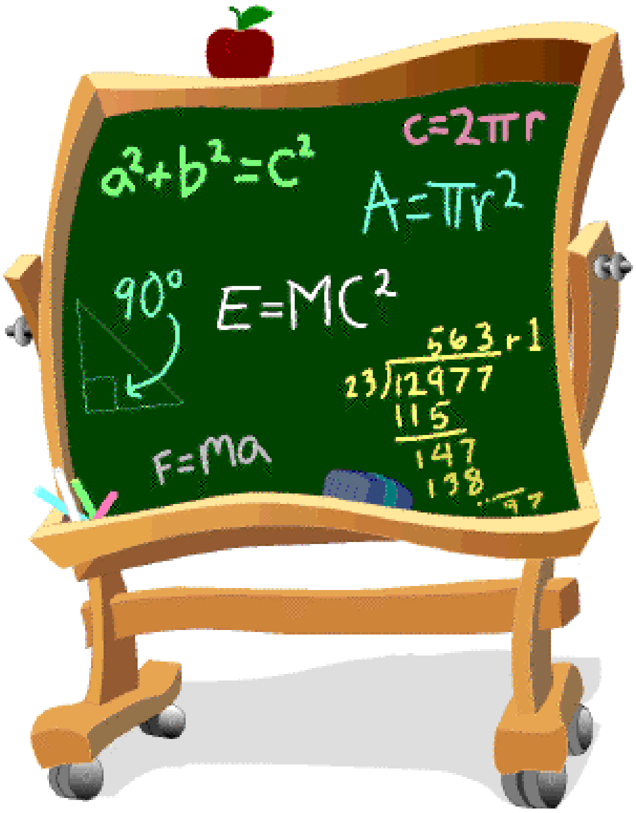
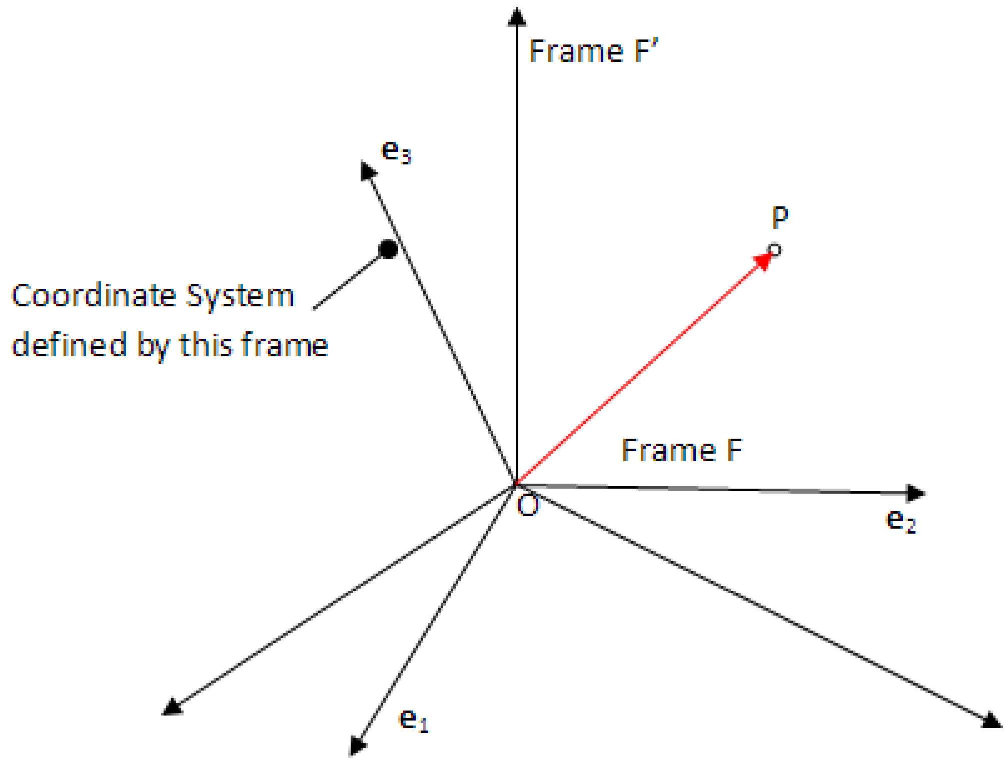
# Math Review Towards Fourier Transform

- Linear Spaces
- Change of Basis
- Cosines and Sines
- Complex Numbers



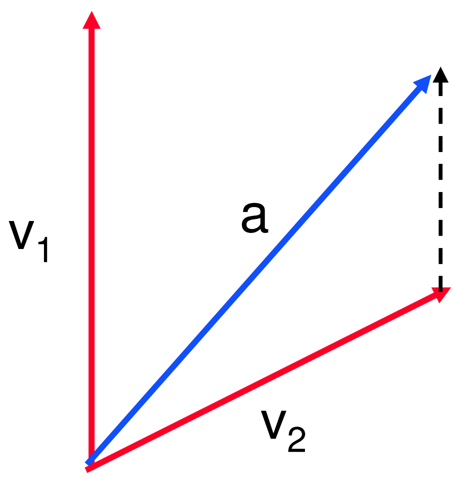
# Part I:
# Vector Spaces and Basis Vectors



# Basis Vectors

- A given vector value is represented with respect to a *coordinate system*.
- A coordinate system is defined by a set of linearly independent vectors forming the system *basis*.
- Any vector value is represented as a linear sum of the basis vectors.

$$a = 0.5 * v_1 + 1 * v_2 \equiv (0.5\ ,\ 1)_v$$




- v1, v2 are basis vectors
- The representation of a with respect to this basis is (0.5,1)

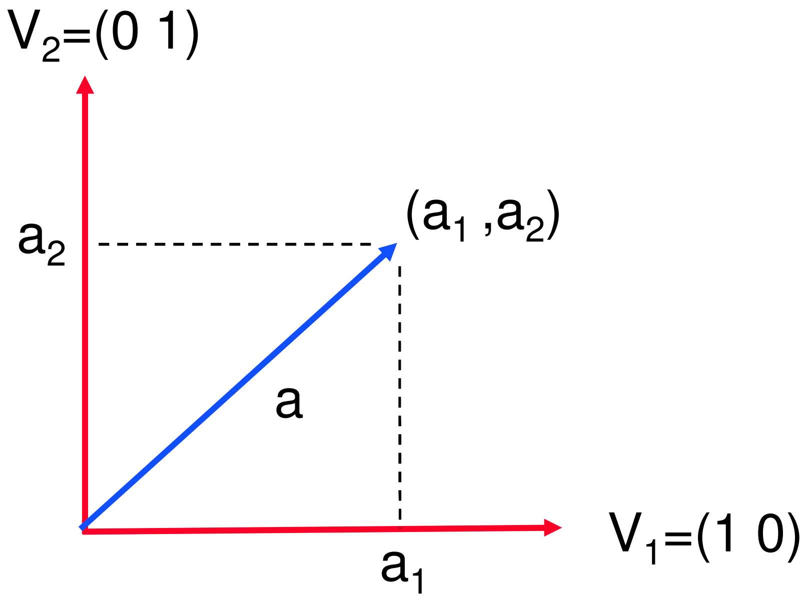
# Orthonormal Basis Vectors

- If the basis vectors are mutually orthogonal and are unit vectors, the vectors form an *orthonormal basis*.
- Example:

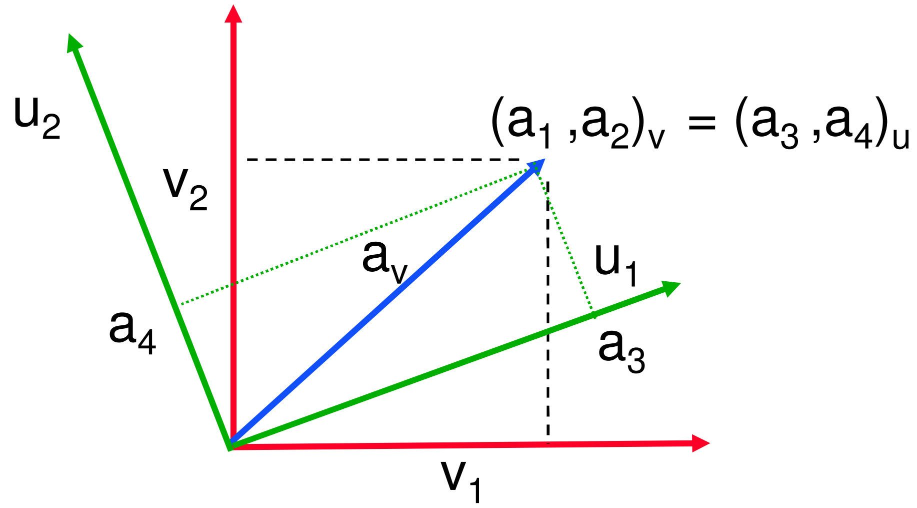
The *standard basis* is orthonormal:

$$V_1=(1\ 0\ 0\ 0\ \ldots)$$
$$V_2=(0\ 1\ 0\ 0\ \ldots)$$
$$V_3=(0\ 0\ 1\ 0\ \ldots)$$

....



