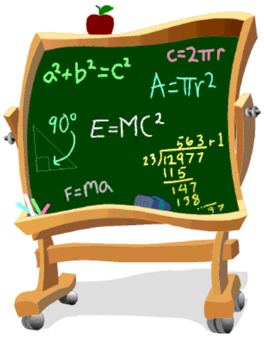
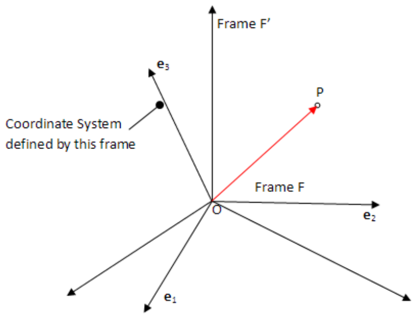
Image Processing

Math Review Towards Fourier Transform

- Linear Spaces
- Change of Basis
- Cosines and Sines
- Complex Numbers



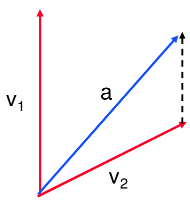
Part I: Vector Spaces and Basis Vectors



Basis Vectors

- A given vector value is represented with respect to a *coordinate system*.
- A coordinate system is defined by a set of linearly independent vectors forming the system *basis*.
- Any vector value is represented as a linear sum of the basis vectors.

$a = 0.5 * v_1 + 1 * v_2 = (0.5 , 1)_v$



- v1, v2 are basis vectors
- The representation of a with respect to this basis is (0.5,1)

Orthonormal Basis Vectors

- If the basis vectors are mutually orthogonal and are unit vectors, the vectors form an *orthonormal basis*.
- Example:

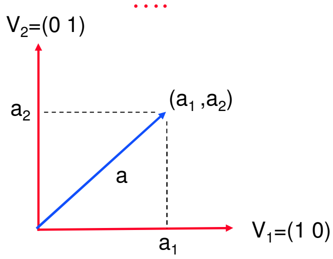
The *standard basis* is orthonormal:

$V_1=(1\ 0\ 0\ 0\ \ldots)$

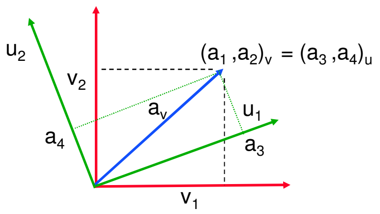
$V_2=(0\ 1\ 0\ 0\ \ldots)$

$V_3=(0\ 0\ 1\ 0\ \ldots)$

....



Change of Basis



Given a vector $\mathbf{a_v}$, represented in orthonormal basis $\{v_i\}$, what is the representation of $\mathbf{a_v}$ in a different orthonormal basis $\{u_i\}$?

$$a_u(i) = \langle a_v, u_i \rangle$$

$$a_v = \sum_i a_u(i)\, u_i$$

where $\langle c, b \rangle = c^T \mathbf{b} = \sum_i c(i)\mathbf{b}(i)$

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Signal (Image) Transforms

1. Basis Functions.
2. Method for finding the transform coefficients given a signal.
3. Method for finding the signal given the transform coefficients.

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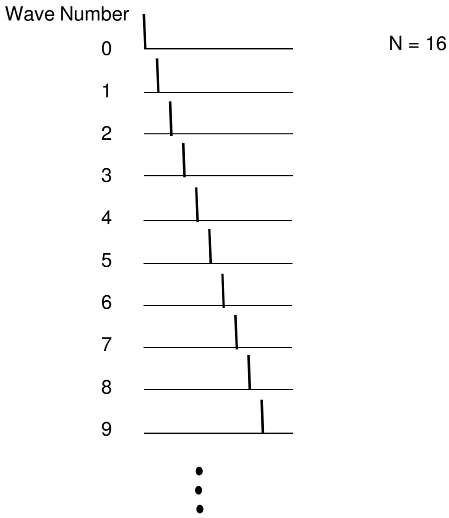
The Orthonormal Standard Basis



