Mathematical Programming manuscript No.
(will be inserted by the editor)

Gideon Weiss

A Simplex Based Algorithm to Solve Separated Continuous Linear Programs*

1 January, 2000,
Revised: April 2000, June 2000, July 2000, November 2000, April 2001, June 2001.
Submitted for Publication to *Mathematical Programming*, August 2001.
Revised for *Mathematical Programming*, November 2002.
—

Abstract. We consider the separated continuous linear programming problem with linear data. We characterize the form of its optimal solution, and present an algorithm which solves it in a finite number of steps, using simplex pivot operations.

Key words. Continuous Linear Programming, Simplex Method

1. Introduction

Bellman [12,13] introduced the following continuous linear programming problem, to model some economic processes:

$$\text{CLP} \qquad \begin{aligned} &\max \int_0^T c'(t)u(t)dt \\ &\text{s.t. } H(t)u(t) + \int_0^t G(s,t)u(s)ds \le a(t), \\ &\qquad u(t) \ge 0, \quad t \in [0,T]. \end{aligned} \tag{1}$$

Work on this problem included: Tyndall [53], Levinson [35], Grinold [26] (duality); Buie and Abraham [19] (solution via discretization); Lehman [33], Drews [24], Hartberger [28], Segers [50], Perold [41,42], Anstreicher [10] (simplex based solution); and Ilyutovich [29,30] (Pontryagin maximum principle approach).

A subclass of CLP is the class of separated continuous linear programs (SCLP). This has been re-introduced by Anderson [1,2] in the context of job

* Research supported in part by US-Israel BSF grant 9400196 and by German-Israel GIF grant I-564-246/06/97

Gideon Weiss: Department of Statistics
The University of Haifa
Mount Carmel 31905, Israel.
e-mail *gweiss@stat.haifa.ac.il*

Mathematics Subject Classification (1991): 90C05, 90C45, 49-99

shop scheduling:

$$\begin{aligned} \max & \int_0^T c'(t)u(t)dt \\ \text{s.t.} & \int_0^t Gu(s)ds \le a(t), \\ & Hu(t) \le b(t), \\ & u(t) \ge 0, \quad t \in [0,T]. \end{aligned} \tag{2}$$

Some special cases of SCLP were solved by Anderson, Nash and Philpott [5–8] Hajek and Ogier [27] and some general results were derived by Anderson, Nash and Perold [4]. This research is summarized in the book of Anderson and Nash [3]. For more recent special cases see Verhaegh [55] and Sourd [52].

Major progress in the theory of SCLP was achieved by Pullan [44–49, 9]. Pullan considered SCLP problems with $a(t)$, $b(t)$ and $c(t)$ piecewise analytic, where the search is for optimal $u(t)$ in the space of measurable bounded functions. He formulated a dual problem:

$$\begin{aligned} \min & \int_0^T a'(t)d\pi(t) + \int_0^T b'(t)r(t)dt \\ \text{s.t.} & -G'\pi(t) + H'r(t) \ge c(t), \\ & r(t) \ge 0, \quad \pi(T) = 0, \quad \pi(t) \text{ non-decreasing}, \quad t \in [0,T]. \end{aligned} \tag{3}$$

and proved strong duality. He also showed that the optimal solution $u(t)$ is piecewise analytic, with a bounded number of (possibly exponentially many) breakpoints, and in particular for the special case that $a(t)$ is piecewise linear and $b(t)$ piecewise constant, the optimal solution $u(t)$ is piecewise constant. For a more general discussion of duality in SCLP, using convex analysis, see a recent paper of Shapiro [51]. Pullan has also proposed a convergent algorithm for the piecewise linear case, based on a sequence of discretizations.

A similar algorithm is suggested in Philpott and Craddock [43]. Luo and Bertsimas [36] have used quadratic programming techniques in conjunction with discretization, and implemented these to a more general problem, namely state constrained SCLP's. An interior point method for solving CLP is suggested by Ito, Kelley and Sachs [31].

Yet in spite of fifty years of research, no clear theory of continuous linear programming has emerged, no satisfactory algorithm has been found, and many questions remained unresolved, chief among them the relation of CLP to LP. Trivially, discretization of the time converts CLP (1), as well as SCLP (2) to an approximating LP. Pullan has used this discretization and a limiting argument in his proof of strong duality. Pullan's dual (3) follows directly from Lagrangian duality theory. However, it is not clear why π may require atomic jumps. Discretization is also the basis for Pullan's [44] convergent (in infinite number of steps) algorithm, as well as most other algorithms. Yet solving a discrete LP approximation has three major faults: (i) the LP problem is big, (ii) the solution is only approximate, and (iii) most important, the discretized solution introduces spurious details and obscures important features of the optimal solution. Finally, the question whether CLP and SCLP are more closely related to optimal control or to LP theory remained unanswered. Perhaps the main achievement of this paper is that we firmly anchor Continuous Linear Programming in LP theory.

We focus on the following problem:

$$\text{SCLP} \qquad \begin{aligned} &\max \int_0^T ((\gamma + (T-t)c)'u(t) + d'y(t))\,dt \\ &\text{s.t.} \int_0^t Gu(s)ds + Fy(t) + x(t) = \alpha + at, \\ &\qquad Hu(t) \qquad\qquad\qquad\qquad = b, \\ &\qquad x(t), u(t) \geq 0, \quad t \in [0,T]. \end{aligned} \tag{4}$$

Here G, H, F are fixed $K \times J$, $I \times J$, $K \times L$ matrices, and the remaining problem data consists of the K-vectors α, a, I-vector b, J-vectors γ, c and L vector d, and we look for controls $u_j(t)$, $j = 1, \ldots, J$, states $x_k(t)$, $k = 1, \ldots, K$, and supplementary states $y_l(t)$, $l = 1, \ldots, L$, as functions of time $0 \leq t \leq T$. In comparison to Anderson's formulation (2) we generalized by the addition of y, but restrict attention to linear functions $a(t), c(t)$ and constant $b(t), d(t)$. In the current paper we make assumption 1 which assures us that the problem has a measurable bounded solution $u(t), 0 < t < T$.

For this problem we develop a theory and algorithm which are entirely analogous to the theory of linear programming and the simplex algorithm (in parametric form). In particular we do:

- Formulate a symmetric dual problem.
- Find that a finite sequence of bases plays the role of a basis for SCLP.
- Define neighboring base-sequences through their validity regions.
- Construct an algorithm to pivot between neighboring base-sequences.
- Find an algorithm to solve SCLP by a sequence of pivots analogous to the parametric simplex algorithm of LP.

The rest of the paper is structured as follows: In Section 2 we motivate continuous linear programming through some applications. In Section 3 we define the main concepts, prove some of their properties and state the main results. In Section 4 we construct our algorithm and verify it, and complete proofs of results stated in Section 3. In Section 5 we illustrate the algorithm by an example. In Section 6 we discuss some insights of our algorithm and some forthcoming extensions.

2. Applications

2.1. Multiclass queueing networks

A multiclass queueing network is a system which consists of servers $i = 1, \ldots, I$, classes $k = 1, \ldots, K$ and processes $j = 1, \ldots, J$. The state of the system is given by the queue length vector $X_k(t)$ of the number of customers in class k. A process j is performed by server $i(j)$ on a customer in class $k(j)$. It requires a random processing time with an average m_j, and at its end the customer moves randomly to class l with probability p_{jl}, or leaves the system with probability $1 - \sum_l p_{jl}$. Customers arrive in the system in a random stream, with average rates a_k.

The MCQN operates as follows: whenever a server is free, it either idles, or it chooses a waiting customer from one of the non-empty classes and a process, and it then serves the customer until the service is complete, at which time the customer leaves the system or moves to another class to be waiting again, and the machine is now free.

Control of the system consists of rules for the choice of the class to serve (sequencing) and of the process to use (routing). This is a stochastic and discrete system, and one is interested in evaluating the performance of various strategies, or in finding optimal strategies for various objectives. In particular, one objective is to minimize waiting costs, where a customer in class k pays at a rate w_k for its wait.

A MCQN can be approximated by a multiclass fluid network. In this the buffer contents are denoted by $x_k(t)$, and are continuous real valued scaled versions of $X_k(t)$. The servers are infinitely divisible between various processes and buffers, and the processing of buffers is at a continuous rate, bounded by the constant $1/m_j$ for process j.

The MCFN optimization problem, for a finite time horizon T, is then the following SCLP: Find processing rates $u_j(t)$ such that:

$$\begin{aligned}
\max \int_0^T & (T-t)c'u(t)dt \\
\text{s.t.} \int_0^t & Gu(s)ds + x(t) = \alpha + at, \\
& Mu(t) \qquad \leq \mathbf{1}, \\
& x(t), u(t) \geq 0, \quad t \in [0,T].
\end{aligned}$$

where:

$$G_{lj} = \begin{cases} 1 & l = k(j) \\ -p_{jl} & l \neq k(j) \end{cases}, \qquad M_{lj} = \begin{cases} m_j & l = i(j) \\ 0 & l \neq i(j) \end{cases}$$

α are the initial buffer contents, and $c' = w'G$. The objective function is equivalent to $\min \int_0^T \sum w_k x_k(t)dt$, as is seen through substituting for x and integration by parts.

Special cases of this are:

A reentrant line, in which all customers move through the buffers in the sequence $k = 1, \ldots, K$, and machine i serves a subset of buffers $k \in C_i$. Here

$$G = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & -1 & 1 \end{bmatrix} \text{ and } M_{ik} = \begin{cases} m_k & k \in C_i \\ 0 & \text{else} \end{cases}.$$

A job shop, in which there are several fixed processing routes, such that jobs on route r move through buffers $k(r,1),\dots,k(r,K_r)$. Here G is of block diagonal form, where block r is that of a re-entrant line of size K_r.

A MCQN without routing, in which there is only one process for each buffer, so that $G = I - P'$ and $M_{ik} = \begin{cases} m_k & k \in C_i \\ 0 & \text{else} \end{cases}$.

Recently Dai [20], Meyn [37], and others have shown that fluid limit models play a major role in the study of stochastic MCQN, both for performance evaluation (including stability) and for optimization. See also recent work of Meyn, [38–40].

Some SCLP for special MCFN were solved by Avram, Bertsimas and Ricard [11] and Weiss [56–58]. The algorithm of the present paper generalizes earlier work of Weiss [56–58].

How to use the SCLP solution, the optimal solution of the MCFN, to control the actual queueing network is still an area of active research. See [14–16, 59, 21, 18].

2.2. Leontief Input Output Economic Systems

The first application of linear programming techniques in the field of economics was to the area of inter-industry or input-output analysis (Leontief [34], Koopmans [32], Dorfman, Samuelson and Solow [23]). These models motivated some of the research into continuous linear programming in the 60s and 70s.

Static Leontief Model Consider $k = 1,\dots,K$ industries, and assume that to produce one unit of product in industry k inputs of A_{kl} units from industry l are required. Then the production of $U = (U_1,\dots,U_K)'$ requires inputs AU, and so production of U yields a surplus of $X = (X_1,\dots,X_K)' = (I-A)U$. For a given demand for consumption a, and with some further limitations on production capacities, of the form $U \le b$, we have the static Leontief model:

$$\begin{aligned} (I-A)U &= a, \\ U &\le b, \\ U \ge 0 & \quad . \end{aligned}$$

More generally, a Leontief matrix G is an $m \times n$ matrix in which every column has at most one positive element (see [17, page 195, exercise 4.32]). For interpretation, each column corresponds to a production process. If G_{ij} is negative $|G_{ij}|$ is the amount of goods of type i consumed by the jth process. If G_{ij} is positive it is the amount of goods of type i produced by the process. In that case, a production vector $U = (U_1,\dots,U_K)'$ will yield a net surplus of $X = (X_1,\dots,X_K)' = G\,U$. Subject to some general capacity constraints one can pose the static Leontief linear program, of satisfying a demand a with minimal cost $c'U$:

$$\begin{aligned} \min \quad & c'U \\ & GU \geq a, \\ & HU \leq b, \\ & U \geq 0\ . \end{aligned}$$

Multi-Period Leontief Model An extension to the static Leontief model is the multiperiod Leontief model, in which the surplus produced in period t by production vector U_t is used (i) to satisfy the demand a_t (ii) create new production capacity, V_t, which uses up BV_t of the surplus (iii) augment the inventory from X_{t-1} to a new level X_t, where initially $X_0 = \alpha$. The problem now is to use the initial and the expanded capacity to produce enough surplus to satisfy the demand with minimal production and inventory costs:

$$\begin{aligned} \min & \sum_{s=1}^{T} c'U_s + w'X_s, \\ & X_0 = \alpha, \\ \text{s.t.}\ & GU_t = BV_t + X_t - X_{t-1} + a_t, \\ & HU_t \leq b + \sum_{s=1}^{t-1} V_s, \\ & X_t, U_t, V_t \geq 0, \quad t = 1, \dots, T. \end{aligned}$$

Continuous Time Leontief Model The multiperiod Leontief model can immediately be converted to a continuous time model, by letting the discrete time periods tend to zero: We let $u(t)$ be the vector of production rates at time t, so for time periods of length Δ we have $U_t = u(t)\Delta$. This yields a continuous linear program, but the capacity expansion introduces constraints which involve both $u(t)$ and $\int_0^t Bu(s)ds$, which are not separated. In the case of no capacity expansion and constant demand rate α, we get an SCLP with linear data,

$$\begin{aligned} \min & \int_0^T (c' + (T-t)w')u(s)ds, \\ \text{s.t.}\ & \int_0^t Gu(s)ds - X(t) = \alpha + a\,t, \\ & Hu(t) \leq b, \\ & X(t), u(t) \geq 0, \quad t = 1, \dots, T. \end{aligned}$$

2.3. Control of City-Wide Rush Hour Vehicle Traffic

Consider a network of main roads and junctions, and take a snapshot at 7:30 in the morning, with a count of the number of vehicles in each directed road section. Use a prediction of the input and output rates of vehicles which enter and which exit each directed road section from and to minor roads. Use a prediction of the routing proportions from each directed road section at the concluding junction. Approximate the vehicles on the road section by a fluid, and assume that fluid is moving from a road section through the concluding junction at a rate proportional to the fraction of time that the junction is open to that road section.