# Change of Basis




Given a vector $\mathbf{a_v}$, represented in orthonormal basis $\{v_i\}$ , what is the representation of $\mathbf{a_v}$ in a different orthonormal basis $\{u_i\}$?

$$a_u(i) = \langle a_v, u_i \rangle$$

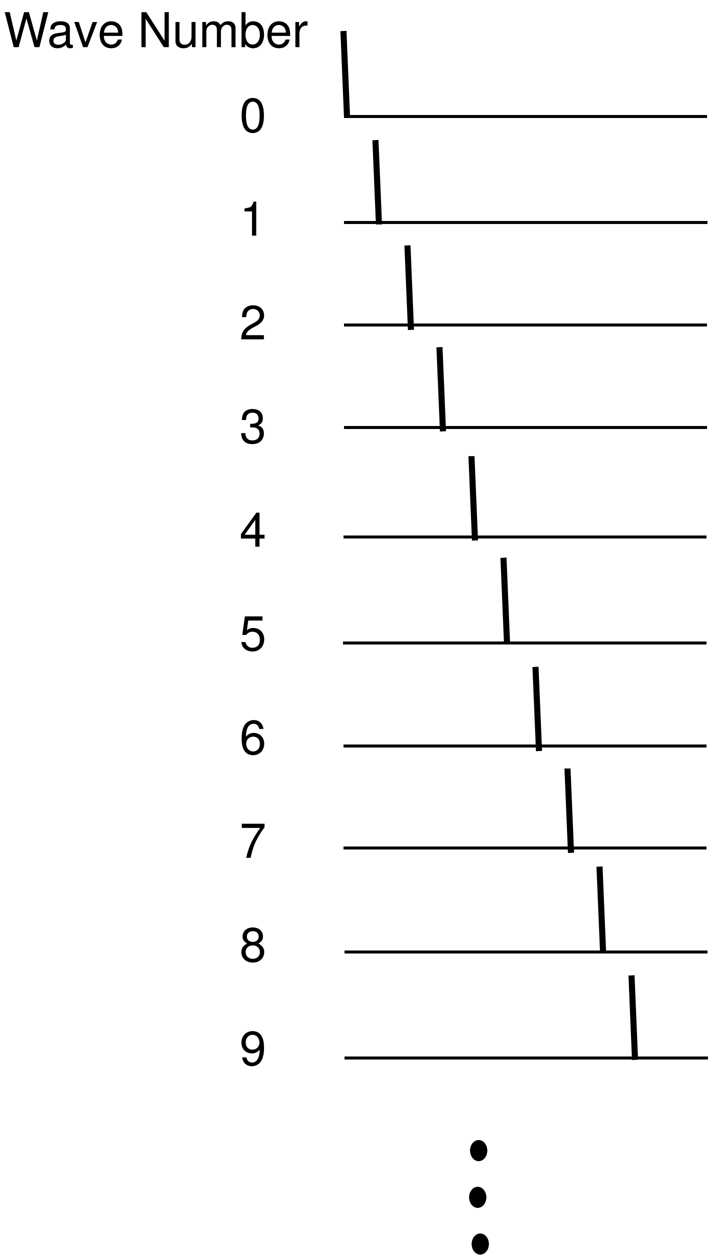
$$a_v = \sum_i a_u(i)\, u_i$$

where $\langle c, b \rangle = c^T \mathbf{b} = \sum_i c(i)\mathbf{b}(i)$

# Signal (Image) Transforms

1. Basis Functions.
2. Method for finding the transform coefficients given a signal.
3. Method for finding the signal given the transform coefficients.

# The Orthonormal Standard Basis

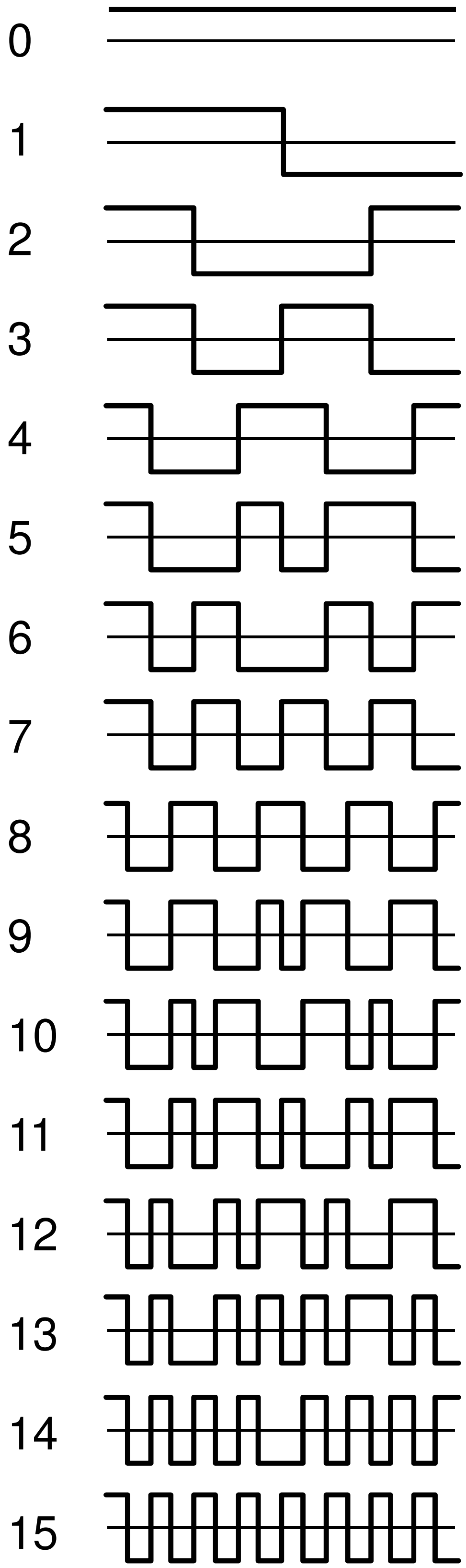



N = 16

**Standard Basis Functions**

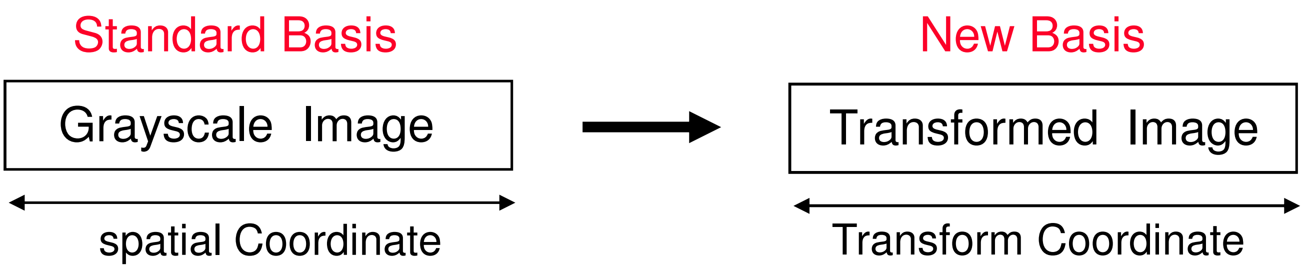
# The Orthonormal Hadamard  Basis

Wave Number




N = 16

# Hadamard Transform




## Standard Basis:

$[\,2\;\;1\;\;6\;\;1\,]_{standard} =$

$$\mathbf{2}[\,1\;0\;0\;0\,] + \mathbf{1}[\,0\;1\;0\;0\,] + \mathbf{6}[\,0\;0\;1\;0\,] + \mathbf{1}[\,0\;0\;0\;1\,]$$

