Fourier Transform - Summary

- F(k) is the Fourier transform of f(x):

$$\tilde{F}\{f(x)\} = F(k)$$

- f(x) is the inverse Fourier transform of F(k):

$$\tilde{F}^{-1}\{F(k)\} = f(x)$$

- f(x) and F(k) are a Fourier pair.

- f(x) is a representation of the signal in the Spatial Domain and F(k) is a representation in the Frequency Domain.

- The Fourier transform F(k) is a function over the complex numbers:

$$F(k) = R_k e^{i\theta_k}$$

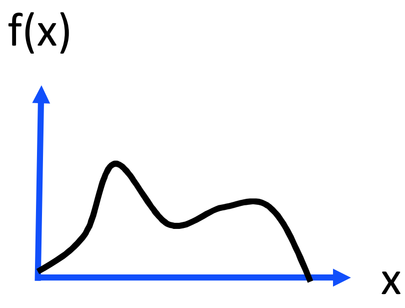
  - R_k tells us how much of frequency k is needed.
  - θ_k tells us the shift of the Sine wave with frequency k.

- Alternatively:

$$F(k) = a_k + ib_k$$

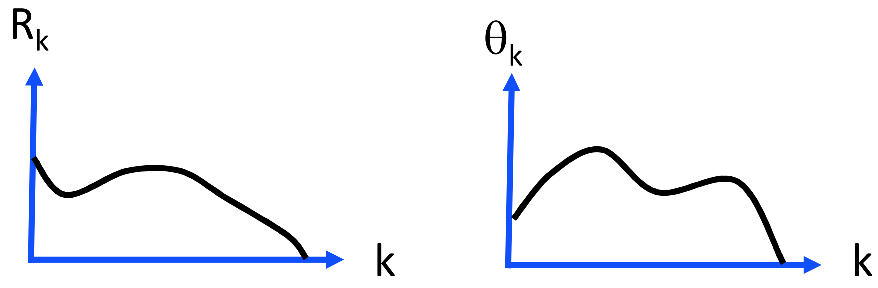
  - a_k tells us how much of cos with frequency k is needed.
  - b_k tells us how much of sin with frequency k is needed.

The Frequency Domain



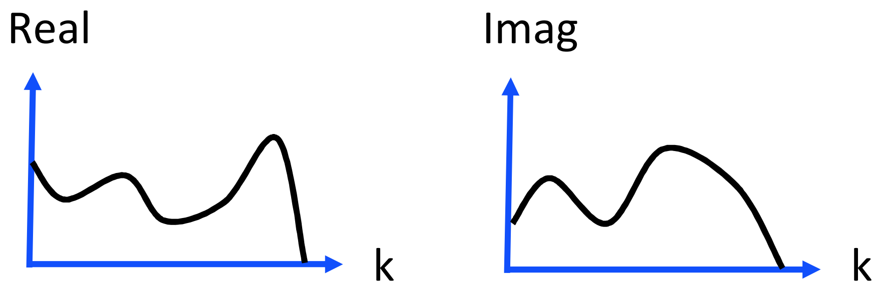
The signal f(x)

$$F(k) = R_k e^{i\theta_k}$$



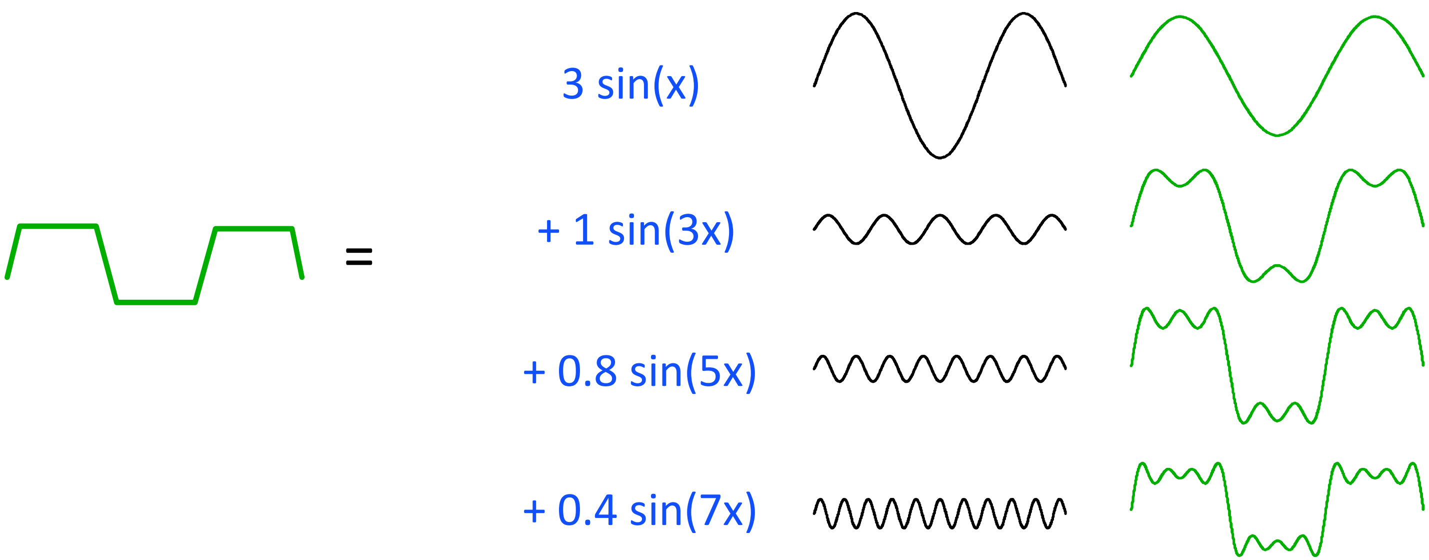
Amplitude (spectrum) and Phase

$$F(k) = a_k + ib_k$$



Real and Imaginary

- R_k - is the amplitude of F(k).
- θ_k - is the phase of F(k).
- $|R_k|^2 = F^*(k)\, F(k)$ - is the power spectrum of F(k) .
- If f(x) has a lot of fine details, $|R_k|^2$ will be high for high k.
- If f(x) is "smooth“, $|R_k|^2$ will be low for high k.