Standard Basis Functions

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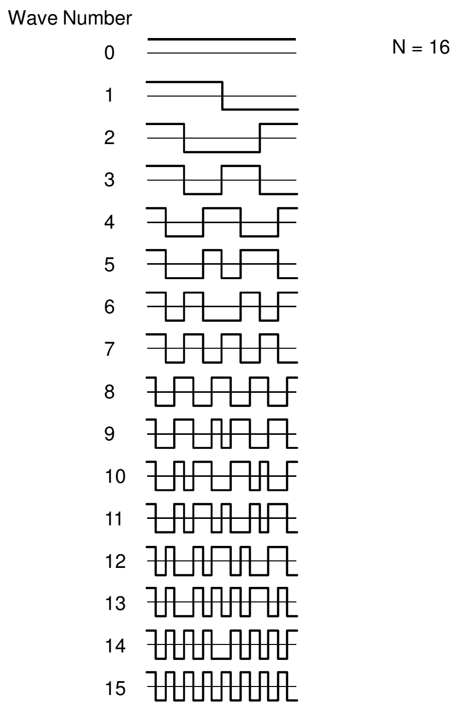
The Orthonormal Hadamard Basis



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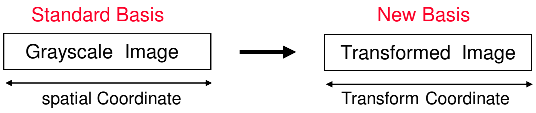
Hadamard Transform



Standard Basis:

$[\,2\ 1\ 6\ 1\,]_{standard} =$

$2\,[\,1\ 0\ 0\ 0\,] + 1\,[\,0\ 1\ 0\ 0\,] + 6\,[\,0\ 0\ 1\ 0\,] + 1\,[\,0\ 0\ 0\ 1\,]$

Hadamard Transform:

$[\,2\ 1\ 6\ 1\,]_{standard} =$

$= 5\,[\,1\ 1\ 1\ 1\,]/2 + -2\,[\,1\ 1\ -1\ -1\,]/2 +$

$-2\,[\,1\ -1\ -1\ 1\,]/2 + 3\,[\,1\ -1\ 1\ -1\,]/2$

$\equiv [\,5\ \ -2\ \ -2\ \ 3\,]_{Hadamard}$

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Finding the transform coefficients

Signal: $X = [\,2\ 1\ 6\ 1\,]_{standard}$

Hadamard Basis:

$T_0 = [\,1\ \ 1\ \ 1\ \ 1\,]/2$
$T_1 = [\,1\ \ 1\ \ -1\ \ -1\,]/2$
$T_2 = [\,1\ \ -1\ \ -1\ \ 1\,]/2$
$T_3 = [\,1\ \ -1\ \ 1\ \ -1\,]/2$

Hadamard Coefficients:

$a_0 = \langle X,T_0\rangle = \langle [\,2\ 1\ 6\ 1\,], [\,1\ 1\ 1\ 1\,]\rangle/2 = 5$
$a_1 = \langle X,T_1\rangle = \langle [\,2\ 1\ 6\ 1\,], [\,1\ 1\ -1\ -1\,]\rangle/2 = -2$
$a_2 = \langle X,T_2\rangle = \langle [\,2\ 1\ 6\ 1\,], [\,1\ -1\ -1\ 1\,]\rangle/2 = -2$
$a_3 = \langle X,T_3\rangle = \langle [\,2\ 1\ 6\ 1\,], [\,1\ -1\ 1\ -1\,]\rangle/2 = 3$

Signal: $[\,2\ 1\ 6\ 1\,]_{Standard} \equiv [\,5\ \ -2\ \ -2\ \ 3\,]_{Hadamard}$

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Reconstructing the Image from the transform coefficients

Transform: $Y = [\,5\ -2\ -2\ \ 3\,]_{Hadamard}$

Hadamard Basis:

$T_0 = [\,1\ \ 1\ \ 1\ \ 1\,]/2$
$T_1 = [\,1\ \ 1\ -1\ -1\,]/2$
$T_2 = [\,1\ -1\ \ -1\ 1\,]/2$
$T_3 = [\,1\ -1\ \ 1\ \ -1\,]/2$