## Hadamard Transform:

$[\,2\;\;1\;\;6\;1\,]_{standard} =$

$$= \mathbf{5}[\,1\;1\;1\;1\,]/2 + \mathbf{-2}[\,1\;1\;-1\;-1\,]/2 +$$

$$\mathbf{-2}[\,1\;-1\;-1\;1\,]/2 + \mathbf{3}[\,1\;-1\;1\;-1\,]/2$$

$$\equiv [\,\mathbf{5}\;\;\mathbf{-2}\;\;\mathbf{-2}\;\;\mathbf{3}\,]_{Hadamard}$$

# Finding the transform coefficients

Signal: $\mathbf{X} = [\,2\ 1\ 6\ 1\,]_{\text{standard}}$

Hadamard Basis:

$$\mathbf{T_0} = [\ 1\ \ 1\ \ 1\ \ 1\ ]\ /2$$

$$\mathbf{T_1} = [\ 1\ \ 1\ \ {-1}\ \ {-1}\ ]\ /2$$

$$\mathbf{T_2} = [\ 1\ \ {-1}\ \ {-1}\ \ 1\ ]\ /2$$

$$\mathbf{T_3} = [\ 1\ \ {-1}\ \ 1\ \ {-1}\ ]\ /2$$

Hadamard Coefficients:

$$a_0 = \mathbf{<X,T_0>} = <[\,2\ 1\ 6\ 1\,],[\,1\ \ 1\ \ 1\ \ 1\,]>/2 = 5$$

$$a_1 = \mathbf{<X,T_1>} = <[\,2\ 1\ 6\ 1\,],[\,1\ \ 1\ \ {-1}\ \ {-1}\,]>/2 = -2$$

$$a_2 = \mathbf{<X,T_2>} = <[\,2\ 1\ 6\ 1\,],[\,1\ \ {-1}\ \ {-1}\ \ 1\,]>/2 = -2$$

$$a_3 = \mathbf{<X,T_3>} = <[\,2\ 1\ 6\ 1\,],[\,1\ \ {-1}\ \ 1\ \ {-1}\,]>/2 = 3$$

Signal: $[\,2\ 1\ 6\ 1\,]_{\text{Standard}} \equiv [\,5\ \ {-2}\ \ {-2}\ \ 3\,]_{\text{Hadamard}}$

# Reconstructing the Image from the transform coefficients

Transform: $\mathbf{Y} = [\ 5\ -2\ -2\ \ 3\ ]_{\text{Hadamard}}$

Hadamard Basis:

$\mathbf{T_0} = [\ 1\ \ 1\ \ 1\ \ 1\ ]\ /2$

$\mathbf{T_1} = [\ 1\ \ 1\ -1\ -1\ ]\ /2$

$\mathbf{T_2} = [\ 1\ -1\ -1\ \ 1\ ]\ /2$

$\mathbf{T_3} = [\ 1\ -1\ \ 1\ -1\ ]\ /2$

Reconstruction;

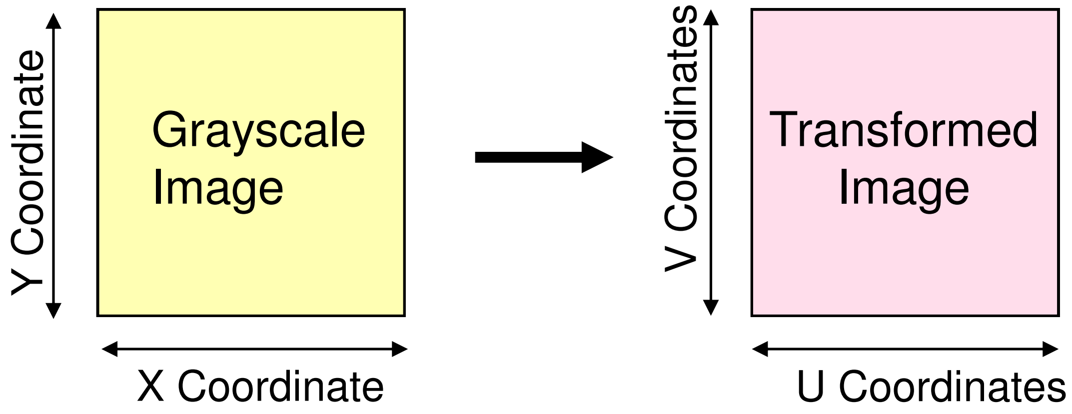
$\Sigma\ \mathbf{Y(i),T_i}\ =$

$\mathbf{5}\ [\ 1\ 1\ 1\ 1\ ]/2\ \ +\ \mathbf{-2}\ [\ 1\ 1\ -1\ -1\ ]\ /2\ +$

$\mathbf{-2}\ [\ 1\ -1\ -1\ 1\ ]\ /2\ +\ \mathbf{3}\ [\ 1\ -1\ \ 1\ -1\ ]\ /2$

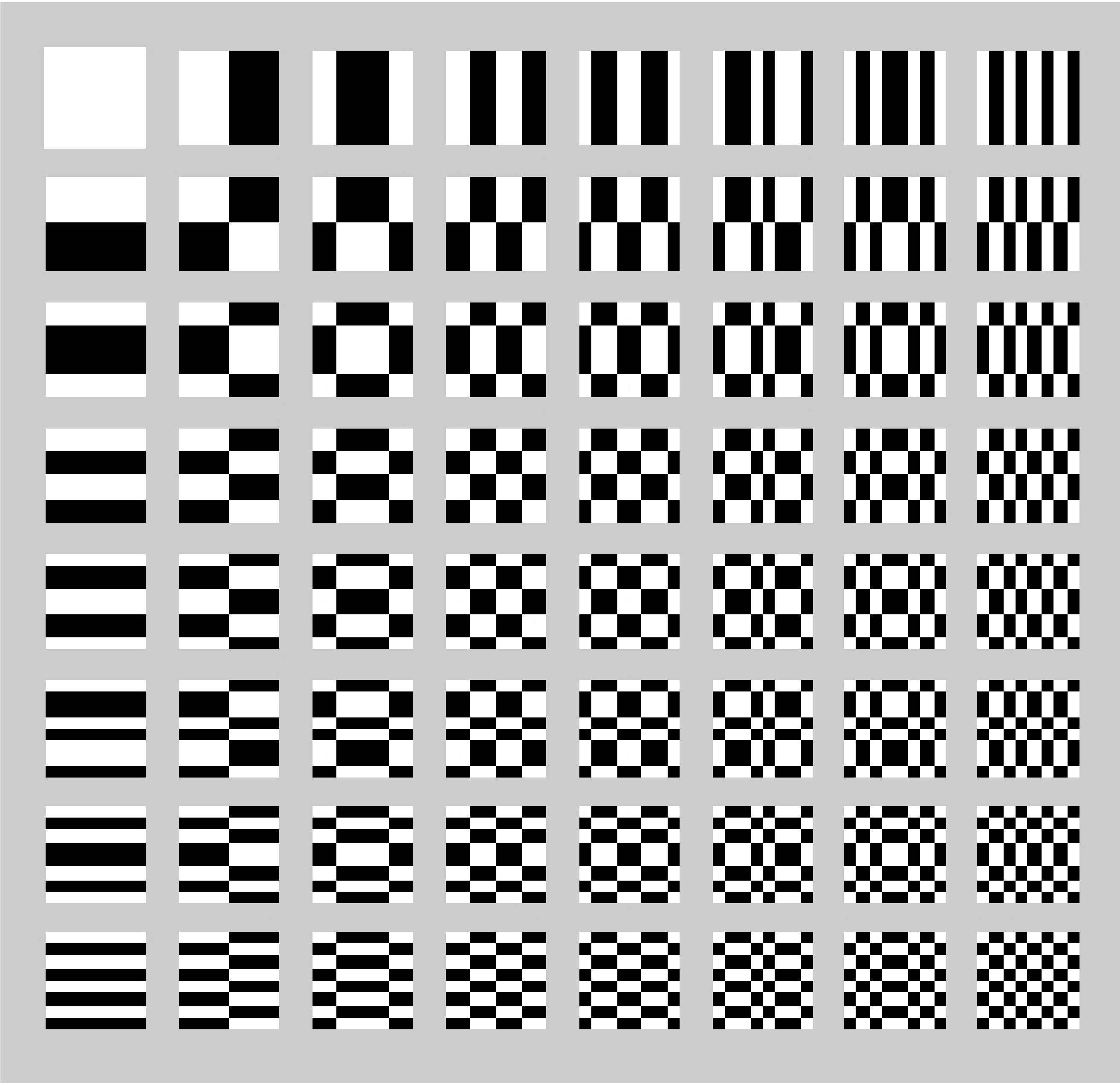
$=\ \ [\ 2\ \ 1\ \ 6\ \ 1\ ]_{\text{standard}}$