Demo

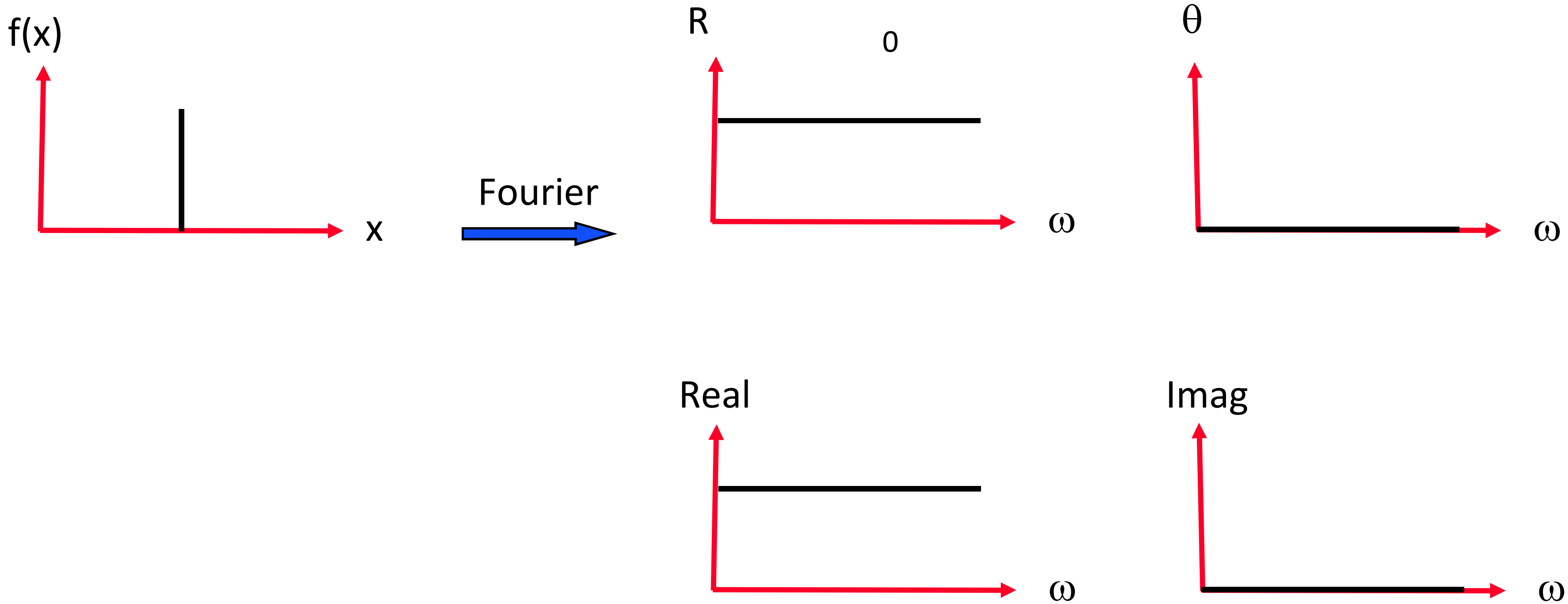
Examples

The Delta Function:

$$f(x) = \delta(x)$$

$$\lim_{x\to 0} \delta(x) = \infty \; ; \quad \int \delta(x)dx = 1$$
$$\int g(x)\delta(x - x_0)dx = g(x_0)$$

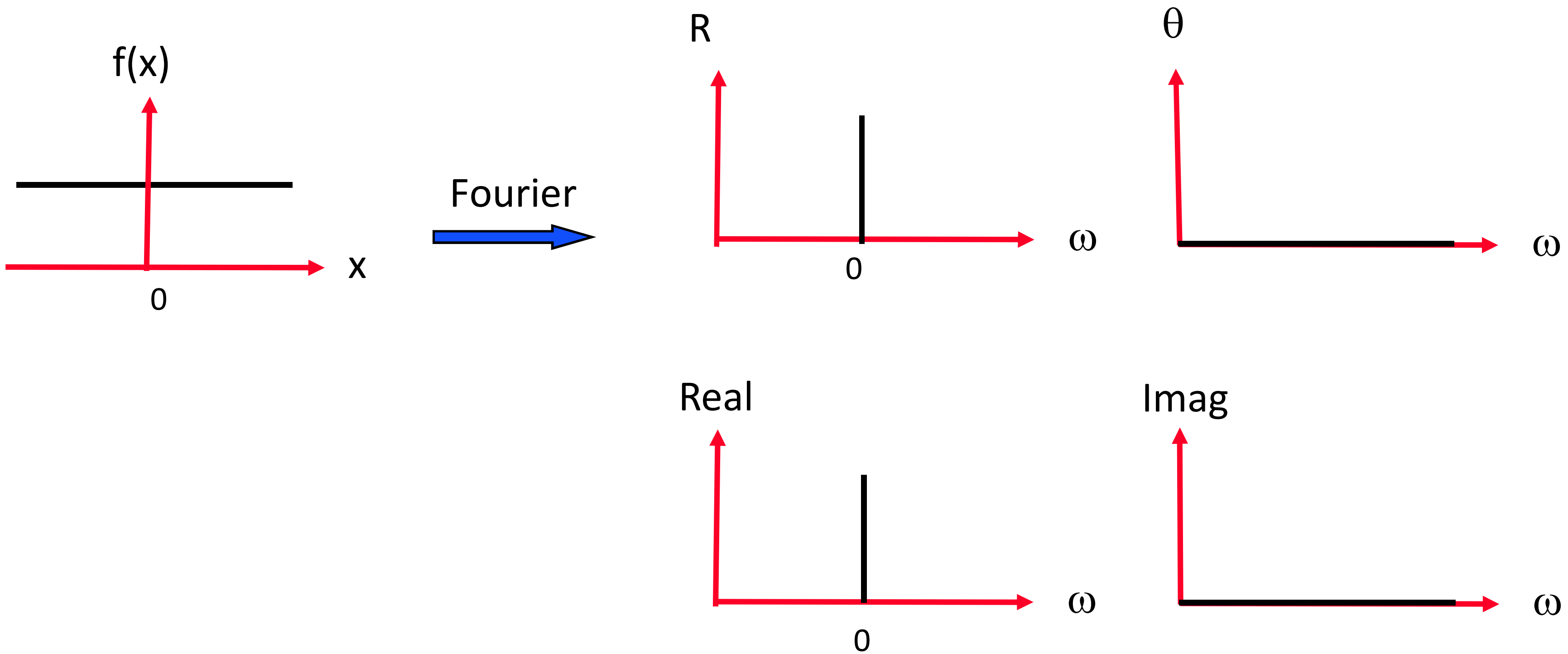
$$F(\omega) = \int_{-\infty}^{\infty} \delta(x) \cdot e^{-i2\pi\omega x}\, dx = 1$$