One can then pose the problem of emptying the rush hour traffic with minimum total waiting times as an SCLP. One complication which one may or may not wish to add to this model is a time delay for passage of vehicles along the road sections.

2.4. SCLP formulation of the general Linear Programming Problem

Consider the following general LP problem with bounded decision variables u:

$$\begin{aligned} \max\ & c'u \\ \text{s.t.}\ & Gu \le a, \\ & 0 \le u \le b. \end{aligned}$$

We formulate the SCLP:

$$\begin{aligned} \max & \int_0^T (T-t)c'u(t)dt \\ \text{s.t.} & \int_0^t Gu(s)ds + x(t) = \alpha + at, \\ & u(t) \le b, \\ & x(t), u(t) \ge 0, \quad t \in [0,T]. \end{aligned}$$

This is a special case of the general SCLP (4) in that H is an identity matrix. To formulate this SCLP we add to the LP data G, a, b, c, an arbitrary positive vector α of initial buffer levels $x(0)$.

Our SCLP algorithm will solve this problem for all $0 < T < \infty$. The solution will drain the initial fluid levels α out of x, until at time horizon $T^{(R)}$ all the buffers are empty, or have positive derivatives. The values of $u(t)$ for $t > T^R$ solve the LP.

3. Theory

3.1. Duality

We formulate a symmetric dual to the SCLP problem (4):

$$\text{SCLP}^* \qquad \begin{aligned} &\min \int_0^T ((\alpha + (T-t)a)'p(t) + b'r(t))dt \\ &\text{s.t.} \int_0^t G'p(s)ds + H'r(t) - q(t) = \gamma + ct, \\ &\quad F'p(t) \qquad\qquad\qquad\quad = d, \\ &\quad p(t), q(t) \geq 0, \quad t \in [0,T]. \end{aligned} \tag{5}$$

where we look for dual controls $p_k(t)$, $k = 1, \dots, K$, dual states $q_j(t)$, $j = 1, \dots, J$, and dual supplementary states $r_i(t)$, $i = 1, \dots, I$ as functions of time $0 \leq t \leq T$. Dual time runs backwards, with primal time t corresponding to dual time $T - t$.

It is immediate to show weak duality:

Proposition 1. *Weak duality holds, the objective value for any feasible solution of the maximization problem (4) is smaller than the objective value of any feasible solution of the minimization problem (5).*

Proof. Assume a pair of feasible solutions to the primal and the dual problem and compare their objective values:

$$\begin{aligned} &\text{Dual objective } = \int_0^T (\alpha + (T-t)a)'p(t)dt + \int_0^T b'r(t)dt \\ &\geq \int_0^T \left(\int_0^{T-t} u(s)'G'ds + y(T-t)'F' \right) p(t)dt + \int_0^T u(T-t)'H'r(t)dt \\ &= \int_0^T u(T-t)' \left(\int_0^t G'p(s)ds \right) dt + \int_0^T u(T-t)'H'r(t)dt + \int_0^T y(T-t)'F'p(t)dt \\ &\geq \int_0^T u(T-t)'(\gamma + ct)dt + \int_0^T y(T-t)'d\,dt = \text{Primal objective} \end{aligned} \tag{6}$$

where the first inequality follows from the primal constraints and from p nonnegative, the equality follows by change in the order of integration, and the second inequality follows from the dual constraints and from u nonnegative. □

Equality in the above will hold if and only if $\int_0^T x(T-t)'p(t)dt = 0$ and $\int_0^T u(T-t)'q(t)dt = 0$. This suggests the following definition:

Complementary Slackness Condition *For almost all t (i.e. excluding only a set of t of measure 0):*

$$\begin{aligned} u_j(t) > 0 &\Rightarrow q_j(T-t) = 0 \\ x_k(t) > 0 &\Rightarrow p_k(T-t) = 0 \end{aligned} \tag{7}$$

As a result of weak duality:

Corollary 1. *Consider a pair of feasible primal and dual solutions of SCLP, SCLP* (4,5). The following are equivalent:*

(i) The two solutions are complementary slack.

(ii) The two solutions are optimal and have the same objective value (no duality gap).

Our algorithm proves by construction the following:

Theorem 1. *Assume that the feasibility and boundedness assumption 1 holds. Then the SCLP SCLP* problems (4,5) possess complementary slack optimal primal and dual solutions (strong duality holds), with continuous piecewise linear* x, y, q, r*, and piecewise constant* u, p*.*

We shall postpone the proof of Theorem 1 to Section 4.6. In anticipation of this form of the solution, we note that it would be determined by the boundary values $x(0), y(0), q(0), r(0)$, and by the piecewise constant rates $u, \dot{x}, \dot{y}, p, \dot{q}, \dot{r}$ (where $\dot{}$ denotes derivative) for $0 < t < T$. In what follows the boundary-LP (8) will determine the boundary values, and the rates-LP (9) will determine the rates, where rates in different intervals of t will correspond to different sign restrictions on $\dot{x}, \dot{q}$. As we shall see in Section 3.2, the entire solution will be specified by a finite sequence of bases $B_1, \dots, B_N$ of the rates-LP (9).

We formulate the boundary-LP and boundary-LP*:

$$\begin{array}{ll} & \max\ d'y^0 \\ \text{Boundary-LP} & \text{s.t. } Fy^0 + x^0 = \alpha, \\ & \qquad x^0 \geq 0, \\ & \\ & \min\ b'r^N \\ \text{Boundary-LP}^* & \text{s.t. } H'r^N - q^N = \gamma, \\ & \qquad q^N \geq 0. \end{array} \tag{8}$$

We denote sign restrictions on a variable v by: “P” for non-negative, “U” for unrestricted, and “Z” for restricted to be 0. We formulate the rates-LP and its dual, the rates-LP*:

$$\begin{array}{ll} & \max\ c'u + d'\dot{y} \\ \text{Rates-LP} & \text{s.t. } Gu + F\dot{y} + \dot{x} = a, \\ & \qquad Hu = b, \\ & \qquad u \geq 0,\ \dot{y} \text{ is “U”}. \\ & \\ & \min\ a'p + b'\dot{r} \\ \text{Rates-LP}^* & \text{s.t. } G'p + H'\dot{r} - \dot{q} = c, \\ & \qquad F'p = d, \\ & \qquad p \geq 0,\ \dot{r} \text{ is “U”} \end{array} \tag{9}$$

The rates-LP and rates-LP* in (9) are not fully specified. In fact (9) represents a family of LP's, such that each interval of the solution will correspond to one of these LP's, which will be fully specified by the sign restrictions on $\dot{x}, \dot{q}$.

We shall assume throughout that $\begin{bmatrix} G & F & I \\ H & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} G' & H' & -I \\ F' & 0 & 0 \end{bmatrix}$ are full rank matrices. Throughout the paper we will mostly make the following two assumptions:

Assumption 1 (Feasibility and Boundedness) *(i) The problems (8) have a solution* x^0, q^N.
(ii) The primal rates-LP of (9) with the additional sign restrictions $\dot{x}_k \geq 0$, $k = 1, \dots, K$, $q_j^N > 0 \Rightarrow u_j = 0$, *and the dual rates-LP* of (9) with the additional sign restrictions* $\dot{q}_j \geq 0$, $j = 1, \dots, J$, $x_k^0 > 0 \Rightarrow p_k = 0$ *are both feasible.*

Our algorithm proves by construction the following:

Theorem 2. *Assumption 1 is sufficient for SCLP, SCLP* (4,5) to have solutions for all time horizons* $0 < T < \infty$.

We postpone the proof of Theorem 2 to Section 4.6.

Assumption 2 (Non-degeneracy) *The column* $\begin{bmatrix} a \\ b \end{bmatrix}$ *is in general position to the matrix* $\begin{bmatrix} G & F & I \\ H & 0 & 0 \end{bmatrix}$ *(it is not a linear combination of any less than* $K + I$ *columns), and the column* $\begin{bmatrix} c \\ d \end{bmatrix}$ *is in general position to the matrix* $\begin{bmatrix} G' & H' & -I \\ F' & 0 & 0 \end{bmatrix}$.

Our algorithm proves by construction the following:

Theorem 3. *Assumption 2 implies that SCLP, SCLP* (4,5) are non-degenerate in the sense that optimal* $u(t), x(t), y(t), p(t), q(t), r(t)$ *are uniquely determined for all* $0 < t < T$, *up to an arbitrary set of values of* t *with Lebesgue measure* 0.

We postpone the proof of Theorem 3 to Section 4.6.

3.2. Base Sequences

A basis B of the problem is a set of variables corresponding to $K+I$ independent columns of $\begin{bmatrix} G & F & I \\ H & 0 & 0 \end{bmatrix}$, which includes all of the variables $\dot{y}_l \in B$, $l = 1, \dots, L$. It is determined by the $K + I - L$ variables $\dot{x}_k, u_j \in B$. The corresponding complementary dual basis B^*, is the set of variables corresponding to $J + L$ independent columns of $\begin{bmatrix} G' & H' & -I \\ F' & 0 & 0 \end{bmatrix}$, which includes all of the variables $\dot{r}_i \in B^*$, $i = 1, \dots, I$, and the $J + L - I$ variables $\dot{q}_j, p_k \in B^*$ so as to satisfy $\dot{x}_k \in B \Rightarrow p_k \notin B^*$, and $\dot{q}_j \in B^* \Rightarrow u_j \notin B$ (it is well known from LP theory that the complementary dual set of columns is indeed independent).

We consider now a sequence of bases $B_1, \dots, B_N$. Their corresponding complementary dual bases are $B_1^*, \dots, B_N^*$. Let $u^n, \dot{x}^n, \dot{y}^n, p^n, \dot{q}^n, \dot{r}^n$ denote the values of the basic primal and dual solutions of (9) corresponding to the basis B_n and its complementary dual B_n^*.

We say that the basis B_n is admissible if u_j^n, p_k^n, $j = 1, \dots, J$, $k = 1, \dots, K$ are non-negative.

We say that the bases B_1, B_N are consistent with the boundary data α, γ if $x_k^0 > 0 \Rightarrow \dot{x}_k \in B_1$ and $q_j^N > 0 \Rightarrow u_j \notin B_N$.

We say that B_n, B_{n+1} are adjacent bases if $B_n \backslash B_{n+1}$ consists of a single variable v_n (and $B_{n+1} \backslash B_n$ consists of a single variable w_n). In that case we can go from B_n to B_{n+1} in a single pivot $B_n \to B_{n+1}$ in which v_n leaves the basis and w_n enters.

Consider now a sequence of adjacent bases, $B_1, \dots, B_N$. We write the following equations for unknown interval lengths τ_n, $n = 1, \dots, N$:

$$\tau_1 + \cdots + \tau_N = T, \tag{10}$$

and for $n = 1, \dots, N-1$:

$$\begin{aligned} \sum_{m=1}^{n} \dot{x}_k^m \tau_m &= -x_k^0, \qquad \text{when } v_n = \dot{x}_k, \\ \sum_{m=n+1}^{N} \dot{q}_j^m \tau_m &= -q_j^N, \qquad \text{when } v_n = u_j. \end{aligned} \tag{11}$$

We also write a set of inequalities:

$$\begin{aligned} \sum_{m=1}^{n} \dot{x}_k^m \tau_m &\geq -x_k^0, \qquad \text{whenever } \dot{x}_k^n < 0, \text{ and } \dot{x}_k^{n+1} > 0 \text{ or } n = N, \\ \sum_{m=n+1}^{N} \dot{q}_j^m \tau_m &\geq -q_j^N, \qquad \text{whenever } \dot{q}_j^n > 0 \text{ or } n = 0, \text{ and } \dot{q}_j^{n+1} < 0. \end{aligned} \tag{12}$$

The equations and inequalities (10–12) can be put together as:

$$\begin{bmatrix} 1 \dots 1 & 0 \\ \mathbf{A} & 0 \\ \mathbf{B} & -I \end{bmatrix} \begin{bmatrix} \tau \\ \sigma \end{bmatrix} = \begin{bmatrix} T \\ g \\ h \end{bmatrix}. \tag{13}$$

The coefficients of A, B are the appropriate values of the $\dot{x}_k^n, \dot{q}_j^n$, and the coefficients of g, h are the appropriate values of $-x_k^0, -q_j^N$. The components of σ are unknown slacks.

Theorem 4. *Consider the SCLP, SCLP* problems (4,5), and assume assumption 2. Let $B_1, \dots, B_N$ be a sequence of bases, which satisfy:*

(i) The bases $B_1, \dots, B_N$ are admissible and adjacent.
(ii) B_1, B_N are consistent with the boundary values.