# Transforms: Change of Basis – 2D Images




The coefficients are arranged in a 2D array.

# Hadamard Basis Functions



size = 8x8

White = +1 Black = -1

# Transforms: Change of Basis

Standard Basis:

$$\begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hadamard Transform:

$$\begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} = 5\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}/2 + -2\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}/2 + -2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}/2 + 3\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}/2$$

$$\equiv \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}_{\text{Hadamard}}$$

# Finding the transform coefficients

Signal:

$$\mathbf{X} = \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix}_{standard}$$

New Basis:

$$\mathbf{T_{11}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}/2 \qquad \mathbf{T_{12}} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}/2$$

$$\mathbf{T_{21}} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}/2 \qquad \mathbf{T_{22}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}/2$$

Signal:

$$\mathbf{X} = a_{11}\mathbf{T_{11}} + a_{12}\mathbf{T_{12}} + a_{21}\mathbf{T_{21}} + a_{22}\mathbf{T_{22}}$$

New Coefficients:

$$a_{11} = \mathbf{<X,T_{11}>} = \text{sum}(\text{sum}(\mathbf{X}.^{*}\mathbf{T_{11}})) = 5$$
$$a_{12} = \mathbf{<X,T_{21}>} = \text{sum}(\text{sum}(\mathbf{X}.^{*}\mathbf{T_{21}})) = -2$$
$$a_{21} = \mathbf{<X,T_{22}>} = \text{sum}(\text{sum}(\mathbf{X}.^{*}\mathbf{T_{22}})) = -2$$
$$a_{22} = \mathbf{<X,T_{12}>} = \text{sum}(\text{sum}(\mathbf{X}.^{*}\mathbf{T_{12}})) = 3$$

$$\mathbf{X} \equiv \begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}_{new}$$

# Standard Basis:

$$\begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}_{\text{Standard}}$$

coefficients

Basis Elements

# Hadamard Transform:

$$\begin{bmatrix} 5 & 3 \\ -2 & -2 \end{bmatrix}_{\text{Hadamard}} \longleftrightarrow \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}/2 & \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}/2 \\ \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}/2 & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}/2 \end{bmatrix}_{\text{Hadamard}}$$

coefficients

Basis Elements

# Continuous images/signals f(x):

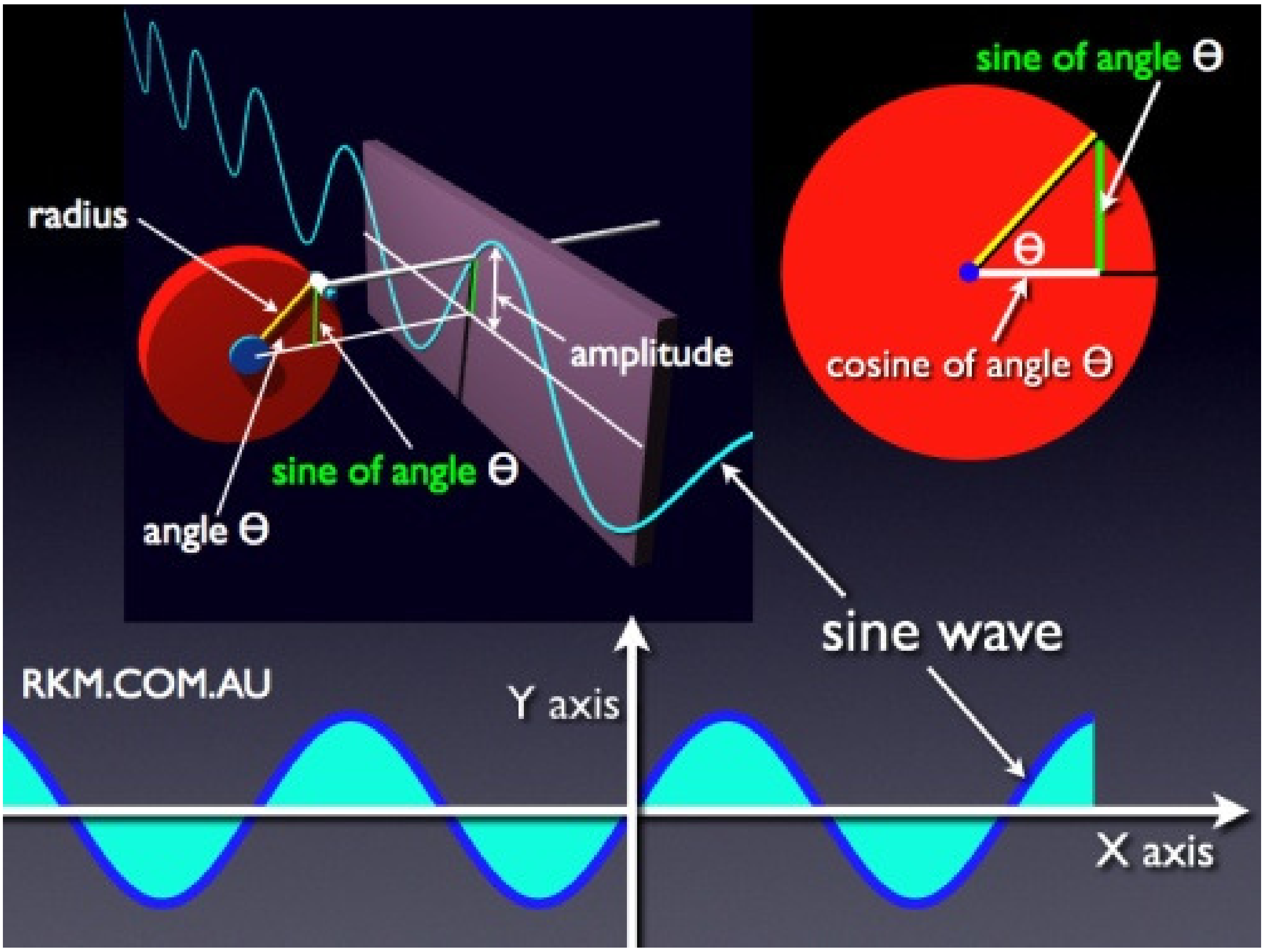
1) The number of Basis Elements $B_i$ is $\infty$.