The Constant Function:

$$f(x) = 1$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i2\pi\omega x}\, dx = \delta(\omega)$$

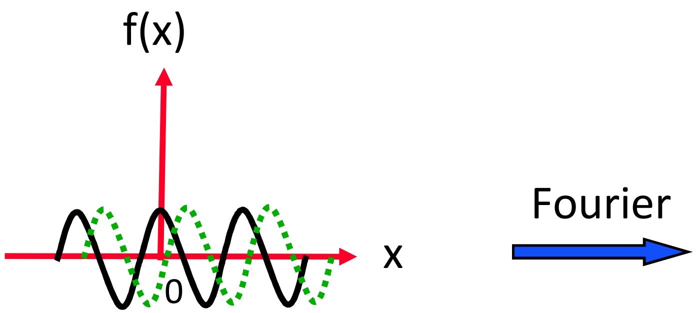


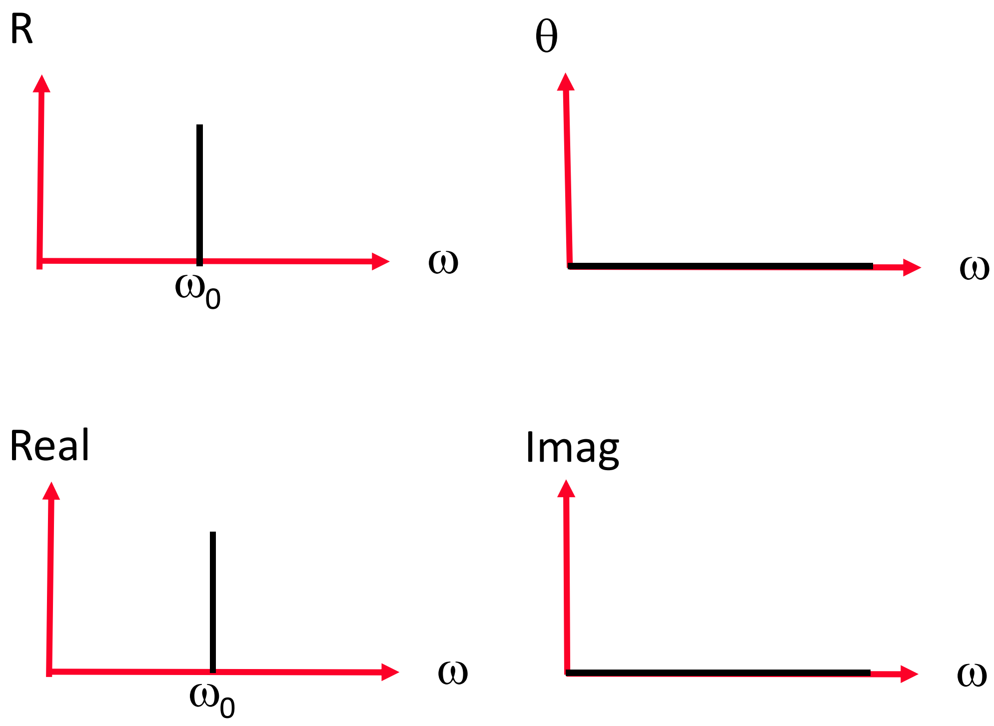
A Basis Function:

$$f(x) = e^{i2\pi\omega_0 x}$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{i2\pi\omega_0 x} e^{-i2\pi\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi(\omega-\omega_0)x}\, dx = \delta(\omega - \omega_0)$$