(iii) The solution of the linear equations (13) is $\tau > 0, \sigma > 0$.

Let $t_0 = 0, t_n = t_{n-1} + \tau_n$, $n = 1, \dots, N$. Let $u(t) = u^n, \dot{x}(t) = \dot{x}^n, \dot{y}(t) = \dot{y}^n, p(T-t) = p^n, \dot{q}(T-t) = \dot{q}^n, \dot{r}(T-t) = \dot{r}^n$, $t_{n-1} < t < t_n$, $n = 1, \dots, N$. Let x^0, q^N be the solutions of (8), and let $x^n = x^{n-1} + \dot{x}^n \tau_n$, $y^n = y^{n-1} + \dot{y}^n \tau_n$, $n = 1, \dots, N$, $q^{n-1} = q^n + \dot{q}^n \tau_n$, $r^{n-1} = r^n + \dot{r}^n \tau_n$, $n = N, \dots, 1$. Let $x(t), y(t), q(T-t), r(T-t)$, for $t_{n-1} < t < t_n$, be linear interpolations between $x^{n-1}, y^{n-1}, q^{n-1}, r^{n-1}$ and x^n, y^n, q^n, r^n, $n = 1, \dots, N$.

Then u, x, y, p, q, r is an optimal solution.

Conversely, for almost all T, α, γ, if assumptions 1, 2 hold, problems (4,5) possess an optimal solution given by such a sequence of bases.

Proof. We postpone the proof of the converse part of the theorem to Section 4.6.

We prove the direct part. By (iii) $\tau > 0$ and by (10) they add up to T. Hence $0 = t_0 < t_1 < \dots < t_N = T$ is a partition of $[0, T]$, $u, \dot{x}, \dot{y}, p, \dot{q}, \dot{r}$ are well defined (at all but the breakpoints) piecewise constant and x, y, q, r are well defined continuous piecewise linear.

By Corollary 1, we need to show that u, x, y, p, q, r satisfy the equality constraints of (4,5), the sign constraints of (4,5), and the complementary slackness (7).

The constraints $Hu = b, F'p = d$ hold for all u^n, p^n. The integrated primal and dual constraints hold at $t = 0$ by (8), and hold for all $0 < t < T$ by integrating both sides of the constraints which involve $\dot{x}, \dot{q}$ in (9) from 0 to t.

Since $B_1, \dots, B_N$ are admissible, $u, p \geq 0$. Next we show that $x, q \geq 0$. Since they are continuous piecewise linear it is enough to check that this holds at the breakpoints. Indeed, it is enough to check that it holds at local minima. By (8), $x_k(0) = x_k(t_0) = x_k^0 \geq 0$ and $q_j(0) = q_j(T - t_N) = q_j^N \geq 0$. Consider x_k, and assume it has a strict local minimum at t_n, for some $0 < n \leq N$. Then $\dot{x}_k^n < 0, \dot{x}_k^{n+1} > 0$, or if $n = N$, $\dot{x}_k^N < 0$. But then, by (iii) $\sigma > 0$, and so the appropriate inequality (12) for k, n holds, and it says exactly that $x_k(t_n) > 0$. A similar argument shows that strict local minima of q_j at $T - t_n$, for some $0 \leq n < N$, are > 0. If x_k has a non-strict local minimum at t, then x_k is constant in a half neighborhood of t, and so $\dot{x}_k = 0$ in that half neighborhood. But as we shall show presently, $\dot{x}_k(s) = 0 \Rightarrow x_k(s) = 0$ at all but the breakpoints (see 14). Hence, by continuity, $x_k(t) = 0$. A similar argument holds for q, so at all non-strict local minima $x, q = 0$.

Finally we show complementary slackness at all but the breakpoints. This is where we need to use the non-degeneracy assumption 2, as a result of which the following strict complementary slackness holds, for all t except the breakpoints:

$$\begin{array}{llll} x_k(t) > 0 \;\Leftrightarrow\; \dot{x}_k(t) \neq 0 & \Leftrightarrow p_k(T-t) = 0, & k = 1, \dots, K, \\ q_j(T-t) > 0 \Leftrightarrow \dot{q}_j(T-t) \neq 0 \Leftrightarrow & u_j(t) = 0, & j = 1, \dots, J, \end{array} \tag{14}$$

To prove (14) we show first that for all t except the breakpoints, $x_k(t) \neq 0 \Rightarrow \dot{x}_k(t) \neq 0$. Assume that $x_k(t) \neq 0$, where $t_n < t < t_{n+1}$. If $x_k(t_n) = 0$, then $x_k(t) \neq 0$ implies by the continuous linearity of x_k that the slope $\dot{x}_k(t) \neq 0$.

If $x_k(t_n) \neq 0$ we proceed by induction on n. For $n = 0$, $x_k(0) > 0$ (it is ≥ 0 by (8)) implies by (ii) that $\dot{x}_k \in B_1$. Hence, by non-degeneracy, $\dot{x}_k(t) \neq 0$. For $n > 1$, if $x_k(t_n) \neq 0$ then by continuity $x_k(s) \neq 0$ for some $t_{n-1} < s < t_n$, which by the induction hypothesis implies $\dot{x}_k(s) \neq 0$, hence $\dot{x}_k \in B_n$. Assume that $\dot{x}_k \notin B_{n+1}$, then $\dot{x}_k$ leaves the basis in the pivot $B_n \to B_{n+1}$. But the corresponding equation (11) for n says in that case that $x_k(t_n) = 0$, which is a contradiction. Hence, if $x_k(t_n) \neq 0$ then $\dot{x}_k \in B_{n+1}$, and so $\dot{x}_k(t) \neq 0$ by non-degeneracy. A similar argument shows that for all t except the breakpoints, $q_j(T-t) \neq 0 \Rightarrow \dot{q}_j(T-t) \neq 0$.

The rest of the proof of (14) is immediate: Assumption 2 implies $\dot{x}_k(t) \neq 0 \Leftrightarrow p_k(T-t) = 0$ and $\dot{q}_j(T-t) \neq 0 \Leftrightarrow u_j(t) = 0$ for all but the breakpoints. Clearly also at all but the breakpoints $\dot{x}_k(t) \neq 0 \Rightarrow x_k(t) > 0$, since we showed that $x_k \geq 0$, so if it has non-zero slope inside an interval, then it must be > 0. □

We shall refer to a sequence of bases $B_1, \dots, B_N$ which satisfies the assumptions of Theorem 4 as *an optimal base-sequence*, and we refer to the solution u, x, y, p, q, r constructed from it as its solution.

Some important properties of the solutions, which are also used by the algorithm, follow almost immediately from Theorem 4. In the following Corollaries 2–5 we assume that $B_1, \dots, B_N$ is an optimal base-sequence (so it satisfies all the assumptions of Theorem 4) and we assume the non-degeneracy assumption 2.

Corollary 2. *$u^n, \dot{x}^n, \dot{y}^n, p^n, \dot{q}^n, \dot{r}^n$ are optimal primal dual complementary slack solutions of the rates-LP and dual rates-LP* (9), with the additional sign restrictions:*

$$\begin{aligned} &\text{if } x_k^{n-1} = 0 \text{ then } \dot{x}_k \text{ is "P"}, \quad \text{if } x_k^{n-1} > 0 \text{ then } p_k \text{ is "Z"}, \\ &\text{if } q_j^n > 0 \text{ then } u_j \text{ is "Z"}, \quad \text{if } q_j^n = 0 \text{ then } \dot{q}_j \text{ is "P"}. \end{aligned} \tag{15}$$

Proof. With these sign restrictions B_n, B_n^* are complementary slack feasible primal and dual bases, hence optimal. □

Corollary 3. *The value of the objective function of the rates-LP (9), $V_n = c'u^n + d'\dot{y}^n$ is strictly decreasing in n.*

Proof. The rates-LP of interval $n+1$ is more restricted than that of interval n: If $\dot{x}_k$ leaves the basis in $B_n \to B_{n+1}$ then $\dot{x}_k$ is "U" in interval n and is "P" in interval $n+1$. If u_j leaves the basis in $B_n \to B_{n+1}$ then u_j is "P" in interval n and is "Z" in interval $n+1$. Furthermore, the bases are different. By non-degeneracy, the objective of the more restricted problem is strictly smaller. □

Corollary 4. *The solution is unique, up to a set of t of measure* 0.

Proof. Since $B_1, \dots, B_N$ is an optimal base-sequence, its solution u, x, y, p, q, r (with continuous piecewise linear x, y, q, r defined for all $0 < t < T$, and with piecewise constant u, p defined for all $0 < t < T$ except the breakpoints) is optimal with no duality gap. Given the solution p, q, r to the dual problem SCLP*, by Corollary 1 any optimal primal solution $\tilde{u}, \tilde{x}, \tilde{y}$ has to be complementary slack with p, q, r, as in (7).

Consider then values of t in the interval $t_{n-1} < t < t_n$. If any of $u_j(t), x_k(t) = 0$, then by the non-degeneracy assumption the corresponding $q_j(T-t), p_k(T-t) \neq 0$, and hence, by the complementary slackness condition (7), for all t in the interval except for a set of t of measure 0, the corresponding $\tilde{u}_j(t), \tilde{x}_k(t) = 0$. Call the points t which are not exceptional, regular points. Note first that for all t in the interval, $\int_{t_{n-1}}^t \tilde{u}_j(s)ds = 0$ whenever $u_j \notin B_n$. Also, for regular t, only j, k for which $u_j, \dot{x}_k \in B_n$ can have $\tilde{u}_j(t), \tilde{x}_k(t) \neq 0$, so only $K + I - L$ of the $\tilde{u}_j, \tilde{x}_k$ are non-zero. But in that case, the equations:

$$\begin{aligned} G\int_{t_{n-1}}^t u(s)ds + Fy(t) + x(t) &= \alpha + a(t - t_{n-1}), \\ H\int_{t_{n-1}}^t u(s)ds \qquad\qquad\qquad &= b(t - t_{n-1}), \end{aligned}$$

have a unique solution, so for all regular t:

$$\int_{t_{n-1}}^t \tilde{u}(s)ds = \int_{t_{n-1}}^t u(s)ds, \quad \tilde{y}(t) = y(t), \quad \tilde{x}(t) = x(t).$$

In fact, by absolute continuity,

$$\int_{t_{n-1}}^t \tilde{u}(s)ds = \int_{t_{n-1}}^t u(s)ds \tag{16}$$

holds for all t, and therefore $\tilde{u}(t) = u(t)$ for almost all t. This proves that u, x, y are unique for almost all t.

The uniqueness of p, q, r for almost all t is analogous.

Note that we can choose arbitrary values for u, p on an arbitrary set of t of measure 0, without affecting (16). Also, on an arbitrary set of measure 0 we can replace the basic columns $\dot{x}_k, \dot{y}_l$ of B_n by any solution of:

$$F\tilde{y}(t) + \tilde{x}(t) = \alpha + a(t - t_{n-1}) - G\int_{t_{n-1}}^t u(s)ds, \quad x(t) \geq 0$$

Hence the solution is indeed only determined for almost all t. Of course, u, x, y, p, q, r is the unique solution with continuous linear x, y, q, r and piecewise constant u, p.
□

Corollary 5. *The matrix* $\begin{bmatrix} 1 \dots 1 \\ \mathbf{A} \end{bmatrix}$ *is invertible.*

Proof. Assume to the contrary that it is not. Since the equations

$$\begin{bmatrix} 1 \dots 1 \\ \mathbf{A} \end{bmatrix} \tau = \begin{bmatrix} T \\ g \end{bmatrix}$$

have a solution $\tau > 0$, this solution is not unique. Let τ_1 be a different solution. Then $\tau' = (1-\alpha)\tau + \alpha\tau_1$ is also a solution, and for α small enough this new solution is > 0. Furthermore, the values of the slacks σ' corresponding to τ' will also be > 0 if α is small enough. Given the new values τ', we can get a new set of breakpoints, $0 = t'_0 < \ldots < t'_N = T$, and a solution u', x', y', p', q', r' of (4,5). But this solution differs from the original solution u, x, y, p, q, r on some intervals (in particular on the parts of the intervals (t_{n-1}, t_n) and (t'_{n-1}, t'_n) which do not overlap), which contradicts the uniqueness Corollary 4. □

3.3. Parametric Validity Regions

The data of our SCLP, SCLP* problems (4,5) can be classified as follows:

(i) G, H, F define the model, and are the *system parameters*, which we shall keep fixed throughout.
(ii) a, b, c, d are (together with G, H, F) the data of the rates-LP (9), and we refer to them as the *exogenous rates*. Perturbation of these will resolve degeneracies (see assumption 2, and Propositions 9,10).
(iii) α, γ are (together with H, F, b, d) the data of the boundary-LP (8), and we refer to them as the *boundary parameters*.
(iv) T is the *time horizon*.

Proposition 2. *Under the non-degeneracy assumption 2:*

(i) Boundary parameters α, γ uniquely determine the boundary values x^0, q^N.
(ii) To boundary values x^0, q^N there corresponds an L dimensional affine-subspace of vectors α, and an I dimensional affine-subspace of vectors γ.

Proof. Consider the boundary-LP from (8) and its dual:

$$\begin{array}{l} \max\ d'y^0 \\ \text{s.t. } Fy^0 + x^0 = \alpha, \quad x^0 \geq 0, \\ \\ \min\ \alpha' p \\ \text{s.t. } F'p = d, \quad p \geq 0. \end{array} \tag{17}$$

To show (i) note that by assumption 2, d is in general position to F', hence the dual to the boundary-LP is non-degenerate, and hence the solution of the boundary-LP, x^0, y^0 is unique. Similarly, q^N, r^N are unique.