$$f(x) = \int_i a_i B_i(x)\, di$$

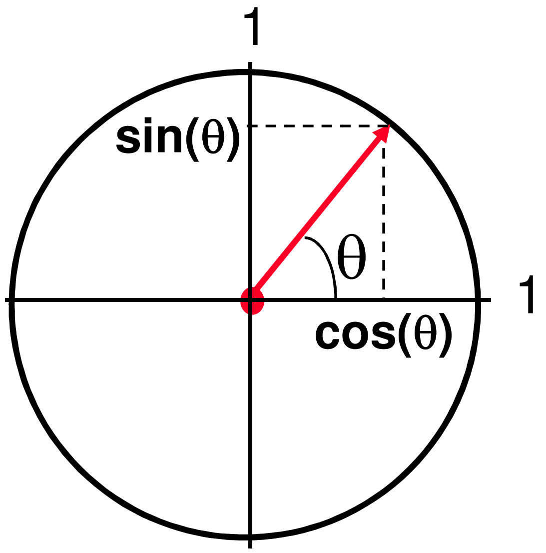
2) The dot product:

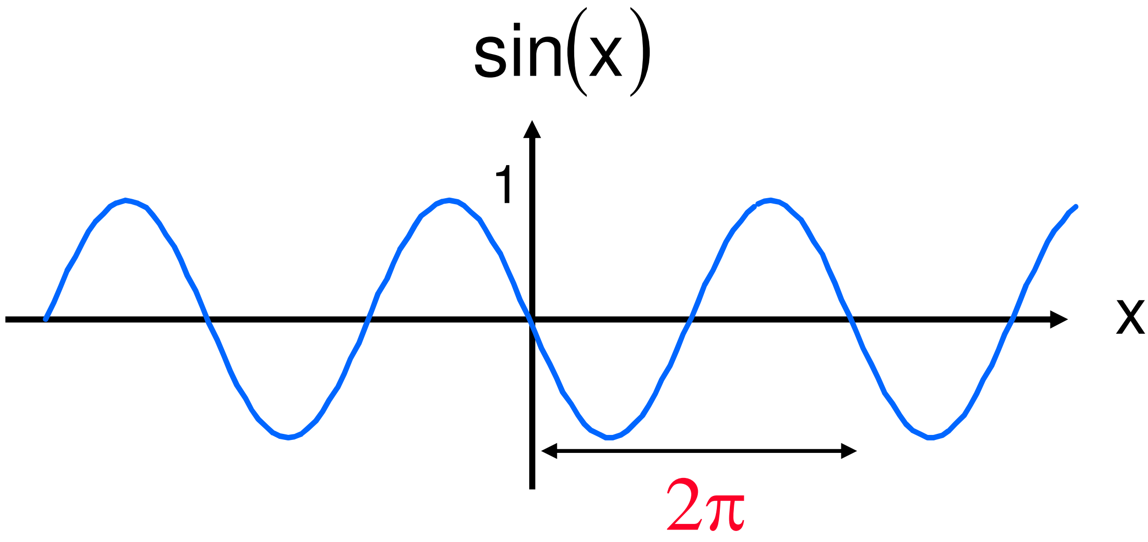
$$<f(x), B_i(x)> = \int_x f(x) B_i(x)\, dx$$

# Part II:
# Sines and Cosines



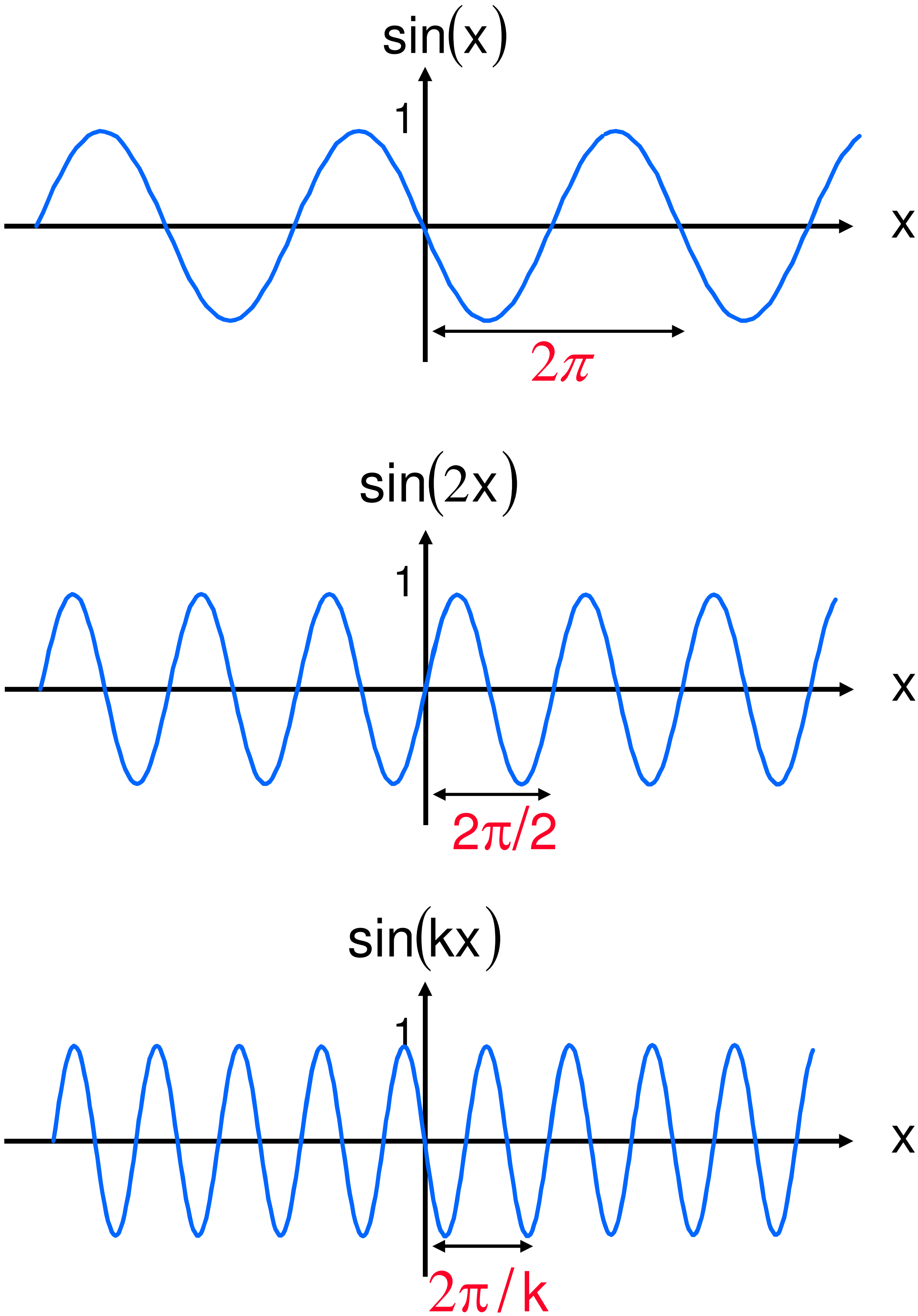
# Wavelength and Frequency of Sine/Cosin



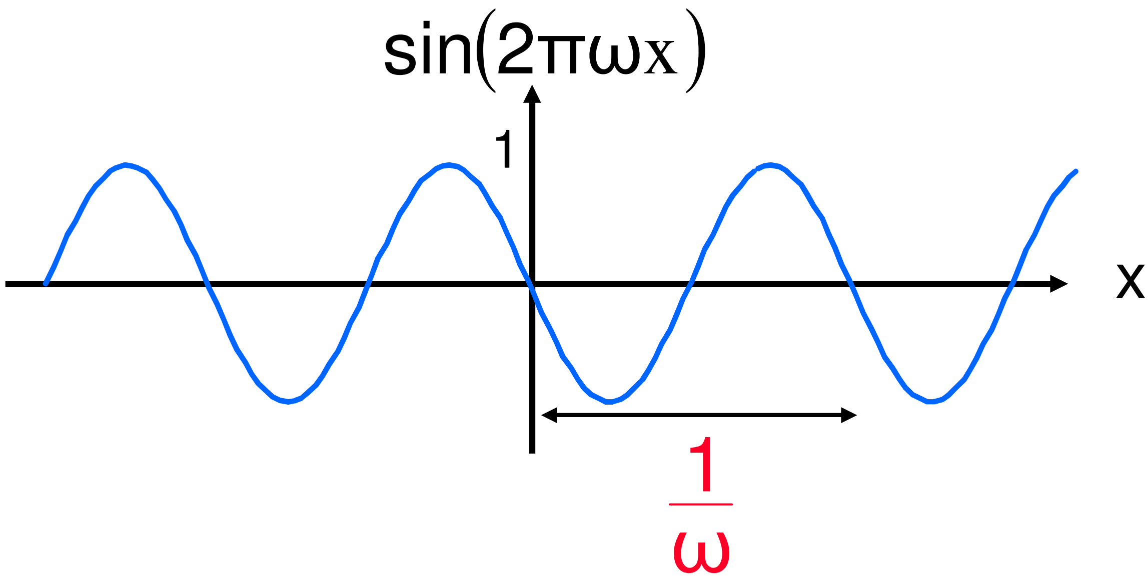




– The wavelength of sin(x) is $2\pi$ .