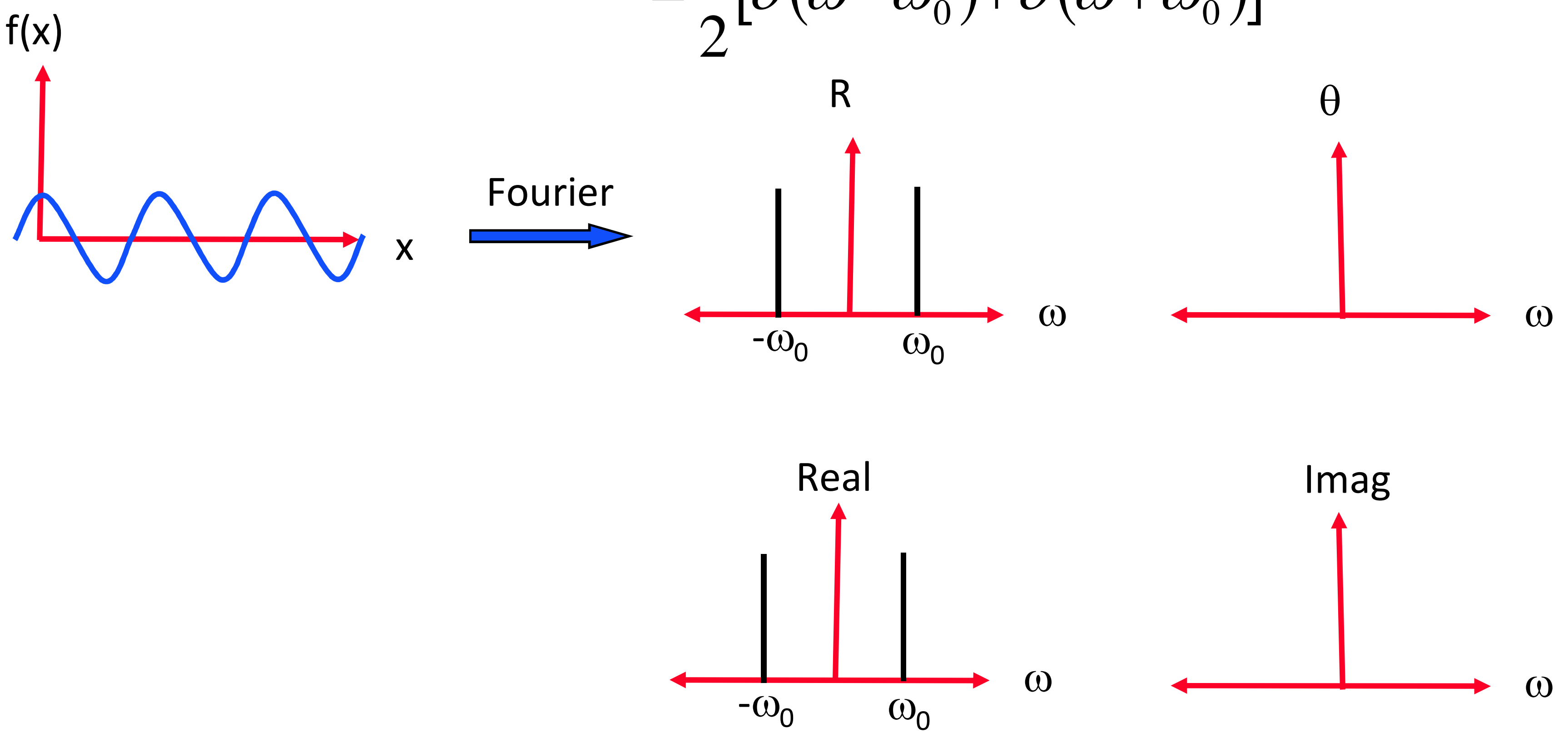




The CosineFunction: $f(x)=\cos(2\pi\omega_0 x)$

$$F(\omega)=\int_{-\infty}^{\infty}\frac{1}{2}\left(e^{i2\pi\omega_0 x}+e^{-i2\pi\omega_0 x}\right)\cdot e^{-i2\pi\omega x}\,dx=$$

$$=\frac{1}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

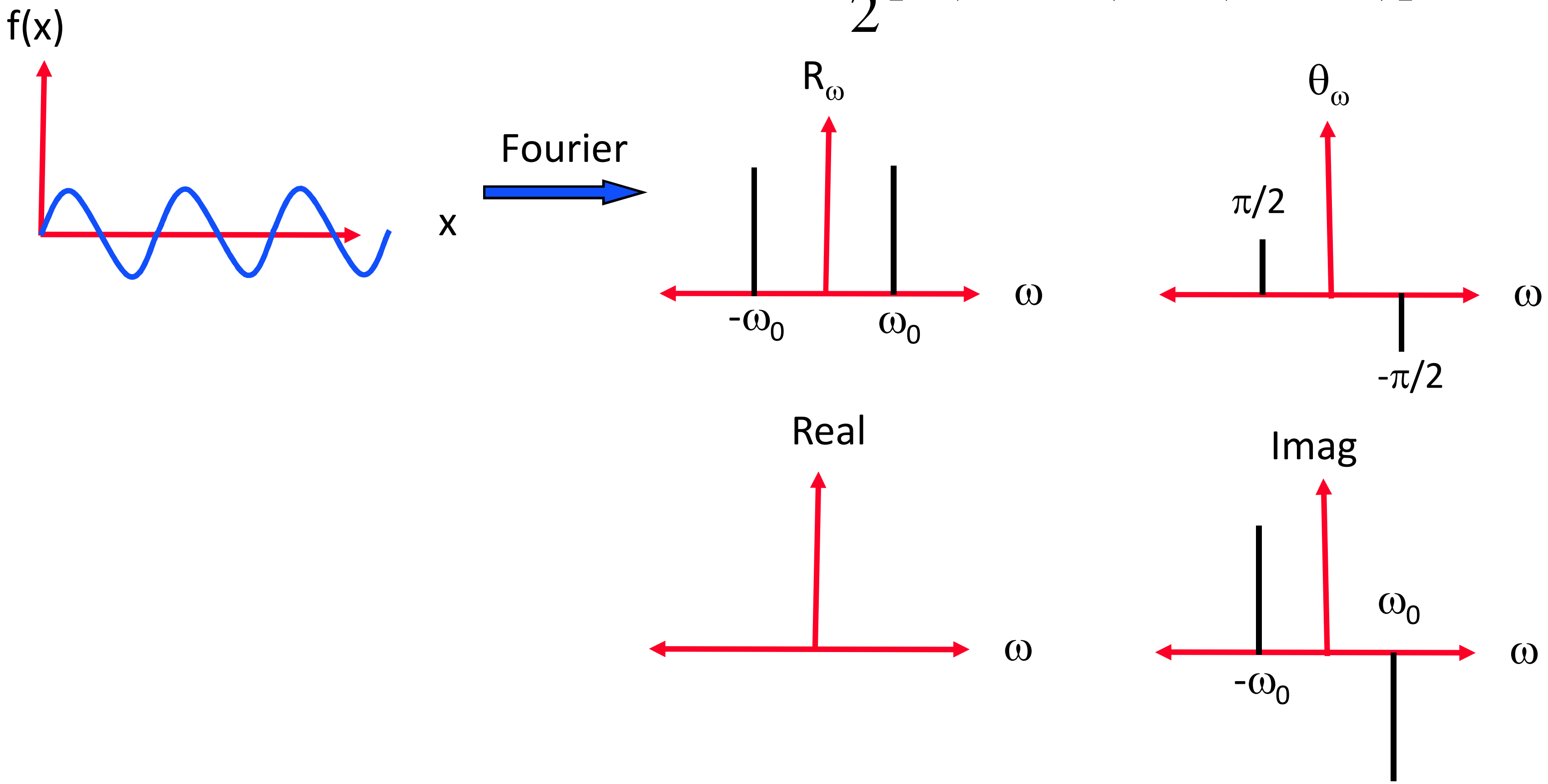


The Sine Function:

$$f(x)=\sin\left(2\pi\omega_0 x\right)$$

$$F(\omega)=\int_{-\infty}^{\infty}\frac{i}{2}\left(e^{-i2\pi\omega_0 x}-e^{i2\pi\omega_0 x}\right)\cdot e^{-i2\pi\omega x}\,dx=$$