To show (ii), assume that x^0, p, are part of the optimal primal and dual solutions of (17) for some α. Then for the choice $\alpha = x^0$, the values p, x^0 and

$y^0 = 0$ are complementary slack feasible primal and dual solutions and hence optimal.

Furthermore, for any y^0 in L-dimensional space, p, x^0, y^0 are optimal for the choice $\alpha = x^0 + Fy^0$. Since F is full rank, this gives an L-dimensional affine-subspace.

Similarly, $\gamma = q^N + H'r^N$ for arbitrary r^N is a an I-dimensional affine-subspace of γ corresponding to q^N. □

Note Solutions with the same x^0, q^N and different α, γ will differ in the values of $y(0), r(0)$ and the resulting values of the objectives, but will have the same $u(t), p(t), x(t), q(t), \dot{y}(t), \dot{r}(t)$, $0 < t < T$. Given x^0, q^N we can always make the 'canonical choice' $\alpha = x^0, \gamma = q^N$, with $y(0), r(0) = 0$.

Theorem 5. *Assume the non-degeneracy assumption 2. Consider a sequence of admissible adjacent bases $B_1, \dots, B_N$. Let $(T, x^0, q^N) \in \mathcal{T}$ be the set of boundary values for which $B_1, \dots, B_N$ is optimal. Then $\mathcal{T}$ is a convex polyhedral cone.*

Proof. Since $B_1, \dots, B_N$ are admissible and adjacent, condition (i) of Theorem 4 holds.

The base-sequence $B_1, \dots, B_N$ (for the fixed values of G, H, F, a, b, c, d) fully determines (irrespective of the values of T, x^0, q^N) all the coefficients of the matrix $\begin{bmatrix} 1 \dots 1 & 0 \\ \mathbf{A} & 0 \\ \mathbf{B} & -I \end{bmatrix}$.

The base-sequence $B_1, \dots, B_N$ also fully determines a matrix E defined as follows: E has as many rows as $\begin{bmatrix} 1 \dots 1 & 0 \\ \mathbf{A} & 0 \\ \mathbf{B} & -I \end{bmatrix}$, and has $1 + K + J$ columns. Each row of E consists of 0's and a single ± 1 entry (we use e_i to denote the ith unit vector). The first row is e_1. For the next $N - 1$ rows, if $\dot{x}_k$ (u_j) leaves the basis in the pivot $B_n \to B_{n+1}$ then the $1 + n$ row is $-e_{1+k}$ ($-e_{1+K+j}$). For the last M rows, if the slack σ_m is a strict local minimum of x_k (q_j) at some time t_n, then the $1 + N + m$ row is $-e_{1+k}$ ($-e_{1+K+j}$).

If $B_1, \dots, B_N$ is not an optimal base-sequence for any (T, x^0, q^N), then $\mathcal{T} = \emptyset$ and there is nothing to prove.

Otherwise, if $B_1, \dots, B_N$ is an optimal base-sequence for at least one (T, x^0, q^N), then by Corollary 5, $\begin{bmatrix} 1 \dots 1 \\ \mathbf{A} \end{bmatrix}$ is invertible, and hence so is $\begin{bmatrix} 1 \dots 1 & 0 \\ \mathbf{A} & 0 \\ \mathbf{B} & -I \end{bmatrix}$.

To verify that $B_1, \dots, B_N$ is an optimal base-sequence for (T, x^0, q^N) it remains to verify conditions (ii), (iii) of Theorem 4.

Condition (ii) holds if and only if

$$\begin{aligned} &x_k^0 = 0 \text{ for all } k \text{ such that } \dot{x}_k \notin B_1, \\ &q_j^N = 0 \text{ for all } j \text{ such that } \dot{q}_j \notin B_N^* \quad (\text{or } u_j \in B_N). \end{aligned} \tag{18}$$

Condition (iii) holds if and only if

$$\begin{bmatrix} \tau \\ \sigma \end{bmatrix} = \begin{bmatrix} 1\dots 1 & 0 \\ \mathbf{A} & 0 \\ \mathbf{B} & -I \end{bmatrix}^{-1} E \begin{bmatrix} T \\ x^0 \\ q^N \end{bmatrix} > 0. \tag{19}$$

These two sets of constraints (18), (19) determine the convex polyhedral cone $\mathcal{T}$. □

We refer to $\mathcal{T}$ as the *validity region of the base-sequence* $B_1, \dots, B_N$. An immediate consequence of the convexity, which we shall exploit in the algorithm is:

Corollary 6. *If $B_1, \dots, B_N$ is the optimal base-sequence for T_1, x_1^0, q_1^N and for T_2, x_2^0, q_2^N then it is the optimal base-sequence for $(1-\theta)(T_1, x_1^0, q_1^N)+\theta(T_2, x_2^0, q_2^N)$, for all $\underline{\theta} < \theta < \overline{\theta}$, where $\underline{\theta} \leq 0$ and $\overline{\theta} \geq 1$.*

We shall also make use of the following:

Corollary 7. *Let $\ell(\theta) = (T, x^0, q^N) + \theta(\delta T, \delta x^0, \delta q^N)$ be a line of boundary values, and $T(\theta) = T + \theta\delta T$. As θ changes, within the validity region of a single base-sequence, each of the interval lengths and slacks τ_n, σ_m, and each of the $x_k(t_n)$, $q_j(T(\theta) - t_n)$ are affine functions of θ.*

Proof. We have that $\ell(\theta)$ is an affine function of θ. The values of τ, σ by equation (19) are affine functions of $\ell(\theta)$, and hence of θ. Finally,

$$\begin{aligned} x_k(t_n) &= x_k^0 + \sum_{m=1}^{n} \dot{x}_k^m \tau_m, \\ q_j(T(\theta) - t_n) &= q_j^N + \sum_{m=n+1}^{N} \dot{q}_j^m \tau_m, \end{aligned} \tag{20}$$

where the coefficients $\dot{x}_k^m, \dot{q}_j^m$ are fixed for all θ in the validity region. Since x_k^0, q_j^N and τ_m are all affine functions of θ, so are $x_k(t_n), q_j(T(\theta) - t_n)$. □

The following Theorem 6 and its Corollary 8 will follow by construction from the algorithm, and they indicate how to construct the algorithm.

Theorem 6. *Assume the non-degeneracy assumption 2. Let $\mathcal{C}$ be the region of boundary values T, x^0, q^N for which the feasibility and boundedness assumption 1 holds. Then $\mathcal{C}$ is tiled by closures of validity regions of admissible adjacent base-sequences.*

Two base-sequences whose validity regions have intersecting boundaries are called *neighboring base-sequences*. Theorem 6 says we can move from any point to any other point in $\mathcal{C}$ through validity regions of neighboring base-sequences. Specifically, moving in a straight line, this implies:

Corollary 8. *Assume the non-degeneracy assumption 2. Consider boundary values T_1, x_1^0, q_1^N and T_2, x_2^0, q_2^N, and assume that at both of these the feasibility and boundedness assumption 1 holds. If T_1, x_1^0, q_1^N and T_2, x_2^0, q_2^N are in the interior of the validity region of some base-sequences, then there exists a partition $0 = \theta^{(0)} < \theta^{(1)} < \ldots < \theta^{(R)} = 1$ such that for each of the intervals $\theta^{(r-1)} < \theta < \theta^{(r)}$ there is a base-sequence $B_1^{(r)}, \ldots, B_{N_r}^{(r)}$ which is optimal for the boundary values $(1-\theta)(T_1, x_1^0, q_1^N) + \theta(T_2, x_2^0, q_2^N)$.*

We postpone the proof of Theorem 6 and Corollary 8 to Section 4.6.

This is exactly how our algorithm works: To solve the problem for boundary-data T_2, x_2^0, q_2^N start from an optimal solution for boundary-data T_1, x_1^0, q_1^N, given by optimal base-sequence $B_1^{(1)}, \ldots, B_{N_1}^{(1)}$, which is also valid for some small $\theta > 0$. Then solve the problem for all values of the parameter $0 < \theta < 1$. Whenever a θ-breakpoint, $\theta^{(r)}$ is reached, the optimal base-sequence has to be changed. Going from base-sequence $B_1^{(r)}, \ldots, B_{N_r}^{(r)}$ to its neighboring base-sequence $B_1^{(r+1)}, \ldots, B_{N_r+1}^{(r+1)}$ is analogous to pivoting in the simplex algorithm of LP. We describe the algorithm of the pivot operation in Section 4.1, and verify it in Section 4.3.

In particular, we can use fixed values of α, γ and solve the problem parametrically with the time horizon T as parameter, where $0 < T < \infty$. In that case, we start from the solution for small $T > 0$, which is given by a single basis, and iterate (using a pivot operation) over time horizon breakpoints $T^{(r)}$. Under assumption 1 this will terminate with a base-sequence $B_1^{(R+1)}, \ldots, B_{N_R+1}^{(R+1)}$ which is optimal for all $T^{(R)} < T < \infty$; see Section 4.4. We also need to solve subproblems, which may arise in the pivot operations; these are discussed in Section 4.5.

4. Algorithm

4.1. Pivoting Between Neighboring Base-Sequences.

Let $B_1, \ldots, B_N$ be a base-sequence with validity region $\mathcal{T}$. Let $\ell(\theta) = (T, x^0, q^N) + \theta(\delta T, \delta x^0, \delta q^N)$ be a line of boundary values, such that components of $\ell(\theta)$ do not change sign in a neighborhood of $\theta = 0$, and such that $\ell(\theta) \in \mathcal{T}$ for a left neighborhood of $\theta < 0$, but $\ell(\theta) \notin \mathcal{T}$ for $\theta > 0$. This means that T, x^0, q^N is on the validity region boundary. It follows from the conditions of Theorem 4, that in the linear equations (13) formulated with $B_1, \ldots, B_N$, some components in the solution $\tau, \sigma \searrow 0$ as $\theta \nearrow 0$. We call this a collision, specifically we say that a collision occurs at t_n if, as $\theta \nearrow 0$, an interval or a slack shrinks to 0 at the breakpoint t_n. We also use the notation $T(\theta) = T + \theta\delta T$.

We now construct the optimal base-sequence for $\theta > 0$. We call this a pivoting operation, because, analogous to pivot in linear programming, it moves us from one extreme point to a neighboring one. To construct the new base-sequence we use the old base-sequence $B_1, \ldots, B_N$ and, according to the type of collision, we delete 0, 1 or > 1 bases from the old sequence, and insert in their place 0, 1 or > 1 new bases. We describe the steps of the pivot operation. We refer to a series of points which we take up in Section 4.3, in order to verify the pivot operation.

Classification of collisions As $\theta \nearrow 0$, two state variables hit zero at t_n (or one hits zero at 0 or T). The various cases are (see Point 4.3.1):

Case i: At collision time t_n intervals between bases B', B'' shrink to 0, B' and B'' are adjacent, and exactly one of the x_k, q_j has a strict local minimum of 0 at t_n.
Case i$_a$: At collision time $t_0 = 0$ the intervals preceding the basis B'' shrink to 0 and exactly one of the q_j has a strict local minimum of 0 at t_0.
Case i$_b$: At collision time $t_N = T$ the intervals succeeding the basis B' shrink to 0 and exactly one of the x_k has a strict local minimum of 0 at t_N.
Case ii: At collision time t_n intervals between bases B', B'' shrink to 0, and $B' \backslash B'' = \{v', v''\}$ where in the optimal solution for $\theta < 0$, between B', B'', v' leaves the basis before v'' (see definition below, (21)).
Case iii: At collision time t_n a slack value of one of the σ hits 0, where $0 < n < N$ and in the pivot $B_n \to B_{n+1}$ v leaves the basis. If the slack that hits 0 is $x_k(t_n) = 0$, let $v' = v$ and $v'' = \dot{x}_k$. If the slack that hits 0 is $q_j(T - t_n) = 0$, let $v' = u_j$ and $v'' = v$.
Case iii$_a$: At collision time t_0 a slack value of one of the σ hits 0, and the slack that hits 0 is $q_j(T) = 0$. In that case set $v' = u_j$.
Case iii$_b$: At collision time t_N a slack value of one of the σ hits 0, and the slack that hits 0 is $x_k(T) = 0$. In that case set $v'' = \dot{x}_k$
Other Cases: In any other case, perturbation of the problem will result in one of the Cases i, i$_a$, i$_b$, ii, iii, iii$_a$, iii$_b$. See Point 4.3.3.