– The frequency is $1/(2\pi)$ .


sin(x)
1
x
2π
sin(2x)
1
x
2π/2
sin(kx)
1
x
2π / k

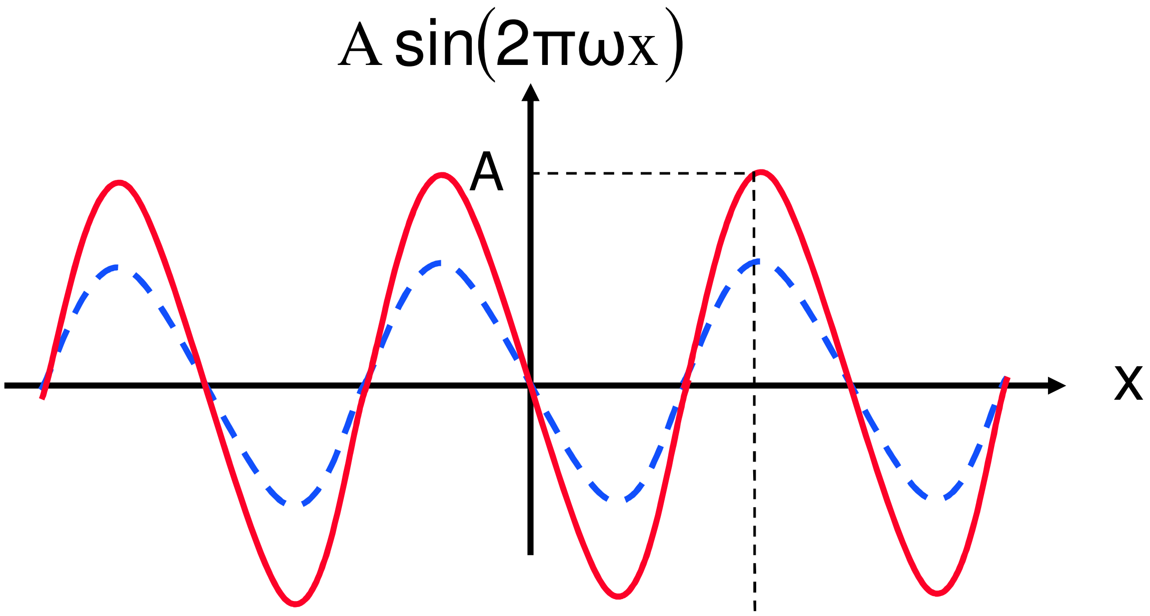
– Define $K=2\pi\omega$



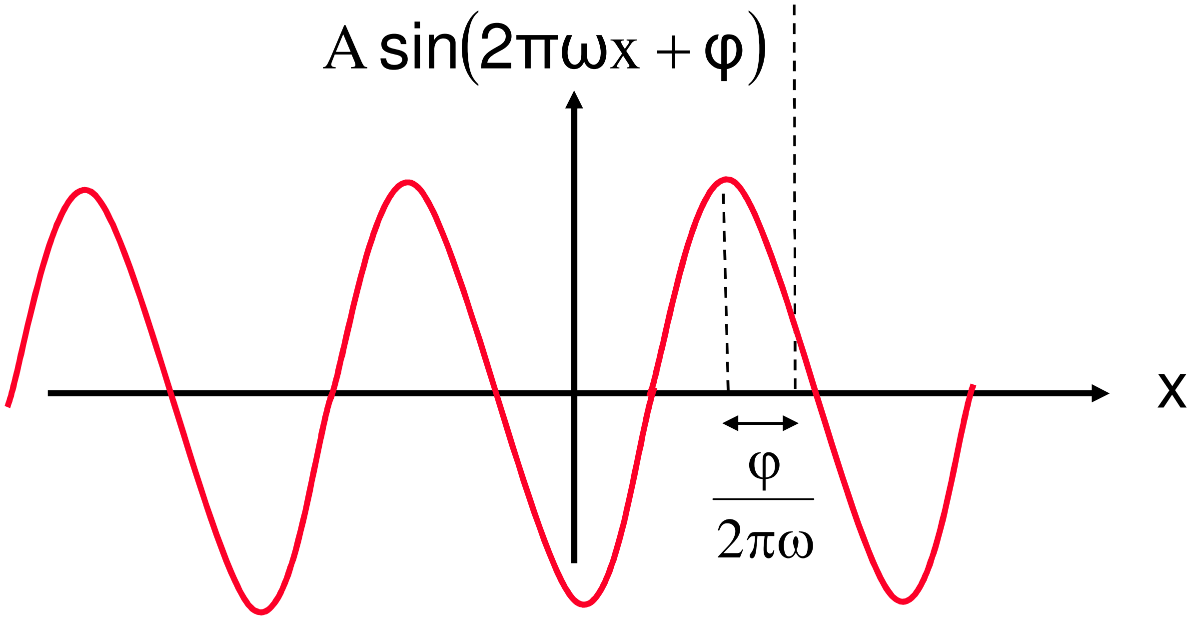

– The wavelength of $\sin(2\pi\omega x)$ is $1/\omega$ .

– The frequency is $\omega$ .

– Changing Amplitude:

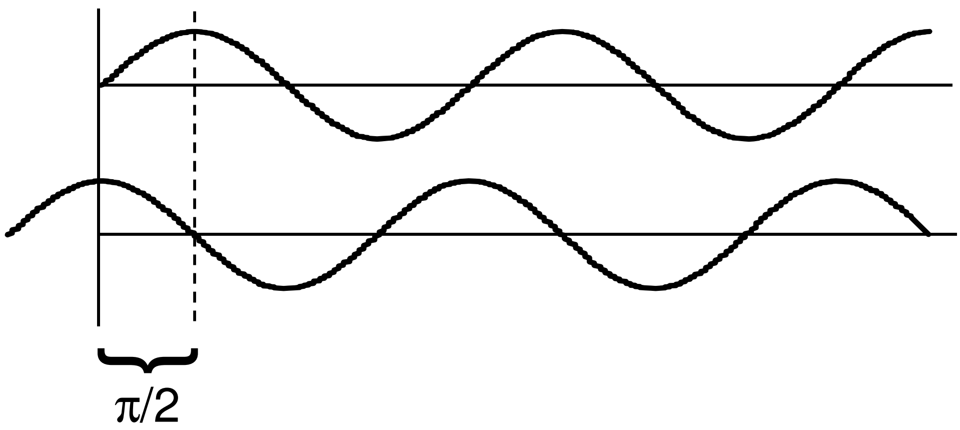



– Changing Phase:



# Sine vs Cosine

$\sin(x) = \cos(x)$ with a phase shift of $\pi/2$.