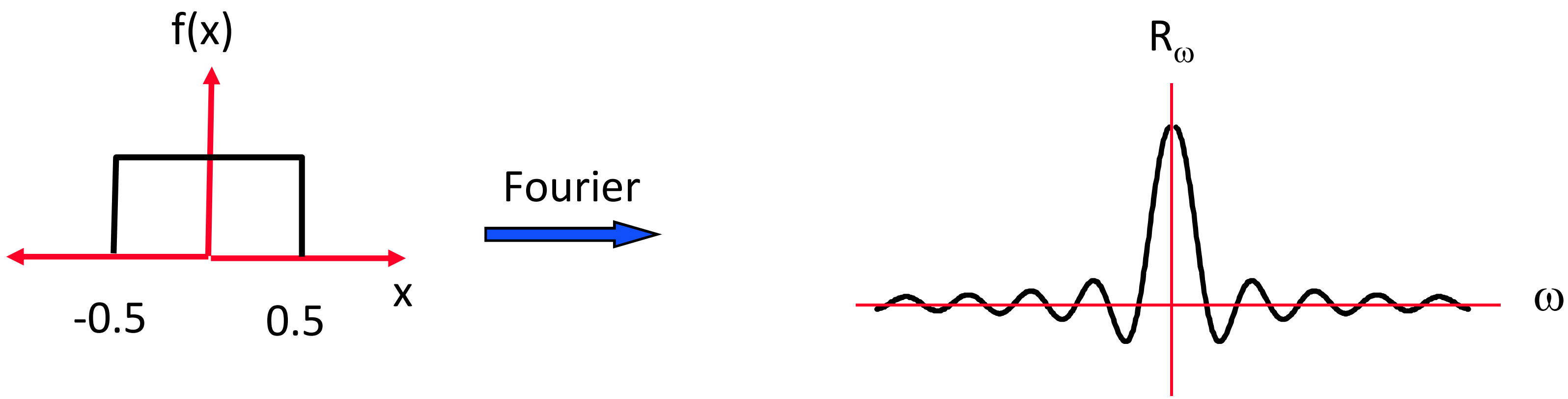
$$=\frac{i}{2}\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$$



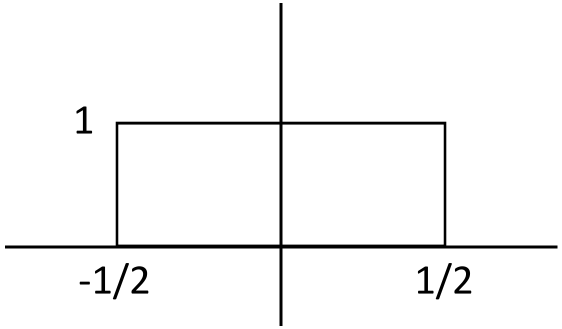
The Window Function (rect):

$$\mathrm{rect}_{1/2}(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$F(\omega) = \int_{-0.5}^{0.5} e^{-i2\pi\omega x}\,dx = \frac{\sin(\pi\omega)}{\pi\omega} = \mathrm{sinc}(\pi\omega)$$



Proof:



f(x) = $rect_{1/2}(x)$ = $\begin{cases} 1 & |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$

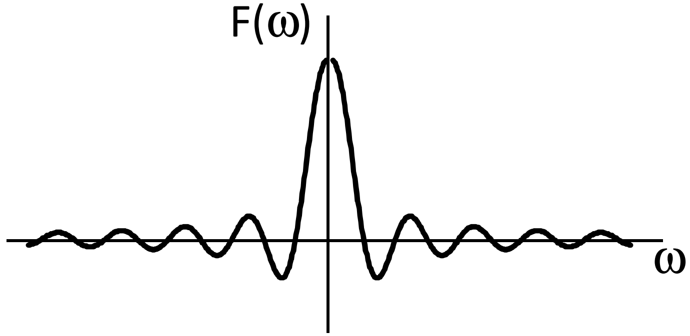
$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} dx = \int_{-1/2}^{1/2} e^{-2\pi i \omega x} dx$$

$$= \frac{1}{-2\pi i \omega} \left[e^{-2\pi i \omega x} \right]_{-1/2}^{1/2}$$

$$= \frac{1}{-2\pi i \omega} \left[e^{-\pi i \omega} - e^{\pi i \omega} \right]$$

$$= \frac{1}{-2\pi i \omega} \left[\cos(\pi\omega) - i\sin(\pi\omega) - \cos(\pi\omega) - i\sin(\pi\omega) \right]$$

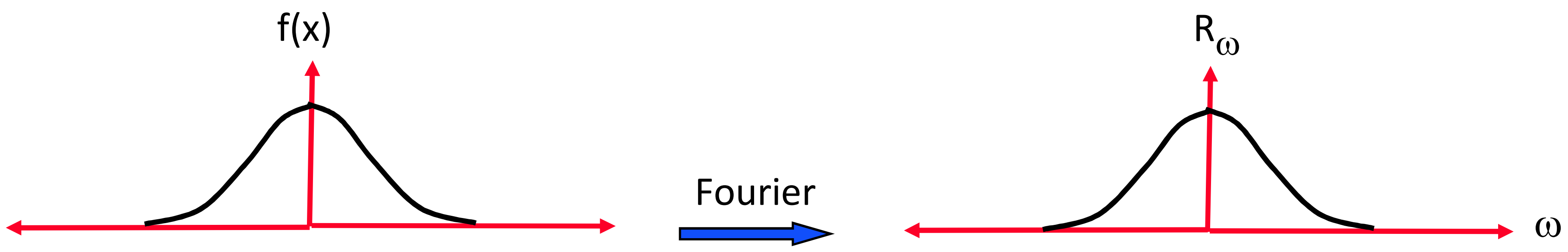
$$= \frac{\sin(\pi\omega)}{\pi\omega} = \boxed{\text{SINC}(\omega)}$$



The Gaussian Function:

$$f(x) = e^{-\pi x^2}$$

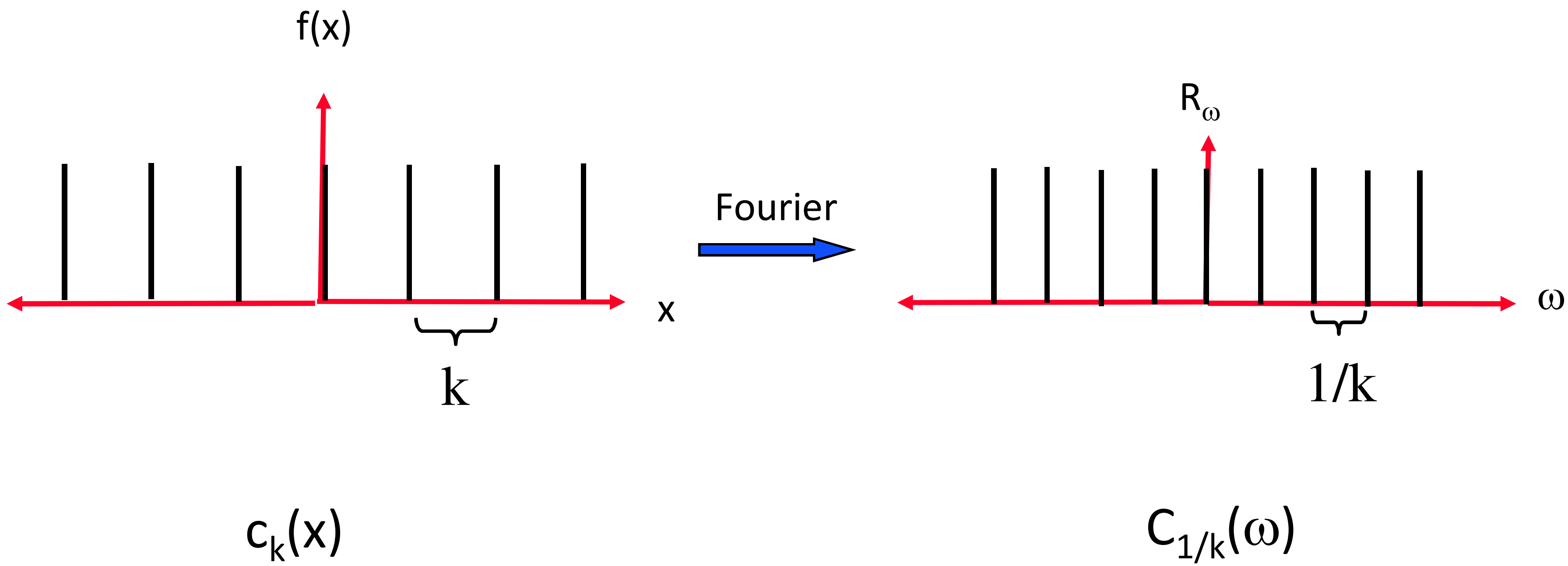
$$F(\omega) = e^{-\pi \omega^2}$$



The Comb Function:

$$c_k(x) = \delta(x \mod k)$$

$$\tilde{F}\{c_k\} = \delta\left(\omega \mod \frac{1}{k}\right) = C_{1/k}(\omega)$$



Properties of The Fourier Transform

- Linearity:

$$\tilde{F}[\alpha f] = \alpha \tilde{F}[f]$$

- Distributive (additivity):

$$\tilde{F}[f_1 + f_2] = \tilde{F}[f_1] + \tilde{F}[f_2]$$

- DC (average):

$$F(0) = \int f(x) e^{0} \, dx$$

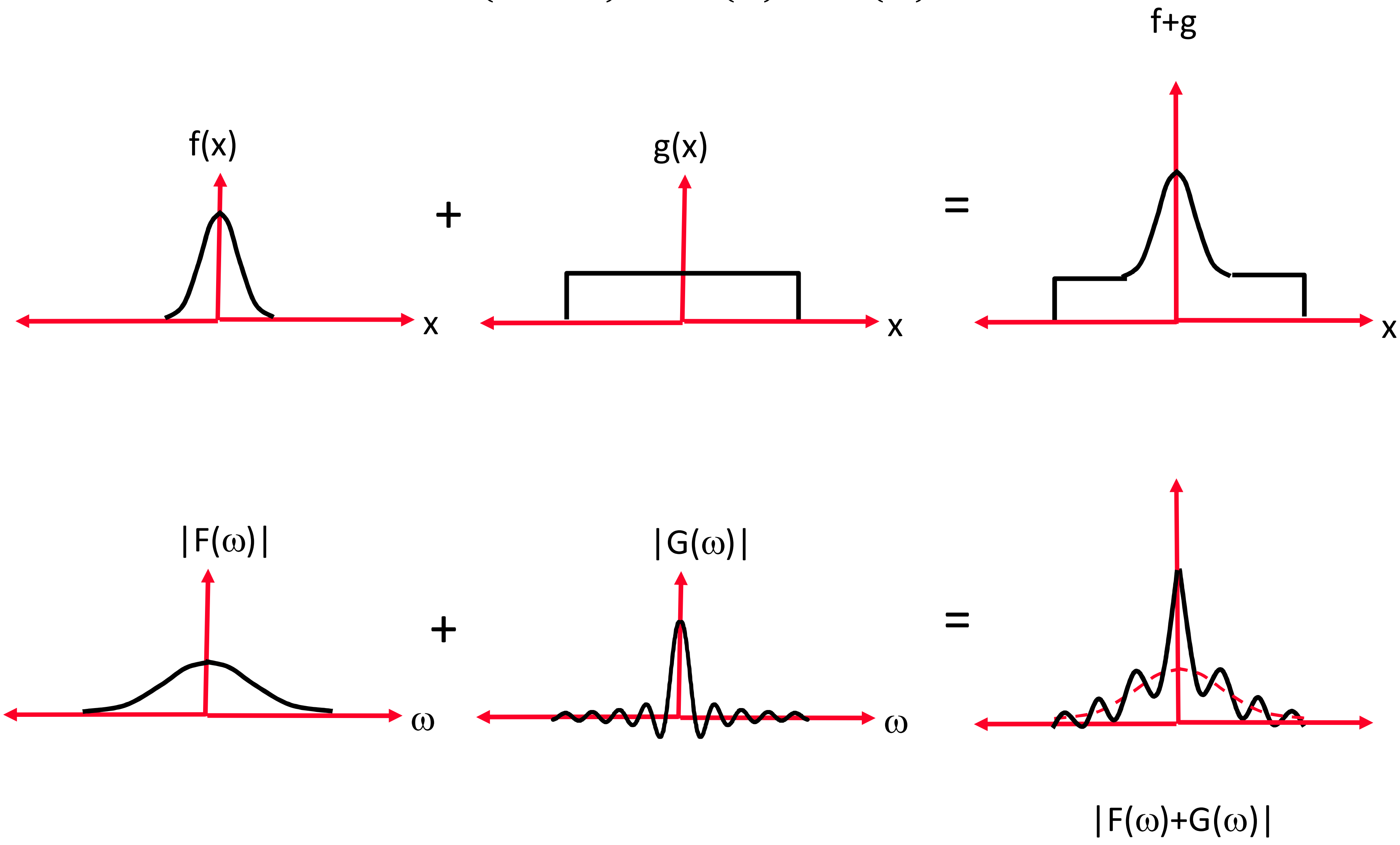
- Symmetric:

  If f(x) is real then,

$$F(\omega) = F^{*}(-\omega) \quad thus \quad |F(\omega)| = |F(-\omega)|$$

Distributive:

$$\tilde{F}\{f + g\} = \tilde{F}\{f\} + \tilde{F}\{g\}$$



Transformations

- Translation:

$$\tilde{F}[f(x-x_0)] = F(\omega)e^{-2\pi i\omega x_0}$$

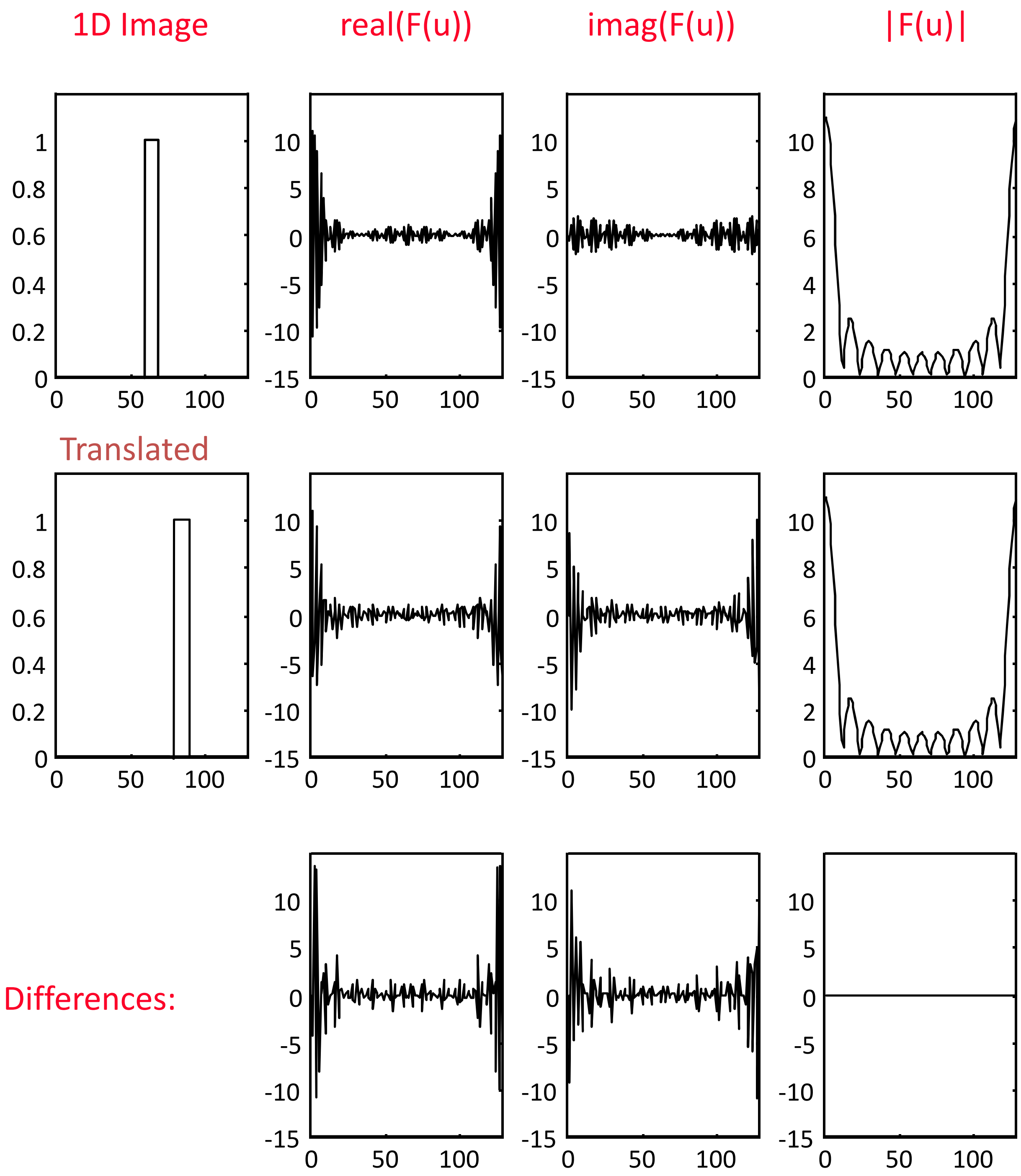
The Fourier Spectrum remains unchanged under translation:

$$|F(\omega)| = \left|F(\omega)e^{-2\pi i\omega x_0}\right|$$

- Scaling:

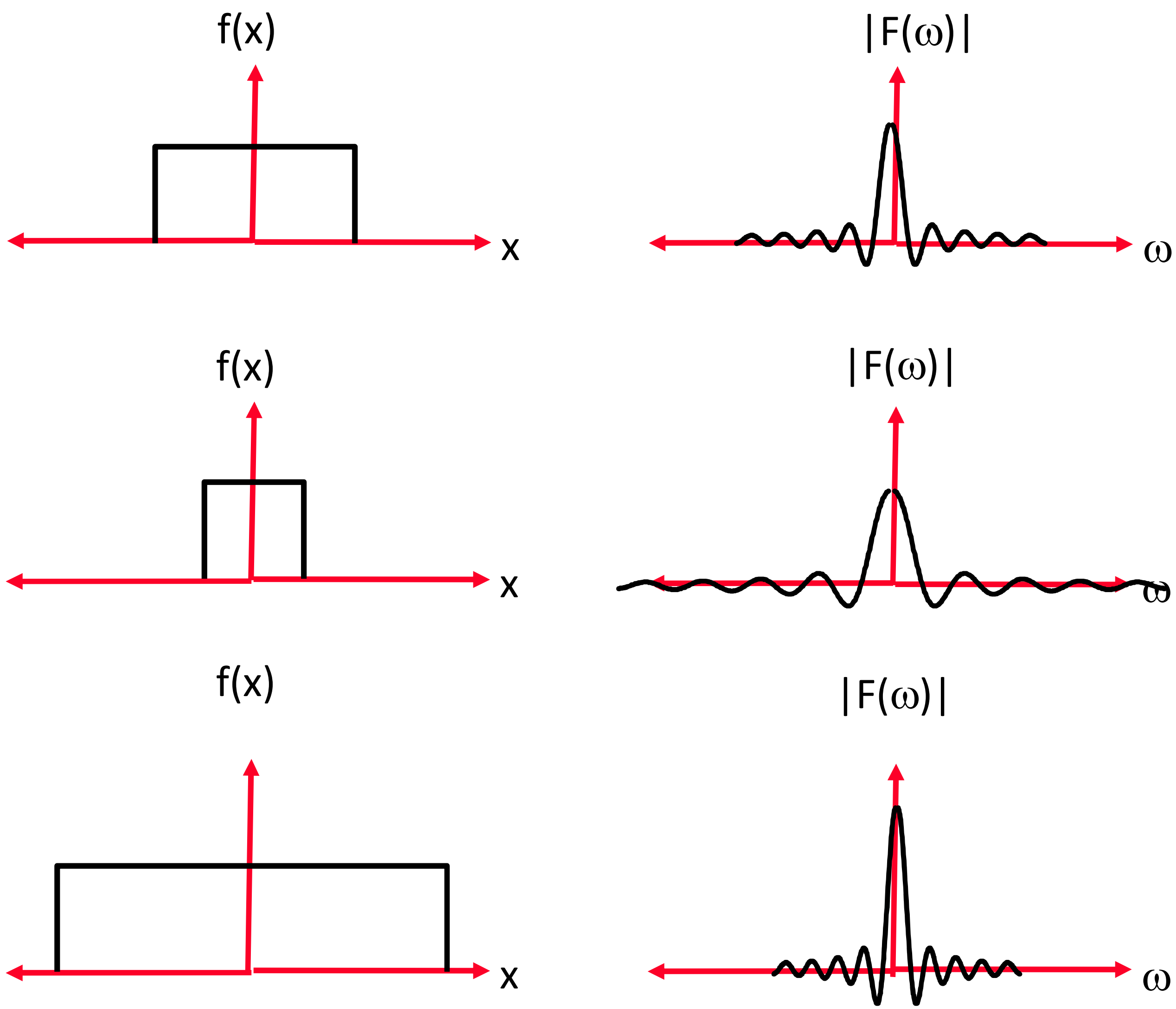
$$\tilde{F}[f(ax)] = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

Example Translation:

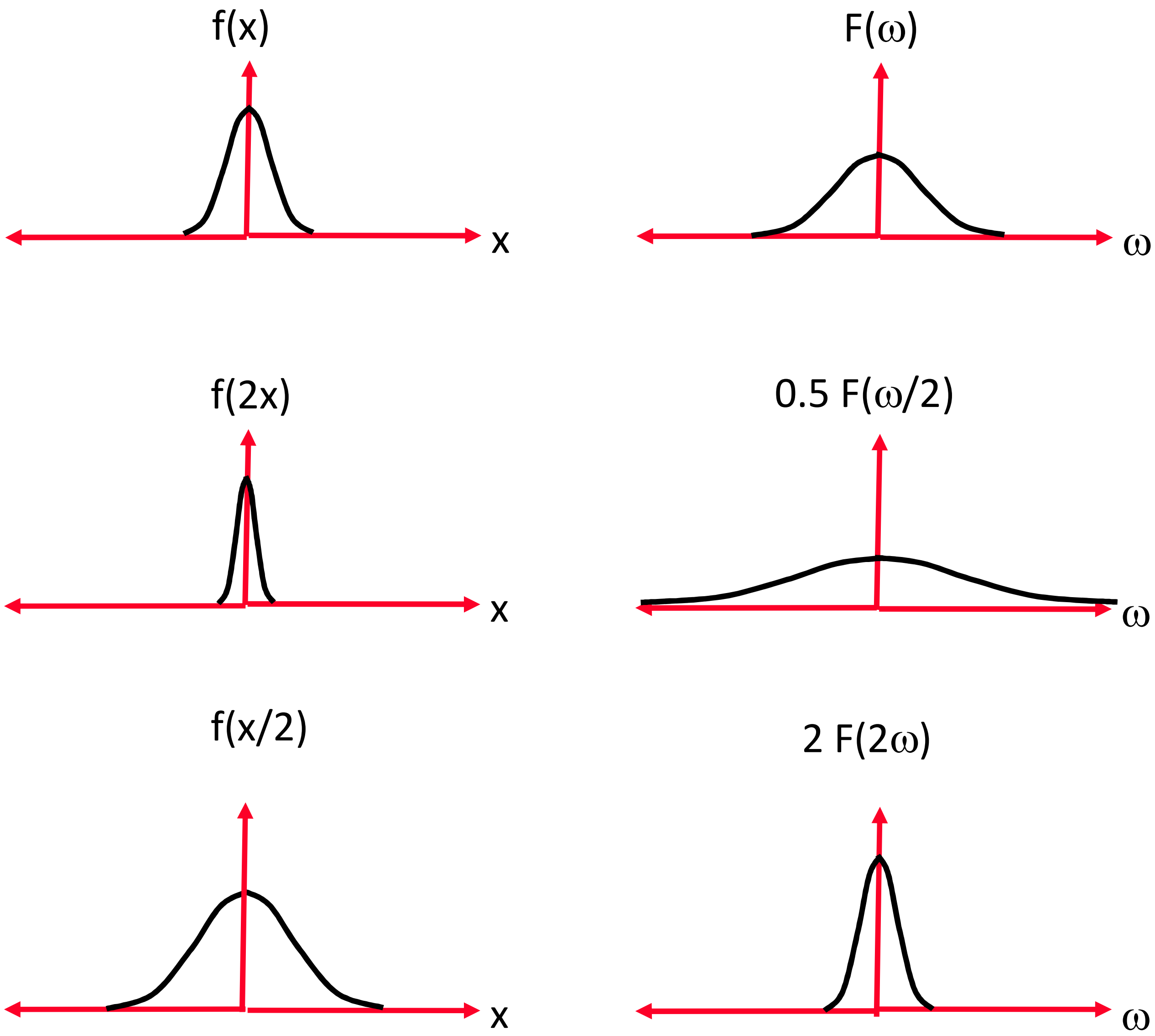


Change of Scale:

$$\text{if } \tilde{F}\{f(x)\} = F(\omega) \text{ then } \tilde{F}\{f(ax)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$



Change of Scale:



End