Reconstruction;

$\Sigma\ Y(i), T_i =$

$5\,[\,1\ 1\ 1\ 1\,]/2 + -2\,[\,1\ 1\ -1\ -1\,]/2 +$

$-2\,[\,1\ -1\ -1\ 1\,]/2 + 3\,[\,1\ -1\ \ 1\ -1\,]/2$

$= [\,2\ \ 1\ \ 6\ \ 1\,]_{standard}$

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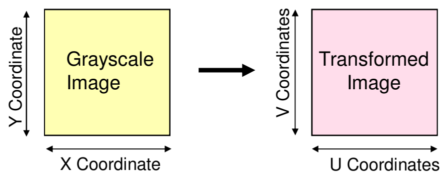
Transforms: Change of Basis – 2D Images



The coefficients are arranged in a 2D array.

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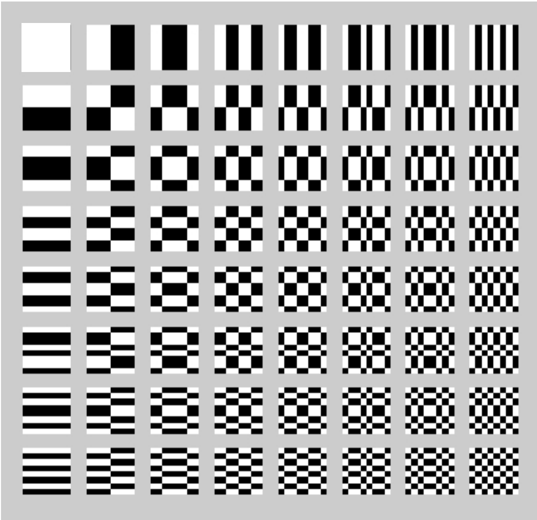
Hadamard Basis Functions



size = 8x8

White = +1 Black = -1

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Transforms: Change of Basis

Standard Basis:

$$\begin{bmatrix}2 & 1\\ 6 & 1\end{bmatrix} = 2\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + 6\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

Hadamard Transform:

$$\begin{bmatrix}2 & 1\\ 6 & 1\end{bmatrix} = 5\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}/2 + -2\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}/2 + -2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}/2 + 3\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}/2$$

$$\equiv \begin{bmatrix}5 & 3\\ -2 & -2\end{bmatrix}_{\text{Hadamard}}$$

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Finding the transform coefficients

Signal: $\mathbf{X} = \begin{bmatrix}2 & 1\\ 6 & 1\end{bmatrix}_{\text{standard}}$

New Basis:

$$\mathbf{T}_{11} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}/2 \qquad \mathbf{T}_{12} = \begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}/2$$

$$\mathbf{T}_{21} = \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}/2 \qquad \mathbf{T}_{22} = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}/2$$

Signal:

$$\mathbf{X} = a_{11}\mathbf{T}_{11} + a_{12}\mathbf{T}_{12} + a_{21}\mathbf{T}_{21} + a_{22}\mathbf{T}_{22}$$

New Coefficients:

$a_{11} = \langle \mathbf{X},\mathbf{T}_{11} \rangle =$ sum(sum(**X**.*$\mathbf{T}_{11}$)) = 5

$a_{12} = \langle \mathbf{X},\mathbf{T}_{21} \rangle =$ sum(sum(**X**.*$\mathbf{T}_{21}$)) = -2

$a_{21} = \langle \mathbf{X},\mathbf{T}_{22} \rangle =$ sum(sum(**X**.*$\mathbf{T}_{22}$)) = -2

$a_{22} = \langle \mathbf{X},\mathbf{T}_{12} \rangle =$ sum(sum(**X**.*$\mathbf{T}_{12}$)) = 3

$$\mathbf{X} \equiv \begin{bmatrix}5 & 3\\ -2 & -2\end{bmatrix}_{\text{new}}$$

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Standard Basis:

$$\underset{\text{coefficients}}{\begin{bmatrix}2 & 1\\ 6 & 1\end{bmatrix}} \longleftrightarrow \left[\begin{matrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} & \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\end{matrix}\right]_{\text{Standard}}$$