We define v' *leaves the basis before* v'' as follows (see Point 4.3.2): Consider the solution for $\theta < 0$. Let t', t'' be the beginning of the interval of the basis B' and the end of the interval of the basis B'' respectively. Depending on the various choices for v', v'':

$$\begin{aligned}
&\text{If } v' = \dot{x}_{\prime},\ v'' = \dot{x}_{\prime\prime} \text{ then } \frac{x_{\prime\prime}(t')}{-\dot{x}_{\prime\prime}^{B'}} \Big/ \frac{x_{\prime}(t')}{-\dot{x}_{\prime}^{B'}} > 1, \\
&\text{if } v' = u_{\prime},\ v'' = u_{\prime\prime} \text{ then } \frac{q_{\prime}(T(\theta) - t'')}{-\dot{q}_{\prime}^{B''^*}} \Big/ \frac{q_{\prime\prime}(T(\theta) - t'')}{-\dot{q}_{\prime\prime}^{B''^*}} > 1, \\
&\text{if } v' = u_{\prime},\ v'' = \dot{x}_{\prime\prime} \text{ then } \frac{x_{\prime\prime}(t')}{-\dot{x}_{\prime\prime}^{B'}} + \frac{q_{\prime}(T(\theta) - t'')}{-\dot{q}_{\prime}^{B''^*}} \Big/ t'' - t' > 1, \\
&\text{if } v' = \dot{x}_{\prime},\ v'' = u_{\prime\prime} \text{ then } t'' - t' \Big/ \frac{x_{\prime}(t')}{-\dot{x}_{\prime}^{B'}} + \frac{q_{\prime\prime}(T(\theta) - t'')}{-\dot{q}_{\prime\prime}^{B''^*}} > 1.
\end{aligned} \tag{21}$$

Here we let $^{B'}$, $^{B''^*}$ denote the values of the variables in the primal basis B' and the dual basis B''^*.

Deletion of old bases In Cases i,i$_a$,i$_b$,ii, delete from the base-sequence those bases whose intervals have shrunk to zero.

Formulation and solution of LP In the Cases ii, iii, formulate rates-LP,LP* with the following sign restrictions:

$$\begin{aligned} \dot{x}_k \in B'\backslash\{v''\} &\Rightarrow \dot{x}_k \text{ is "U"}, \quad p_k \text{ is "Z"}, \\ \text{else } &\Rightarrow \dot{x}_k \text{ is "P"}, \quad p_k \text{ is "P"}, \\ u_j \in \overline{B''}\backslash\{v'\} &\Rightarrow u_j \text{ is "Z"}, \quad \dot{q}_j \text{ is "U"}, \\ \text{else } &\Rightarrow u_j \text{ is "P"}, \quad \dot{q}_j \text{ is "P"}, \end{aligned} \tag{22}$$

In Case iii$_a$, formulate rates-LP,LP* with the following sign restrictions:

$$\begin{aligned} x_k^0 > 0 &\Rightarrow \dot{x}_k \text{ is "U"}, \quad p_k \text{ is "Z"}, \\ \text{else } &\Rightarrow \dot{x}_k \text{ is "P"}, \quad p_k \text{ is "P"}, \\ u_j \in \overline{B''}\backslash\{v'\} &\Rightarrow u_j \text{ is "Z"}, \quad \dot{q}_j \text{ is "U"}, \\ \text{else } &\Rightarrow u_j \text{ is "P"}, \quad \dot{q}_j \text{ is "P"}, \end{aligned} \tag{23}$$

In Case iii$_b$, formulate rates-LP,LP* with the following sign restrictions:

$$\begin{aligned} \dot{x}_k \in B'\backslash\{v''\} &\Rightarrow \dot{x}_k \text{ is "U"}, \quad p_k \text{ is "Z"}, \\ \text{else } &\Rightarrow \dot{x}_k \text{ is "P"}, \quad p_k \text{ is "P"}, \\ q_j^N > 0 &\Rightarrow u_j \text{ is "Z"}, \quad \dot{q}_j \text{ is "U"}, \\ \text{else } &\Rightarrow u_j \text{ is "P"}, \quad \dot{q}_j \text{ is "P"}. \end{aligned} \tag{24}$$

In the Cases ii, iii, iii$_b$, the rates-LP (22,24) can be solved from initial basis B' by the dual simplex method, where initially only the variable v'' violates the sign restrictions, and leaves the basis in the first pivot operation.

In the Cases ii, iii, iii$_a$, the rates-LP (22,23) can be solved from initial dual basis B''^* by the primal simplex method, where initially only the variable v'^* violates the dual sign restrictions, and leaves the dual basis in the first pivot operation.

Let D be the basis of the optimal solution, see Point 4.3.4.

Formulation of subproblem If D is adjacent to B' and B'', (to B'' in Case iii$_a$, to B' in Case iii$_b$) we say that this is a simple pivot. If D is not adjacent to one or to both of B', B'' we need to solve a subproblem. We describe the formulation and solution of subproblems in Section 4.5, see also Point 4.3.5. The solution of the subproblem is a sequence of adjacent admissible bases $B', D_1, \ldots, D_M, B''$ ($D_1, \ldots, D_M, B''$ in Case iii$_a$, $B', D_1, \ldots, D_M$ in Case iii$_b$), in which v'' leaves the basis before v'.

Insertion of new bases In Cases i,i$_a$,i$_b$ no bases need to be inserted. In Cases ii,iii insert D or $D_1, \ldots, D_M$ between B', B''. In Case iii$_a$ insert D or $D_1, \ldots, D_M$ before B''. In Case iii$_b$ insert D or $D_1, \ldots, D_M$ after B' (see Point 4.3.6).

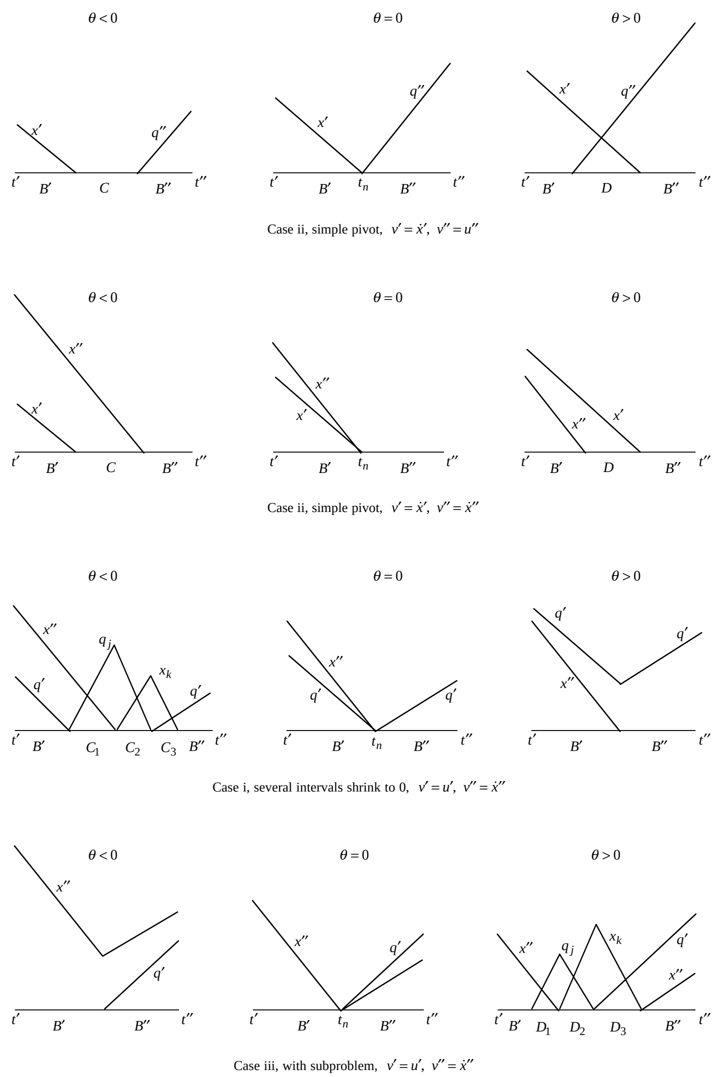

Fig. 1. Illustrations of several cases of pivot operations at $0 < t_n < T$

4.2. Illustration of the pivoting operation

Before we go into the formal proofs, the following gallery of illustrations, in Figures 1, 2 describe various cases of the pivoting operations. For the rate variables involved in the collision, v', v'', we denote by z', z'' the corresponding primal or dual state variables, the correspondence being $z = x_k, q_j$ if $v = \dot{x}_k, u_j$. In each figure we plot the relevant state variables, including the state variables z', z'', and others, as a function of time, in the vicinity of the collision time t_n. Each illustration of a particular pivot operation illustrates a different case, and con-

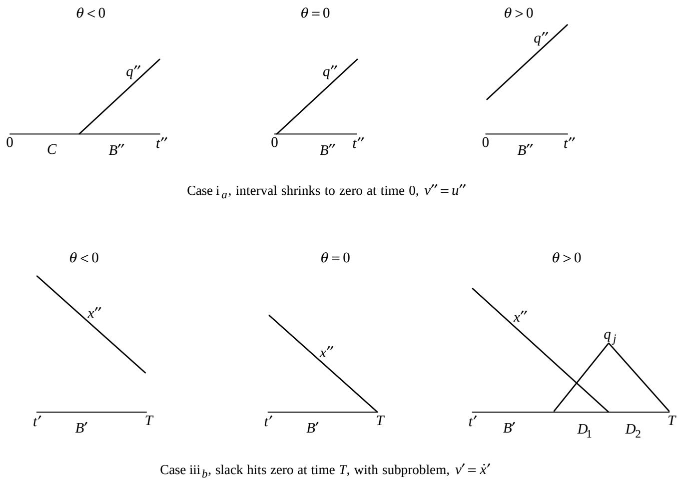

Fig. 2. Illustrations of several cases of pivot operations at 0 or T

sists of three parts: $\theta < 0$ is a picture of the solution in the vicinity of t_n with the old base-sequence, before the collision, $\theta = 0$ is the solution at the collision after some intervals or slacks disappeared, and $\theta > 0$ is the solution with the new base-sequence, after the collision. Note the symmetry: If we started from the new bases sequence and let θ decrease we would do the reversed pivot and end up with the old sequence. In the reverse pivot the roles of v', v'' are exchanged. The reversed pivot for Case i is Case iii, and vice versa, while the reverse pivot for Case ii is Case ii.

4.3. Verification of the pivoting operation

We need to do the following things to verify the algorithm:

- Show that a perturbation will lead to one of the classified collision cases, Sections 4.3.1, 4.3.3.
- Explain the definition (21), Section 4.3.2.
- Show that the formulated rates-LP has a solution, Section 4.3.4.
- Describe formulation and solution of subproblems. We postpone this to Section 4.5, but see first Section 4.3.5.
- Show that the new base-sequence is optimal for $\theta > 0$, Section 4.3.6.

4.3.1. Classification of Collisions As $\theta \nearrow 0$ some of the $\tau, \sigma \searrow 0$ and a collision occurs. We study such collisions now. We start with a definition of a *single collision*.

Consider the solution at $\theta = 0$, with a collision at t_n. For $0 < t_n < T$, let B', B'' be the bases of the non-zero intervals preceding and following t_n. For $t_n = 0$ let B'' be the non-zero interval following t_n. For $t_n = T$, let B' be the non-zero interval preceding t_n. Note, when M intervals shrink to zero all $M+1$ breakpoints at their ends converge to t_n. We will denote the beginning of the interval of B' and the end of the interval of B'' by t', t'' respectively.

We say that x_k hits zero at t_n, if $\dot{x}_k \in B'$ and $x_k(t_n) = 0$. We say that q_j hits zero at t_n, if $\dot{q}_j \in B''^*$ and $q_j(T - t_n) = 0$.

If x_k hits zero at t_n then $x_k(t) > 0, t' < t < t_n$ and decreases to 0, and the value of $\dot{x}_k$ in the basic solution B' is $\dot{x}_k^{B'} < 0$. Note that x_k cannot hit zero at $t_n = 0$. If $0 < t_n < T$, then either $\dot{x}_k \in B' \backslash B''$, or else $\dot{x}_k \in B' \cap B''$ and x_k has a strict local minimum of 0 at t_n.

If q_j hits zero at t_n then $q_j(T - t) > 0, t_n < t < t''$ and decreases to 0, and the value of $\dot{q}_j$ in the dual basic solution B''^* is $\dot{q}_j^{B''^*} < 0$. Note that q_j cannot hit zero at $t_n = T$. If $0 < t_n < T$, then either $u_j \in B' \backslash B''$, or else $u_j \notin B' \cup B''$ and q_j has a strict local minimum of 0 at $T - t_n$.

Recall our assumption that x^0, q^N do not change sign in neighborhood of $\theta = 0$. If we would allow x_k^0 to decrease to 0, or to start increasing from 0 this would constitute x_k hitting 0 at t_0. Similarly, if we would allow q_j^N to decrease to 0, or to start increasing from 0 this would constitute q_j hitting 0 at T_N.

A single collision occurs at $\theta = 0$ if exactly one of the following happens at a single time t_n: Two of the x_k, q_j hit zero at t_n $(0 < t_n < T)$, or one of the q_j hits zero at $t_n = 0$, or one of the x_k hits zero at $t_n = T$.

A multiple collision occurs in all other cases.

Proposition 3. *Cases i,i$_a$,i$_b$,ii,iii,iii$_a$,iii$_b$ cover all the cases in which a single collision occur as* $\theta \nearrow 0$.

Proof. We first see that if a single slack shrinks to zero, or if a single interval shrinks to zero, this is always a case of a single collision, and it always falls within one of the Cases i,i$_a$,i$_b$,ii,iii,iii$_a$,iii$_b$.

Single slack shrinks to zero: Assume one of the slacks, $\sigma_m = x_k(t_n)$ or $\sigma_m = q_j(T(\theta) - t_n) \searrow 0$ as $\theta \nearrow 0$. This is a collision at t_n. If $0 < n < N$, let B', B'' be the bases before and after t_n, and let $v = B' \backslash B''$. Then the state variables corresponding to σ_m and to v hit zero at t_n, and this is a single collision; it is Case iii. If $n = 0$, $\sigma_m = q_j(T)$, or if $n = N$, $\sigma_m = x_k(T)$, the state variable corresponding to σ_m hits zero, and this is a single collision; it is Case iii$_a$ or iii$_b$.