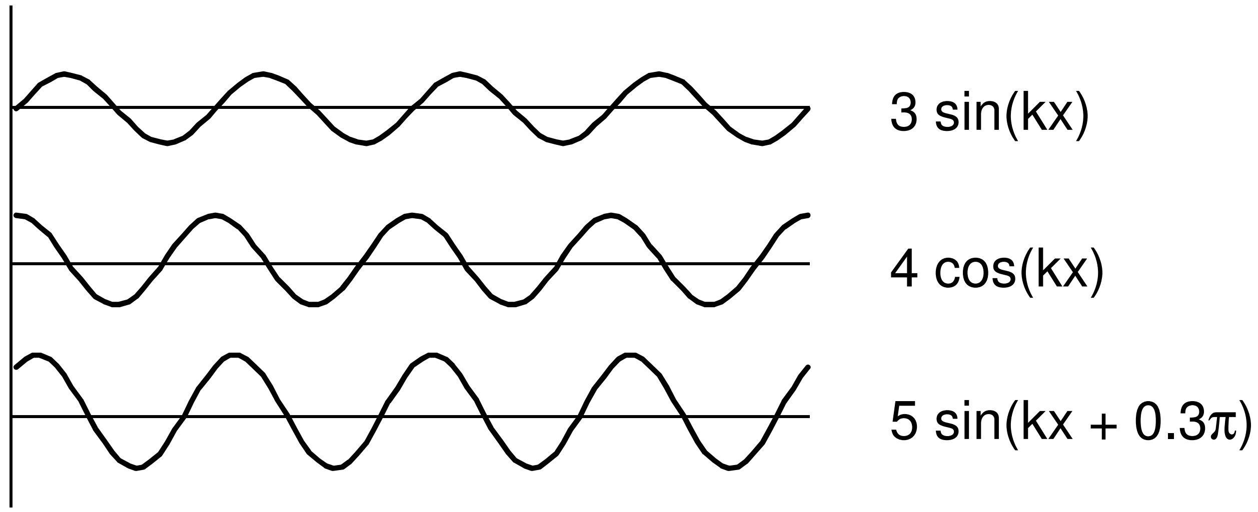



$\sin(x) + \cos(x) = ?$

# Sine vs Cosine

sin(x) + cos(x) = sin(x) scaled by $\sqrt{2}$ with a phase shift of $\pi/4$.

3 sin(kx) + 4 cos(kx) = sin(kx) with amplitude scaled by 5 and phase shift of $0.3\pi$



# Combining Sine and Cosine

If we add a Sine wave to a Cosine wave with the same frequency we get a scaled and shifted (Co-) Sine wave with the same frequency:

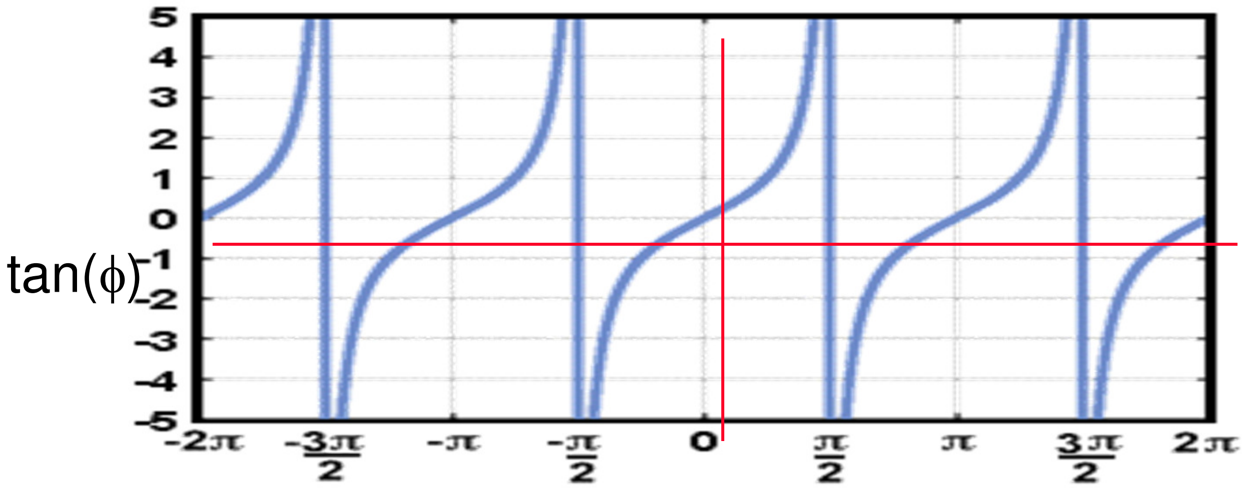
$$a \sin(kx) + b \cos(kx) = R \sin(kx + \phi)$$

$$\text{where} \quad R = \sqrt{a^2 + b^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

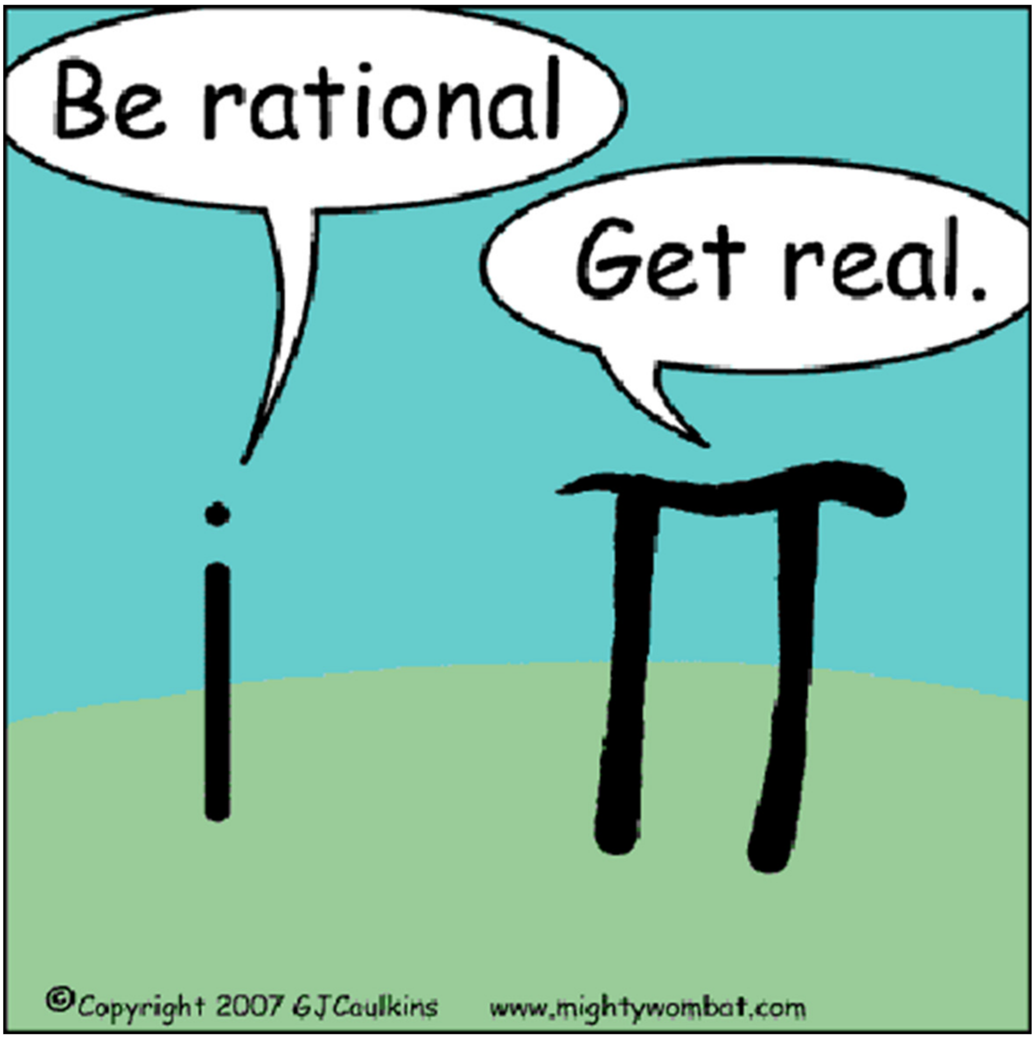
(prove it!)

What is the result if a=0?