Basis Elements

Hadamard Transform:

$$\underset{\text{coefficients}}{\begin{bmatrix}5 & 3\\ -2 & -2\end{bmatrix}_{\text{Hadamard}}} \longleftrightarrow \left[\begin{matrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}/2 & \begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}/2\\ \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}/2 & \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}/2\end{matrix}\right]_{\text{Hadamard}}$$

Basis Elements

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Continuous images/signals f(x):

1) The number of Basis Elements B_i is ∞.

$$f(x) = \int_i a_i B_i(x)\, di$$

2) The dot product:

$$\langle f(x), B_i(x) \rangle = \int_x f(x) B_i(x)\, dx$$

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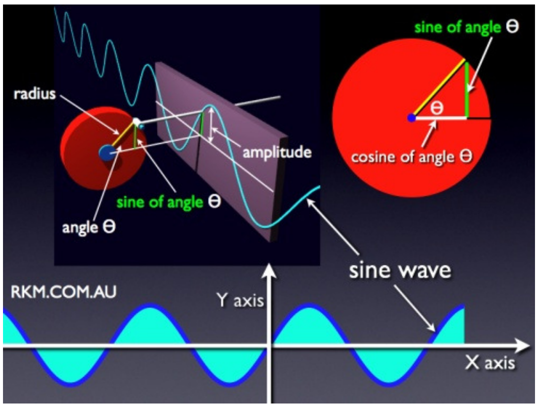
Part II: Sines and Cosines



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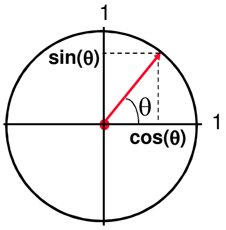
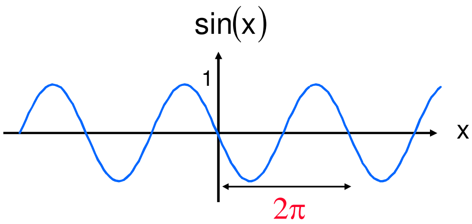
Wavelength and Frequency of Sine/Cosin





– The wavelength of sin(x) is 2π .

– The frequency is $1/(2\pi)$.

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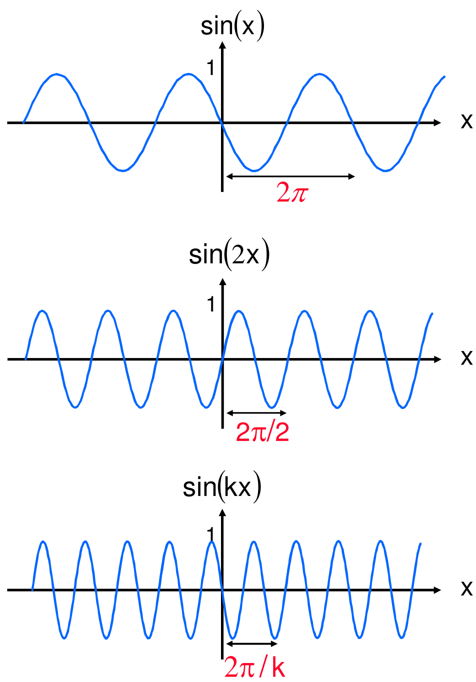

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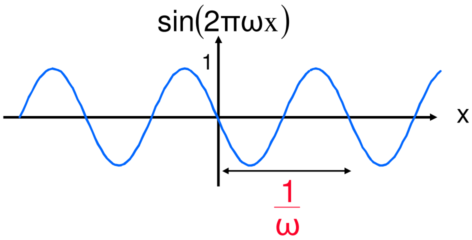
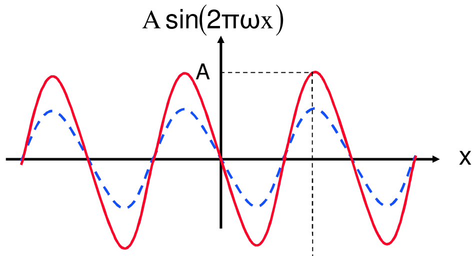
– Define $K = 2\pi\omega$