Single interval shrinks to zero: Assume the single interval of the basis C shrinks to zero at $0 < t_n < T$. Let B', C, B'' be the consecutive bases in the base-sequence, and $v' = B' \backslash C$, $v'' = C \backslash B''$, and let z', z'' be the corresponding primal or dual state variables. Then z', z'' hit zero at t_n, and this is a single collision. If $B' \backslash B'' = \{v', v''\}$ it is Case ii. If $B' \backslash B'' = v'$ $(B' \backslash B'' = v'')$ then the other state variable z' (or z'') has a strict local minimum of zero at t_n. This is Case i. Note that before the collision z' (or z'') is 0 in the interval of C, and > 0 in the intervals of B', B''. By Corollary 3 $B' \neq B''$, so $|B' \backslash B''|$ is either 2 or 1.

If C is the first (last) interval, then collision occurs at 0 (at T), with B'', v'', z'' (B', v', z') as before. In that case $z'' = q_{\prime\prime}$, and $q_{\prime\prime}$ hits zero at 0 ($z' = x_{\prime}$ and $x_{\prime}$ hits zero at T), and this is a single collision; it is Case i$_a$ (Case i$_b$).

We next discuss the case of several consecutive intervals which shrink to zero. In this case, we can have a single collision, which is of one of the Cases ii,iii,iii$_a$,iii$_b$, or it can be a case of multiple collisions.

Consecutive intervals shrinks to zero: Let $B', C_1, \dots, C_M, B''$ be consecutive bases in the base-sequence $B_1, \dots, B_N$ such that the intervals of the bases $C_1, \dots, C_M$ shrink to zero, and let $t'_0 < \cdots < t'_M$ be the breakpoints of the pivots $B' \to C_1$, $C_1 \to C_2, \dots, C_M \to B''$. Let $v_0, \dots, v_M$ be the variables that leave the basis, $w_0, \dots, w_M$ those that enter the basis, in those pivots.

Consider any $\dot{x}_k \in \{v_0, \dots, v_M, w_0, \dots, w_M\}$. It may enter and leave the basis several times in the pivots $B' \to C_1$, $C_1 \to C_2, \dots, C_M \to B''$. We have $\dot{x}_k \in B' \backslash B''$ if it leaves one more time than it enters, and we have $\dot{x}_k \in B'' \backslash B'$ if it enters one more time than it leaves. We have $\dot{x}_k \in B' \cap B''$ and x_k has a strict local minimum of 0 at the collision time if $\dot{x}_k$ leaves and enters the basis the same number of times, starting with leaving. If $\dot{x}_k$ enters and leaves the basis the same number of times, starting with entering, then $x_k(t) > 0$ for $\theta < 0$ in the intervals between $\dot{x}_k$ entering and leaving, but these sections of positive x_k disappear as $\theta \nearrow 0$, and $\dot{x}_k \notin B' \cup B''$. In summary, x_k hits zero at this collision if $\dot{x}_k$ enters or leaves the basis at least once, and in the first of these times it leaves the basis.

Similarly, for any $u_j \in \{v_0, \dots, v_M, w_0, \dots, w_M\}$, q_j hits zero at this collision if u_j enters or leaves the basis at least once, and in the last of these times it leaves the basis.

Hence a single collision occurs if and only if out of all $\dot{x}_k, u_j \in \{v_0, \dots, v_M, w_0, \dots, w_M\}$ exactly two satisfy the above conditions, which can happen in one of two ways: If $|B' \backslash B''| = 2$, and none of x_k, q_j have a strict local minimum of 0 at the collision, this is Case ii. If $|B' \backslash B''| = 1$, and in addition exactly one of x_k, q_j has a strict local minimum of 0 at the collision, this is Case i.

We can exclude the case that $|B' \backslash B''| = 1$ and none of x_k, q_j have a strict local minimum of 0, see Proposition 4.

If the intervals that shrink to zero belong to consecutive bases $C_1, \dots, C_M$ which are the initial or the last bases in $B_1, \dots, B_N$, then a single collision, of Case i$_a$ (Case i$_b$) occurs if all but one of the variables leaving and entering the basis in the M pivots leave and enter the same number of times, and one variable $u_{\prime\prime}$ leaves once more than it enters (one variable $\dot{x}_{\prime}$, leaves once more than it enters) in which case $q_{\prime\prime}$ hits zero at 0 ($x_{\prime}$ hits zero at T).

Multiple collisions: In all the other cases, multiple collisions occur. These include the following: Collisions occur at > 1 different times, say $t_{n'}, t_{n''}$ such that when $\theta = 0$, $t_{n'} < t_{n''}$. Collisions occur at a single time when several intervals shrink to zero, and $|B' \backslash B''| > 2$, or $|B' \backslash B''| = 2$ and one of the x_k, q_j has a strict local minimum of 0 at the collision, or if more than one of the x_k, q_j has a strict local minimum of 0 at a collision. □

Proposition 4. *If one or more intervals shrink to* 0 *between* B', B'', *and* $|B' \backslash B''| = 1$, *then at least one of the* x_k, q_j *has a strict local minimum of* 0 *at the collision.*

Proof. Assume the contrary. Then the sequence of bases $B_1, \dots, B', B'', \dots, B_N$ is an optimal base-sequence for θ in a neighborhood of 0, and this contradicts the uniqueness Corollary 4. □

4.3.2. Order of leaving the basis. We explain the definition in (21). In collision Case ii, one or more consecutive intervals, corresponding to single basis C or to bases $C_1, \dots, C_M$, between B', B'', shrink to 0 as $\theta \nearrow 0$, and $B' \backslash B'' = \{v', v''\}$. Denote the breakpoints of the pivots $B' \to C_1$, $C_1 \to C_2, \dots, C_M \to B''$ by $t'_0 < t'_1 < \cdots < t'_M$. If a single interval with basis C shrinks to zero, and if $v' = B' \backslash C$, $v'' = C \backslash B''$, then v' leaves the basis at t'_0, v'' leaves the basis at t'_1, and it is natural to say that v' leaves the basis before v''. However, if $M > 1$ intervals shrink to zero, we cannot exclude the possibility that v', v'' leave and re-enter the basis several times within this sequence of pivots, and it is no longer possible to say which precedes which in the sequence of breakpoints $t'_0, t'_1, \dots, t'_M$.

To cover such cases, we define v' leaves before v'' by (21). To justify the definition we now show that (21) is independent of the value of $\theta < 0$ within the validity region (Proposition 6) and that it coincides with the natural definition for the case that a single interval shrinks to zero (Proposition 7). We note first that:

Proposition 5. *At $\theta = 0$, the inequalities in (21) become equalities.*

Proof. Recall that at $\theta = 0$, t' is the beginning of the interval of B', the collision breakpoint t_n is the value to which all the breakpoints $t'_0, t'_1, \dots, t'_M$ converge, which is the end of the interval of B' and beginning of the interval of B'', and t'' is the end of the interval of B'', with $t' < t_n < t''$. So for $\theta = 0$ we have:

$$\begin{aligned}
&\text{If } v' = \dot{x}_{\prime} \text{ then } \frac{x_{\prime}(t')}{-\dot{x}_{\prime}^{B'}} = t_n - t',\\
&\text{If } v'' = \dot{x}_{\prime\prime} \text{ then } \frac{x_{\prime\prime}(t')}{-\dot{x}_{\prime\prime}^{B'}} = t_n - t',\\
&\text{if } v' = u_{\prime} \text{ then } \frac{q_{\prime}(T(\theta) - t'')}{-\dot{q}_{\prime}^{B''*}} = t'' - t_n,\\
&\text{if } v'' = u_{\prime\prime} \text{ then } \frac{q_{\prime\prime}(T(\theta) - t'')}{-\dot{q}_{\prime\prime}^{B''*}} = t'' - t_n.
\end{aligned}$$

It is now easy to check the claim by substituting in (21). □

Proposition 6. *The inequalities (21) hold for all θ in the validity region.*

Proof. By Corollary 7, the left hand side in (21) is a ratio of two affine functions of θ if $\ell(\theta)$ remains in one validity region. Hence it is monotone in θ. By Proposition 5 this left hand side equals the right hand side value of 1 at $\theta = 0$. Hence, for all $\theta < 0$ in the validity region, either (21) holds, or it holds as an equality, or the reversed inequality holds.

We can exclude the case that it holds as an equality for all $\theta < 0$ in the validity region. If it did then $B_1, \dots, B', B'', \dots, B_N$ would be an optimal base-sequence for a neighborhood of $|\theta| \geq 0$, which would contradict the uniqueness of the solution for $\theta < 0$.

Thus either (21) holds for a neighborhood of $\theta < 0$ and v' leaves before v'', or the reversed inequality holds, and we have v'' leaves the basis before v'.

So whenever we have $|B' \backslash B''| = 2$, equation (21) defines an order in which the two variables leave the basis. □

Proposition 7. *If a single interval of basis C shrinks to 0, and if $v' = B' \backslash C$, $v'' = C \backslash B''$, then (21) holds for $\theta < 0$.*

Proof. For $\theta < 0$ for which $B_1, \dots, B_N$ is the optimal base-sequence, we have the breakpoints $t' < t'_0 < t'_1 < t''$, which are respectively the beginning of the interval of B', the breakpoint of the pivot $B' \to C$, the breakpoint of the pivot $C \to B''$, and the end of the interval of B''.

Consider the case that $v' = \dot{x}_{\prime}$ and $v'' = \dot{x}_{\prime\prime}$ (see 2nd illustration of Figure 1). Then $x_{\prime}(t'_0) = 0$ while $x_{\prime\prime}(t'_0) > 0$, hence,

$$x_{\prime}(t') = -\dot{x}_{\prime}^{B'}(t'_0 - t'),$$
$$x_{\prime\prime}(t') = -\dot{x}_{\prime\prime}^{B'}(t'_0 - t') + x_{\prime\prime}(t'_0)$$

which implies (21).

The case that $v' = u_{\prime}$ and $v'' = u_{\prime\prime}$ is similar.

Consider the case that $v' = \dot{x}_{\prime}$ and $v'' = u_{\prime\prime}$ (see top illustration of Figure 1), so $x_{\prime}(t'_0) = 0$ and $q_{\prime\prime}(T(\theta) - t'_1) = 0$, hence:

$$x_{\prime}(t') = -\dot{x}_{\prime}^{B'}(t'_0 - t'),$$
$$q_{\prime\prime}(T(\theta) - t'') = -\dot{q}_{\prime\prime}^{B''^*}(t'' - t'_1),$$

so that

$$\frac{x_{\prime}(t')}{-\dot{x}_{\prime}^{B'}} + \frac{q_{\prime\prime}(T(\theta) - t'')}{-\dot{q}_{\prime\prime}^{B''^*}} = (t'_0 - t') + (t'' - t'_1) < t'' - t'$$

which implies (21).

The case that $v' = u_{\prime}$ and $v'' = \dot{x}_{\prime\prime}$ is considerably harder. We have:

$$x_{\prime\prime}(t') = -\dot{x}_{\prime\prime}^{B'}(t'_0 - t') - \dot{x}_{\prime\prime}^{C}(t'_1 - t'_0),$$
$$q_{\prime}(T(\theta) - t'') = -\dot{q}_{\prime}^{B''^*}(t'' - t'_1) - \dot{q}_{\prime}^{C^*}(t'_1 - t'_0).$$

If we can show that:

$$\frac{-\dot{x}_{\prime\prime}^{C}}{-\dot{x}_{\prime\prime}^{B'}} + \frac{-\dot{q}_{\prime}^{C^*}}{-\dot{q}_{\prime}^{B''^*}} > 1. \tag{25}$$

then (21) will follow.

We now prove (25). Assume to the contrary that $\frac{-\dot{x}^C_{\prime\prime}}{-\dot{x}^{B'}_{\prime\prime}} + \frac{-\dot{q}^{C^*}_{\prime}}{-\dot{q}^{B''^*}_{\prime}} = R \leq 1$. Formulate the following subproblem: Eliminate all $\dot{x}_k, x_k, p_k$ for which $\dot{x}_k \in B' \cap B''$. Eliminate all $\dot{q}_j, q_j, u_j$ for which $u_j \notin B' \cup B''$. Set $x^0_k = 0$ and $q^N_j = 0$ for all remaining x_k, q_j except $x_{\prime\prime}, q_{\prime}$. Set

$$x^0_{\prime\prime} = -\dot{x}^C_{\prime\prime}/R, \qquad q^N_{\prime} = -\dot{q}^{C^*}_{\prime}/R.$$

and set $T = 1$. Then this problem has both B', B'' and C as optimal base-sequences, which contradicts the uniqueness Corollary 4. □

It is interesting to note that:

Proposition 8. *In collision Case iii, (21) holds for $\theta < 0$.*

Proof. We verify all the possible combinations of v', v''. We now have for $\theta \leq 0$ two intervals, of the bases B', B'', with the endpoints $t' < t_n < t''$.