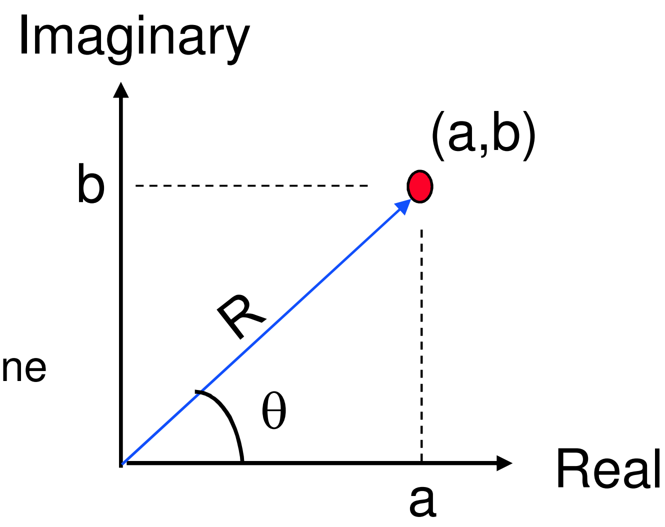
What is the result if b=0?



# Part III:
# Complex Numbers



# Complex Numbers




The Complex Plane

- Two kind of representations for a point (a,b) in the complex plane
  - The Cartesian representation:

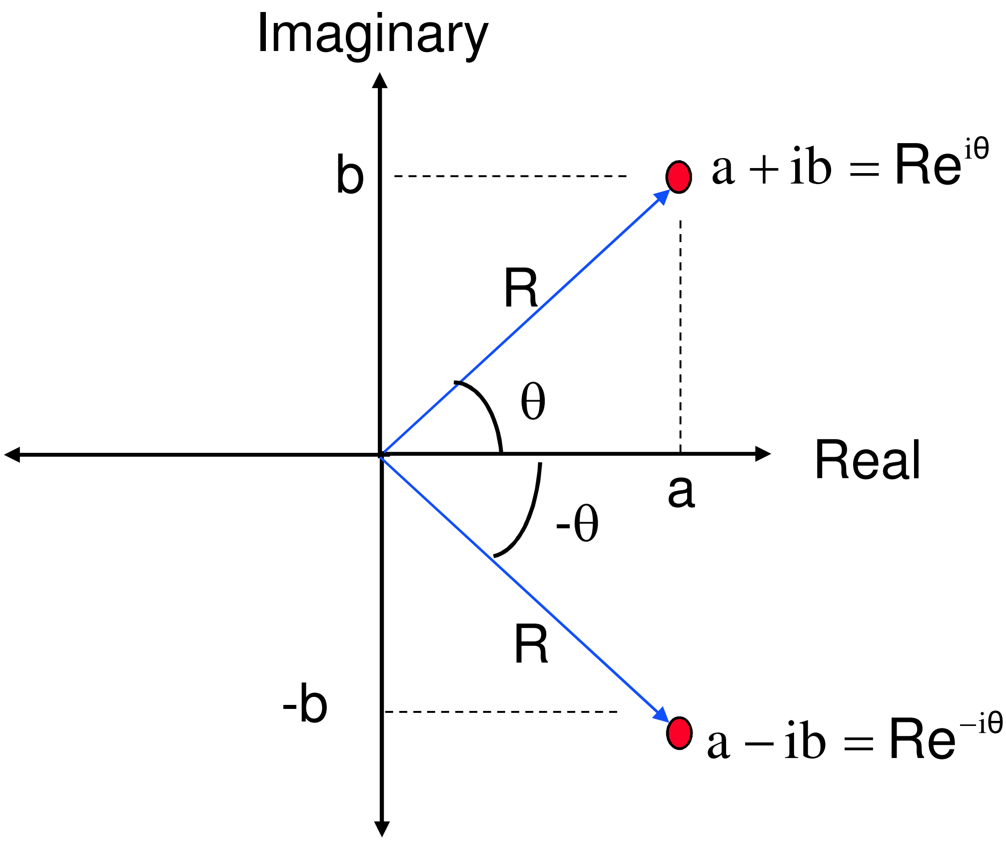
$$Z = a + ib \qquad where \quad i^2 = -1$$

  - The Polar representation:

$$Z = Re^{i\theta}$$ (Complex exponential)

- Conversions:
  - Polar to Cartesian: $Re^{i\theta} = R\cos(\theta) + iR\sin(\theta)$
  - Cartesian to Polar $a + ib = \sqrt{a^2 + b^2}\, e^{i\tan^{-1}(b/a)}$

- Conjugate of Z is $Z^*$:
  - Cartesian rep. $(a + ib)^* = a - ib$
  - Polar rep. $(Re^{i\theta})^* = Re^{-i\theta}$



# Algebraic operations:

- addition/subtraction:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

- multiplication:

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)$$

$$Ae^{i\alpha}\ Be^{i\beta}=ABe^{i(\alpha+\beta)}$$

- inner Product:

$$\langle (a+ib),(c+id)\rangle=(a+ib)^*(c+id)=(a-ib)(c+id)$$

$$<Ae^{i\alpha},Be^{i\beta}> = Ae^{-i\alpha}Be^{i\beta} = ABe^{i(\beta-\alpha)}$$

- norm:

$$\|a+ib\|^2=(a+ib)^*(a+ib)=a^2+b^2$$

$$\|Re^{i\theta}\|^2=(Re^{i\theta})^*Re^{i\theta}=Re^{-i\theta}Re^{i\theta}=R^2$$

# The (Co-) Sinusoid

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

- The (Co-)Sinusoid as complex exponential:

$$\cos(x) = \text{Real}(e^{ix})$$
$$\sin(x) = \text{Imag}(e^{ix})$$

Or

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

- What about generalization?

$$S \sin(kx) + C \cos(kx) = ?$$

Scaling and phase shifting can be represented as a multiplication with $Z = \mathrm{Re}^{i\theta}$

$$\cos(kx) \longrightarrow R\cos(kx+\theta)$$

$$\sin(kx) \longrightarrow R\sin(kx+\theta)$$

$$e^{ikx} \longrightarrow Re^{i(kx+\theta)}$$

$$= Re^{i\theta}\, e^{ikx}$$

$$= Ze^{ikx}$$

We saw that :

S sin(kx) + C cos(kx) = R sin(kx + θ)

where $R = \sqrt{S^2 + C^2}$ and $\theta = \tan^{-1}\left(\frac{C}{S}\right)$

$$
\begin{aligned}
R\sin(kx+\theta) &= \mathrm{Imag}(R\,e^{i\theta}\,e^{ikx}) \\
&= \mathrm{Imag}(Z\,e^{ikx})
\end{aligned}
$$

$$
\begin{aligned}
R\sin(kx+\theta) &= \frac{1}{2i}(R\,e^{i\theta}e^{ikx} - R\,e^{-i\theta}e^{-ikx}) \\
&= \frac{1}{2i}(Z\ e^{ikx} - Z^{*}e^{-ikx})
\end{aligned}
$$

S sin(kx) + C cos(kx) = R sin(kx + θ)

$$R\sin(kx+\theta) = \mathrm{Imag}(R\,e^{i\theta}\,e^{ikx})$$
$$= \mathrm{Imag}(Z\,e^{ikx})$$

$$R\sin(kx+\theta) = \frac{1}{2i}(R\,e^{i\theta}e^{ikx} - R\,e^{-i\theta}e^{-ikx})$$
$$= \frac{1}{2i}(Z\;e^{ikx} - Z^{*}\,e^{-ikx})$$

# The 1D Continuous Fourier Transform

The Continuous Fourier Transform finds the $F(\omega)$ given the (cont.) signal $f(x)$:

$$F(\omega) = \int_x f(x)\ e^{-i 2 \pi \omega x}\ dx$$

$B_\omega(x)=e^{i2\pi\omega x}$ is a complex wave function for each $\omega$ .

The Inverse Continuous Fourier Transform composes a signal $f(x)$ given $F(\omega)$:

$$f(x) = \int_\omega F(\omega)\ e^{i 2 \pi \omega x}\ d\omega$$



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