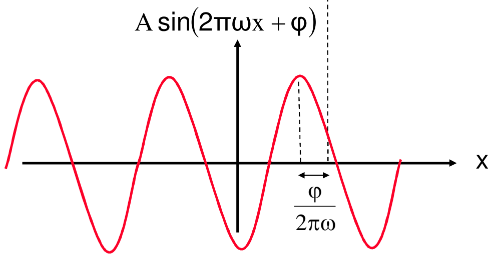
– The wavelength of $\sin(2\pi\omega x)$ is $1/\omega$.

– The frequency is ω .

– Changing Amplitude:

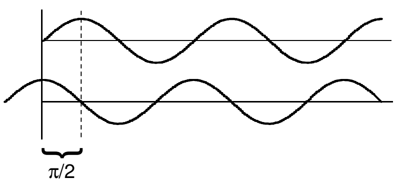


– Changing Phase:



Sine vs Cosine

$\sin(x) = \cos(x)$ with a phase shift of $\pi/2$.

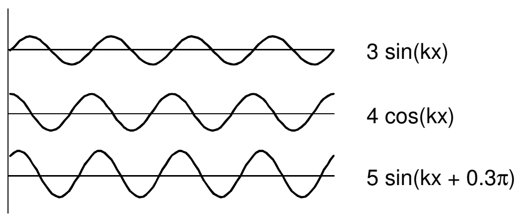


$\sin(x) + \cos(x) = ?$

Sine vs Cosine

$\sin(x) + \cos(x) = \sin(x)$ scaled by $\sqrt{2}$ with a phase shift of $\pi/4$.

$3\sin(kx) + 4\cos(kx) = \sin(kx)$ with amplitude scaled by 5 and phase shift of 0.3π



Combining Sine and Cosine

If we add a Sine wave to a Cosine wave with the same frequency we get a scaled and shifted (Co-) Sine wave with the same frequency:

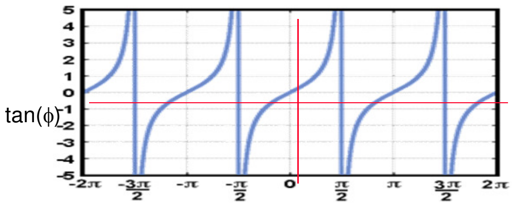
$$a \sin(kx) + b \cos(kx) = R \sin(kx + \phi)$$

$$\text{where} \quad R = \sqrt{a^2 + b^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

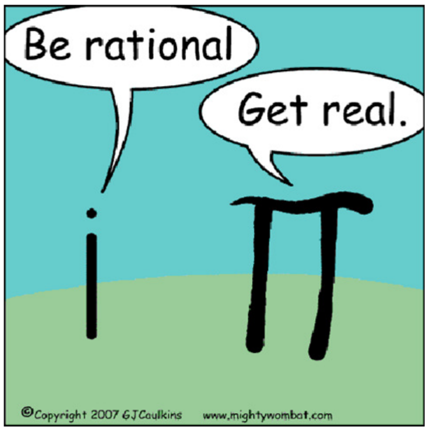
(prove it!)

What is the result if a=0?

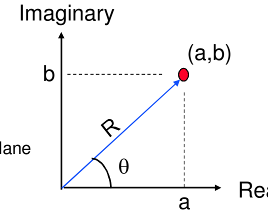
What is the result if b=0?



Part III: Complex Numbers



Complex Numbers



The Complex Plane

- Two kind of representations for a point (a,b) in the complex plane
  - The Cartesian representation:

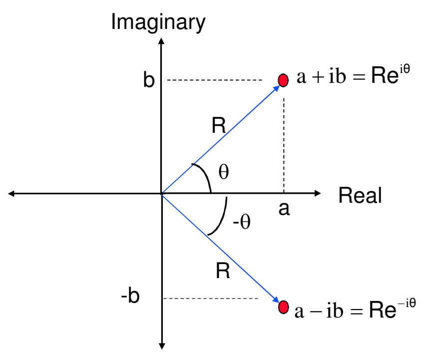
    $Z = a + ib \qquad where \quad i^2 = -1$