If the slack that shrinks to zero is $x_{\prime\prime}(t_n)$, then $v'' = \dot{x}_{\prime\prime}$, and $v' = B' \backslash B''$ (see bottom illustration of Figure 1). For $v' = \dot{x}_{\prime}$:

$$\begin{aligned} x_{\prime}(t') &= -\dot{x}^{B'}_{\prime}(t_n - t'), \\ x_{\prime\prime}(t') &= -\dot{x}^{B'}_{\prime\prime}(t_n - t') + x_{\prime\prime}(t_n) \end{aligned}$$

which implies (21). For $v' = u_{\prime}$:

$$\begin{aligned} q_{\prime}(T(\theta) - t'') &= -\dot{q}^{B''^*}_{\prime}(t'' - t_n), \\ x_{\prime\prime}(t') &= -\dot{x}^{B'}_{\prime\prime}(t_n - t') + x_{\prime\prime}(t_n) \end{aligned}$$

again implies (21).

If the slack that shrinks to zero is $q_{\prime}(t_n)$, then $v' = u_{\prime}$, and $v'' = B' \backslash B''$. For $v'' = u_{\prime\prime}$:

$$\begin{aligned} q_{\prime\prime}(T(\theta) - t'') &= -\dot{q}^{B''^*}_{\prime\prime}(t'' - t_n), \\ q_{\prime}(T(\theta) - t'') &= -\dot{q}^{B''^*}_{\prime}(t'' - t_n) + q_{\prime\prime}(T(\theta) - t_n) \end{aligned}$$

which implies (21). For $v'' = \dot{x}_{\prime\prime}$:

$$\begin{aligned} x_{\prime\prime}(t') &= -\dot{x}^{B'}_{\prime\prime}(t_n - t'), \\ q_{\prime}(T(\theta) - t'') &= -\dot{q}^{B''^*}_{\prime}(t'' - t_n) + q_{\prime\prime}(T(\theta) - t_n) \end{aligned}$$

again implies (21).

Note that we get the combination $v'' = \dot{x}_{\prime\prime}$, $v' = u_{\prime}$ twice, once for the slack $x_{\prime\prime}(t_n) > 0$ and once for $q_{\prime}(t_n) > 0$. We do not get the combination $v'' = u_{\prime\prime}$, $v' = \dot{x}_{\prime}$ (which was the hard one to verify in Proposition 7) at all. □

4.3.3. Perturbation of Collisions

Proposition 9. *Consider an SCLP problem which satisfies assumption 2. Let $\ell(\theta)$ be a line of boundary values, and $B_1, \dots, B_N$ a base-sequence such that $\ell(\theta)$ intersects the validity region of $B_1, \dots, B_N$ for $\underline{\theta} < \theta < \overline{\theta}$. Perform the following perturbation to the system data: To each of the coefficients of a, b, c, d add a small random quantity, identically, independently, and uniformly drawn from $(-\epsilon, \epsilon)$. Then for every δ there exists an ϵ_0 such that the perturbed problem with $\epsilon < \epsilon_0$ will satisfy:*

(i) $B_1, \dots, B_N$ is the optimal base-sequence of the perturbed problem for $\ell(\theta)$, $\tilde{\underline{\theta}} < \theta < \tilde{\overline{\theta}}$.
(ii) $|\tilde{\underline{\theta}} - \underline{\theta}| < \delta$, $|\tilde{\overline{\theta}} - \overline{\theta}| < \delta$,
(iii) The perturbed problem has single collisions at $\tilde{\underline{\theta}}$ and $\tilde{\overline{\theta}}$ with probability 1.

Proof. We denote perturbed terms by ˜ and the perturbation sizes by δ.

The perturbation of a, b, c, d will change all the values of the basic primal and dual variables $u_j^n, \dot{x}_k^n, \dot{y}_l^n, p_k^n, \dot{q}_j^n, \dot{r}_i^n$ linearly and continuously and leave the non-basic variables as zero. If the perturbation is small enough it will not change the optimality of each of these basic solutions, under the sign restrictions of each B_n.

The values τ, σ are obtained from

$$\begin{bmatrix} \tau \\ \sigma \end{bmatrix} = \begin{bmatrix} 1 \dots 1 & 0 \\ \mathbf{A} & 0 \\ \mathbf{B} & -I \end{bmatrix}^{-1} E\, \ell(\theta), \tag{26}$$

where the coefficients of $\mathbf{A}, \mathbf{B}$ consist of various $\dot{x}_k^n, \dot{q}_j^n$, and the inverse exists for the unperturbed data. Hence, the perturbation of τ, σ is continuous in the perturbations of a, b, c, d and it is an affine function of θ.

For fixed θ, if the unperturbed $\tau, \sigma > 0$, then for small enough perturbations the solution remains > 0. At $\underline{\theta}$ or $\overline{\theta}$ some of the τ, σ equal 0. For the perturbed problem, those components will become $\neq 0$ but as τ, σ are affine functions of θ, a change in θ will restore those components to 0, and the size of the change is continuous in the size of the perturbations. It follows that $|\tilde{\underline{\theta}} - \underline{\theta}|$, $|\tilde{\overline{\theta}} - \overline{\theta}|$ are continuous functions of the perturbations of a, b, c, d. This implies (i) and (ii).

We turn to (iii). If for the unperturbed validity region of $B_1, \dots, B_N$ the collisions at $\underline{\theta}, \overline{\theta}$ are single, then they remain so after a small enough perturbation. It remains to consider the case that the unperturbed validity region has a multiple collision.

For concreteness, assume that as $\theta \nearrow \overline{\theta}$, three of the x_k, say $x_\prime, x_{\prime\prime}, x_{\prime\prime\prime}$, hit zero together at t_n. The proof for all the other situations of multiple collisions is similar, and will be skipped.

Note that because all the primal (dual) basic variables are obtained from multiplying (a, b) $((c, d)$ $)$ by the inverse of the basis matrix, which is non-singular, the correlation matrix between the resulting perturbations of the various primal and dual variables is always strictly positive definite, and hence every linear combination of these perturbations which has constant coefficients is $\neq 0$ with probability 1.

From this it follows that if the unperturbed $x_{\prime}(t), x_{\prime\prime}(t), x_{\prime\prime\prime}(t)$ are > 0, then the perturbations of these values at t have strictly positive definite correlations, and hence every linear combination of these perturbations which has constant coefficients is $\neq 0$ with probability 1.

Consider now the values $x_{\prime}(t_{n-1}), x_{\prime\prime}(t_{n-1}), x_{\prime\prime\prime}(t_{n-1}) > 0$, and the slopes $\dot{x}^n_{\prime}, \dot{x}^n_{\prime\prime}, \dot{x}^n_{\prime\prime\prime} > 0$, for the unperturbed a, b, c, d, at $\overline{\theta}$. Then the three ratios $x_{\prime}(t_{n-1})/-\dot{x}^n_{\prime}, x_{\prime\prime}(t_{n-1})/-\dot{x}^n_{\prime\prime}, x_{\prime\prime\prime}(t_{n-1})/-\dot{x}^n_{\prime\prime\prime}$ are all equal to $\tau = t_n - t_{n-1}$. After the perturbation, they are no longer equal. If we change $\overline{\theta}$ we can get any two of them to be equal, but not all three. The value of $\tilde{\overline{\theta}}$ will then be the minimal value at which two of $x_{\prime}(t), x_{\prime\prime}(t), x_{\prime\prime\prime}(t)$ hit zero simultaneously, and as $\theta \nearrow \tilde{\overline{\theta}}$ the perturbed problem will (with probability 1) have a single collision. □

Proposition 10. *Consider an SCLP problem which satisfies assumption 2, and a line of boundary values $\ell(\theta)$. Perform the following perturbation to the system data: To each of the coefficients of a, b, c, d add a small random quantity, identically, independently, and uniformly drawn from $(-\epsilon, \epsilon)$. Then if ϵ is small all the collision along $\ell(\theta)$ will be single collisions.*

Proof. Consider all sequences of admissible adjacent bases consistent with the boundary values of the line $\ell(\theta)$, and with strictly decreasing objectives $V = c'u + d'\dot{y}$. There is a finite number of them (see Theorem 8). The intersection of the validity region of each one of them with the line $\ell(\theta)$ is either empty, or consists of a single point, or consists of an interval. Since there is a finite number of base-sequences, we can choose a small enough perturbation of a, b, c, d such that: The sequences with empty validity region remain so, the sequences with interval validity regions remain so, and some of the sequences with single point validity region now have empty validity regions, while others have interval validity regions. With probability 1 all these new validity regions end with single collisions. □

4.3.4. Solution of the rates-LP

Proposition 11. *Consider the rates-LP and dual rates-LP*, under the sign restrictions (22-24). In Cases ii,iii the rates-LP and its dual rates-LP* under sign restrictions (22) are both feasible. In Case iii_a, the rates-LP under sign restrictions (23) is feasible. In Case iii_b, the dual rates-LP* under sign restrictions (24) is feasible.*

Proof. By Corollary 2, B', B'' are optimal, and hence both primal feasible and dual feasible, for the rates-LP with the corresponding sign restrictions (15).

The primal sign restrictions of (22,23) are less tight than those of B'' as given in (15), hence the rates-LP under sign restrictions (22,23) are primal feasible.

The dual sign restrictions of (22,24) are less tight than those of B'^* as given in (15), hence the dual rates-LP* under sign restrictions (22,24) are dual feasible. □

It follows from Proposition 11 that the solution of the LP is always possible in Cases ii, iii. In Cases iii$_a$, iii$_b$ we need additional assumptions to insure that the rates-LP (23,24) have a solution. This is always the case when one is solving a subproblem, since all the bases which occur in the subproblem are sandwiched between the bases B', B'' of the calling problem (between D, B'' if the calling problem has collision Case iii$_a$, between B', D if the calling problem has collision Case iii$_b$). Finally:

Proposition 12. *Under assumption 1, the rates-LP and dual rates-LP*, under the sign restrictions (22-24) are both feasible, and hence have complementary slack optimal solutions.*

Proof. This follows immediately from the fact that the sign restrictions of assumption 1 are at least as restrictive as (22-24). □

4.3.5. Subproblem Solutions The formulation and solution of a subproblem is described in Section 4.5. The solution consists of a sequence of bases, $B', D_1, \dots, D_M, B''$ ($D_1, \dots, D_M, B''$ in Case iii$_a$, $B', D_1, \dots, D_M$ in Case iii$_b$, but the details of the following discussion are similar for these two cases, and will be skipped).

It follows from Proposition 16 that this sequence of bases is an optimal solution for the original SCLP problem (4) solved for time $t' < t < t''$ where $t'' - t'$ is small, with the following boundary values of $x^0 = x(t')$, $q^N = q(T - t'')$: For $\dot{x}_k \in B' \cap B''$, and for $u_j \notin B' \cup B''$, x_k^0 and q_j^N are large. All the other variables except z', z'' (which are the states corresponding to v', v'') are x_k^0, $q_j^N = 0$. Finally, the values of z', z'' at the boundaries satisfy the conditions for v'' leaves the basis before v', that is conditions (21) with reversed inequalities.

It also follows that the intervals $\tau_1, \dots, \tau_M$ of this solution equal 0 if the values of z', z'' at the boundaries satisfy the conditions (21) as equalities.

Finally, it follows from Proposition 16 and Corollary 7 that any changes in $t'' - t'$, or in the boundary values of z', z'' cause $\tau_1, \dots, \tau_M$ to change in the same direction, and in fixed proportions.

4.3.6. Validity of the new base-sequence

Theorem 7. *The new sequence of bases is an optimal base-sequence for $\theta > 0$.*

Proof. We first prove the theorem under the assumption that the pivot is simple and no subproblem was solved. The proof is in four steps (a)–(d). We then repeat the same steps modifying the proof to the case that the pivot operation involves a subproblem.

(a) The new base-sequence is different from $B_1, \dots, B_N$: This is clearly so in Cases i, i$_a$, i$_b$, iii, iii$_a$, iii$_b$. In Case ii this follows from the fact that between B', B'' in the old base-sequence v' leaves the basis before v'', while in the new sequence v'' leaves the basis before v'.

(b) The matrix $\begin{bmatrix} 1 \dots 1 \\ \mathbf{A} \end{bmatrix}$ in the linear equations (13) formulated with the new base-sequence is invertible: The new base-sequence is optimal for $\theta = 0$, that is for boundary values T, x^0, q^N, where the equations (13) formulated with the new base-sequence have solutions $\tau, \sigma \geq 0$. If the pivot operation is simple (no subproblem) then exactly one of τ, σ equals 0 in this solution. In fact, in Cases ii,iii,iii$_a$,iii$_b$ the interval of the new inserted basis D is zero at $\theta = 0$. In the Cases i,i$_a$,i$_b$ there is one x_k, q_j which has a strict local minimum of 0 at the collision, and σ for this local minimum is 0 at $\theta = 0$.

If $\begin{bmatrix} 1 \dots 1 \\ \mathbf{A} \end{bmatrix}$ is not invertible, we would have alternative solutions $(\tau, \sigma) + \delta(\tau_1, \sigma_1)$. These would be > 0 if δ is small enough, and if we choose the sign of δ so as to ensure that the 0 component of τ, σ has positive component for $\delta(\tau_1, \sigma_1)$. But this would contradict the uniqueness of the optimal solution for $\theta = 0$.

(c) The linear equations (13) formulated with the new base-sequence have solutions τ, σ for each $\theta > 0$ by (b), and if θ is small enough then all τ, σ which were > 0 for $\theta = 0$ remain > 0.