  - The Polar representation:

    $Z = Re^{i\theta}$ (Complex exponential)

- Conversions:
  - Polar to Cartesian: $Re^{i\theta} = R\cos(\theta) + iR\sin(\theta)$
  - Cartesian to Polar $a + ib = \sqrt{a^2 + b^2}\ e^{i\tan^{-1}(b/a)}$

- Conjugate of Z is Z^*:
  - Cartesian rep. $(a + ib)^* = a - ib$
  - Polar rep. $(Re^{i\theta})^* = Re^{-i\theta}$



Algebraic operations:

- addition/subtraction:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

- multiplication:

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)$$

$$Ae^{i\alpha}\ Be^{i\beta}=ABe^{i(\alpha+\beta)}$$

- inner Product:

$$\langle(a+ib),(c+id)\rangle=(a+ib)^*(c+id)=(a-ib)(c+id)$$

$$\langle Ae^{i\alpha},Be^{i\beta}\rangle = Ae^{-i\alpha}Be^{i\beta} = ABe^{i(\beta-\alpha)}$$

- norm:

$$\|a+ib\|^2=(a+ib)^*(a+ib)=a^2+b^2$$

$$\|Re^{i\theta}\|^2=(Re^{i\theta})^* Re^{i\theta}=Re^{-i\theta}Re^{i\theta}=R^2$$

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The (Co-) Sinusoid

$$e^{i\theta}=\cos(\theta)+i\sin(\theta)$$

- The (Co-)Sinusoid as complex exponential:

$$\cos(x)=\text{Real}(e^{ix})$$
$$\sin(x)=\text{Imag}(e^{ix})$$

Or

$$\cos(x)=\frac{e^{ix}+e^{-ix}}{2}$$
$$\sin(x)=\frac{e^{ix}-e^{-ix}}{2i}$$

- What about generalization?

$S\sin(kx)+C\cos(kx)=\ ?$

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Scaling and phase shifting can be represented as a multiplication with $Z=Re^{i\theta}$

$\cos(kx) \longrightarrow R\cos(kx+\theta)$

$\sin(kx) \longrightarrow R\sin(kx+\theta)$

$$e^{ikx} \longrightarrow Re^{i(kx+\theta)}$$
$$= Re^{i\theta}e^{ikx}$$
$$= Ze^{ikx}$$

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We saw that :

$S\sin(kx)+C\cos(kx)=R\sin(kx+\theta)$

where $R=\sqrt{S^2+C^2}$ and $\theta=\tan^{-1}\left(\frac{C}{S}\right)$

$$R\sin(kx+\theta)=\text{Imag}(R\,e^{i\theta}e^{ikx})$$
$$=\text{Imag}(Ze^{ikx})$$

$$R\sin(kx+\theta)=\tfrac{1}{2i}(Re^{i\theta}e^{ikx}-Re^{-i\theta}e^{-ikx})$$
$$=\tfrac{1}{2i}(Z\ e^{ikx}-Z^* e^{-ikx})$$

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S sin(kx) + C cos(kx) = R sin(kx + θ)

$$R\sin(kx+\theta) = \text{Imag}(R\,e^{i\theta}\,e^{ikx}) = \text{Imag}(Z e^{ikx})$$

$$R\sin(kx+\theta) = \frac{1}{2i}(R\,e^{i\theta}e^{ikx} - R\,e^{-i\theta}e^{-ikx}) = \frac{1}{2i}(Z\;e^{ikx} - Z^{*}\,e^{-ikx})$$

The 1D Continuous Fourier Transform

The Continuous Fourier Transform finds the F(ω) given the (cont.) signal f(x):

$$F(\omega) = \int_x f(x)\; e^{-i2\pi\omega x}\,dx$$

$B_\omega(x)=e^{i2\pi\omega x}$ is a complex wave function for each ω .

The Inverse Continuous Fourier Transform composes a signal f(x) given F(ω):

$$f(x) = \int_\omega F(\omega)\; e^{i2\pi\omega x}\,d\omega$$