(d) It remains to show that for $\theta > 0$ the single component of τ, σ which was 0 for $\theta = 0$, is now > 0. Assume contrary to what we want to show that this one component of τ, σ is ≤ 0 when $\theta > 0$. Then if we choose $\theta < 0$ small enough we would have all $\tau, \sigma \geq 0$. But then the new base-sequence would be optimal for small $\theta < 0$, which contradicts the uniqueness. This completes the proof for simple pivots.

We turn now to a pivot operation which required the solution of a subproblem. Let $B', D_1, \dots, D_M, B''$ be the solution of the subproblem, with $B' \backslash B'' = \{v', v''\}$ (the discussion of subproblem in Cases iii$_a$, iii$_b$ is similar). Solving the linear equations (13) for new sequence of bases $B_1, \dots, B', D_1, \dots, D_M, B'', \dots, B_N$ at $\theta = 0$ has the intervals $\tau'_1, \dots, \tau'_M$ of the bases $D_1, \dots, D_M$ equal to 0, while all other τ, σ are > 0.

The equations for these intervals $\tau'_1, \dots, \tau'_M$ are equivalent to the equations for $\tau'_1, \dots, \tau'_M$ when the SCLP problem is solved only for a small interval $t' < t < t''$ under the boundary values described in Section 4.3.5. Consider then the solution of the linear equations (13) for new sequence of bases $B_1, \dots, B', D_1, \dots, D_M, B'', \dots, B_N$, as a function of θ. For θ small enough all τ, σ except $\tau'_1, \dots, \tau'_M$ remain > 0. From the discussion Point 4.3.5 it follows that any change in θ will change all of the $\tau'_1, \dots, \tau'_M$ in the same direction and in fixed proportions. So letting $\theta \nearrow$ will either leave all $\tau'_1, \dots, \tau'_M$ unchanged, or they all become negative, or they all become positive.

It is now easy to check that (a), (b), (c), and (d) continue to hold for pivots with subproblems. □

4.4. Algorithm for SCLP, solving for time horizons $0 < t < T$

We shall assume assumptions 1 and 2. We then show that the SCLP problem (4) can be solved for given x^0, q^N (or α, γ), for all T, by starting from $T = 0$, and performing a sequence of pivot operations at the boundary time horizon values $0 = T^{(0)} < T^{(1)} < \cdots < T^{(R)}$. For $0 < T < T^{(1)}$ the optimal base-sequence has a single basis determined by the boundary values. The algorithm terminates when the validity region beyond $T^{(R)}$ includes all $T^{(R)} < T < \infty$. We supply the details now.

Proposition 13. *Assume assumptions 1 and 2. Then the single basis B_0 which is optimal for the rates-LP (9) with added sign restrictions:*

$$\begin{aligned} x_k^0 > 0 &\Rightarrow \dot{x}_k \text{ is "}U\text{"}, \quad p_k \text{ is "}Z\text{"}, \\ \text{else } &\Rightarrow \dot{x}_k \text{ is "}P\text{"}, \quad p_k \text{ is "}P\text{"}, \\ q_j^N > 0 &\Rightarrow u_j \text{ is "}Z\text{"}, \quad \dot{q}_j \text{ is "}U\text{"}, \\ \text{else } &\Rightarrow u_j \text{ is "}P\text{"}, \quad \dot{q}_j \text{ is "}P\text{"}. \end{aligned} \tag{27}$$

is an optimal base-sequence for a range of time horizons $0 < T < T^{(1)}$.

Proof. By assumption 1 the rates-LP with additional sign restrictions (27) has an optimal solution. The basis B_0 is therefore admissible. It is also consistent with the boundary values x^0, q^N. The linear equations (13) for the base-sequence B_0 consist of the single equation:

$$\tau_1 = T,$$

and of inequalities for slacks σ:

$$\begin{aligned} \dot{x}_k^{B_0} \tau_1 \geq -x_k^0 \quad \text{when } \dot{x}_k^{B_0} < 0, \\ \dot{q}_j^{B_0^*} \tau_1 \geq -q_j^N \quad \text{when } \dot{q}_j^{B_0^*} < 0, \end{aligned}$$

and these will have positive τ, σ as long as

$$0 < T < T^{(1)} = \min\{-x_k^0/\dot{x}_k^{B_0}, -q_j^N/\dot{q}_j^{B_0^*} \; : \; \dot{x}_k^{B_0} < 0, \dot{q}_j^{B_0^*} < 0\}.$$

By Theorem 4, B_0 is an optimal base-sequence for $0 < T < T^{(1)}$. □

Hence we have an optimal base-sequence and solution for an initial range of $T > 0$.

Given a base-sequence $B_1^{(r)}, \dots, B_{N_r}^{(r)}$ which is optimal for $T > T^{(r-1)}$, we iterate as follows:

Formulate the linear equations (13) for $B_1^{(r)}, \dots, B_{N_r}^{(r)}$, solve them for τ, σ as an affine function of T, and find the validity region boundary $T^{(r)}$, which is the earliest $T > T^{(r-1)}$ at which some τ, σ shrink to zero.

Perform a pivot operation to obtain a new base-sequence $B_1^{(r+1)}, \dots, B_{N_{r+1}}^{(r+1)}$ which is optimal for $T > T^{(r)}$.

Terminate when τ, σ remain > 0 for all $T > T^R$.

Proposition 14. *In the solution with validity region $T^{(R)} < T < \infty$ one of the bases is B_∞ which is the solution of the rates-LP,LP* with the added sign restrictions: All $\dot{x}_k, \dot{q}_j \geq 0$. The interval for this basis has length $T - \hat{T}$ where the constant $\hat{T}$ is $\leq T^{(R)}$, while all the other intervals have fixed lengths.*

Proof. Denote the value of the objective of the rates-LP with $\dot{x}, \dot{q} \geq 0$ by $V_\infty = c'u^{B_\infty} + d'\dot{y}^{B_\infty}$. Let $B_1, \dots, B_N$ be any optimal base-sequence, for some T. Recall (Corollary 3) that $V_n = c'u^n + d'\dot{y}^n$ is decreasing in n. We show:

(i) For all n such that $V_n > V_\infty$ there is a $\dot{x}_k \in B_n$ with value $\dot{x}_k^n < 0$.
(ii) For all n such that $V_n < V_\infty$ there is a $\dot{q}_j \in B_n^*$ with value $\dot{q}_j^n < 0$.

Assume for B_n all $\dot{x}_k^n \geq 0$. Then B_n is optimal for the rates-LP with sign restrictions:

$$\begin{aligned}&\dot{x}_k \text{ is "P"}\\ &\text{if } q_j^n > 0 \text{ then } u_j \text{ is "Z", else } u_j \text{ is "P".}\end{aligned}$$

But this is at least as restrictive as the rates-LP of B_∞. Hence $V_n \leq V_\infty$. This proves (i). (ii) is similar.

Next we show by induction that each of the intervals of the bases B_n with $V_n > V_\infty$ is bounded. For the first interval we get the bounds $t_1 \leq b_1 = \max_k\{-x_k^0/\dot{x}_k^1\}$, and $x_k(t_1) \leq z_1 = \max\{x_k^0\} + \max\{0, b_1\, \dot{x}_k^1\}$. For the n'th interval we have by induction: $t_n - t_{n-1} \leq b_n = \max_k\{-z_{n-1}/\dot{x}_k^n\}$, and $x_k(t_n) \leq z_n = z_{n-1} + \max_k\{0, b_n \dot{x}_k^n\}$. Similarly, each of the intervals of the bases B_n with $V_n < V_\infty$ is bounded.

Since the number of bases in any base-sequence is bounded by the finite number of different bases (see Theorem 8), this shows that for large enough T the optimal base-sequence must contain the basis B_∞.

Consider then optimal base-sequences for T large enough, so that they contain B_∞. For such sequences, if $\dot{x}_k^{B_\infty} > 0$ then for T large enough $x_k(t) > 0$ for all intervals following B_∞, and similarly if $\dot{q}_j^{B_\infty^*} > 0$ then for T large enough $q_j(T-t) > 0$ for all intervals preceding B_∞.

Consider then finally the base-sequence $B_1, \dots, B_N$ which is valid for $T^{(R)} < T < \infty$. It must contain within it the basis B_∞. Also, for those x_k for which $\dot{x}_k^{B_\infty} > 0$ it must be that $x_k(t) > 0$ in all the intervals that follow B_∞. Similarly, for those q_j for which $\dot{q}_j^{B_\infty^*} > 0$ it must be that $q_j(T-t) > 0$ in all the intervals that precede B_∞. Hence, changing the length of the interval of B_∞ leaves all the other intervals unchanged. As a result all the intervals except that of B_∞ have fixed length for all T, and the length of the interval of B_∞ equals T minus the constant sum of the other intervals. This sum, denoted $\hat{T}$, is clearly $\leq T^{(R)}$ □

Corollary 9. *Assume that B_∞ consists only of u_j. Let the optimal base-sequence for $\hat{T}$ be $B_1, \dots, B_N$, which contain the basis B_∞. Then for all $T > \hat{T}$ the bases which follow B_∞ remain fixed with fixed interval lengths. Similarly, assume that the complementary dual basis ${B_\infty}^*$ consists only of p_k. Let the optimal base-sequence for $\hat{T}$ be $B_1, \dots, B_N$, which contain the basis B_∞. Then for all $T > \hat{T}$ the bases which precede B_∞ remain fixed with fixed interval lengths.*

Proof. Immediate, all the equations for τ, σ for these intervals are the same for all $T > \hat{T}$. □

4.5. Formulation and solution of subproblems

We now consider the case where we have bases B', B'' with $B' \backslash B'' = \{v', v''\}$, and D is the optimal basis of the rates-LP with sign restrictions (22), so that $v' \in D$, while $v'' \notin D$, but D is not adjacent to B', B''. Let z', z'' denote the corresponding state variables.

We formulate and solve an SCLP subproblem. It involves a subset of the variables of the calling problem. It is solved for boundary values $\ell(\theta)$, where $0 < \theta < 1$, where at $\theta = 0$ the optimal base-sequence is D, and at $\theta = 1$ the optimal base-sequence is B', B''.

The formulation of the subproblem is:

For all $\dot{x}_k \in B' \cap B''$, exclude $x_k, \dot{x}_k, p_k$ from the problem, by erasing the appropriate rows of G, F, a, and for all $u_j \notin B' \cup B''$, exclude $q_j, \dot{q}_j, u_j$ from the problem, by erasing the appropriate columns of G, H, c'.

Set the boundary values of all the remaining variables except v', v'' to x^0_k, $q^N_j = 0$.

Set the boundary values of z', z'' for $\theta = 0$:

$$\begin{array}{lll} \text{If } v' = \dot{x}_{\prime},\ v'' = \dot{x}_{\prime\prime} & \text{then} & x^0_{\prime\prime} = 0,\ x^0_{\prime} = -\dot{x}^D_{\prime}, \\ \text{if } v' = u_{\prime},\ v'' = u_{\prime\prime} & \text{then} & q^N_{\prime\prime} = -\dot{q}^D_{\prime\prime},\ q^N_{\prime} = 0, \\ \text{if } v' = \dot{x}_{\prime},\ v'' = u_{\prime\prime} & \text{then} & q^N_{\prime\prime} = -\dot{q}^D_{\prime\prime},\ x^0_{\prime} = -\dot{x}^D_{\prime}, \\ \text{if } v' = u_{\prime},\ v'' = \dot{x}_{\prime\prime} & \text{then} & x^0_{\prime\prime} = 0,\ q^N_{\prime} = 0, \end{array} \tag{28}$$

Set the boundary values of z', z'' for $\theta = 1$:

$$\begin{array}{lll} \text{If } v' = \dot{x}_{\prime},\ v'' = \dot{x}_{\prime\prime} & \text{then} & x^0_{\prime\prime} = -\dot{x}^{B'}_{\prime\prime},\ x^0_{\prime} = -\dot{x}^{B'}_{\prime}, \\ \text{if } v' = u_{\prime},\ v'' = u_{\prime\prime} & \text{then} & q^N_{\prime\prime} = -\dot{q}^{B''}_{\prime\prime},\ q^N_{\prime} = -\dot{q}^{B''}_{\prime}, \\ \text{if } v' = \dot{x}_{\prime},\ v'' = u_{\prime\prime} & \text{then} & q^N_{\prime\prime} = -\dot{q}^{B''}_{\prime\prime},\ x^0_{\prime} = -\dot{x}^{B'}_{\prime}, \\ \text{if } v' = u_{\prime},\ v'' = \dot{x}_{\prime\prime} & \text{then} & x^0_{\prime\prime} = -\dot{x}^{B'}_{\prime\prime}\ q^N_{\prime} = -\dot{q}^{B''}_{\prime}, \end{array} \tag{29}$$

Set the time horizons $T(0) = 1$ and $T(1) = 2$.

To each basis of the subproblem corresponds a basis of the calling problem, in which we re-introduce the set of excluded variables. We shall use the notation B', B'', D, D_m to denote both the bases of the subproblem, and the corresponding bases of the calling problem.

It is seen immediately that:

Proposition 15. *D is the optimal base-sequence for $\theta = 0$ but not for $\theta > 0$, and B', B'' is the optimal base-sequence for $\theta = 1$ but not for $\theta < 1$. At $\theta = 0$ two collision occur: z'' hits zero at $t = 0$ and z' hits zero at $t = 1$.*

The first iteration in the solution of the subproblem involves two pivot operations, for the collisions at $t = 0$ and at $t = 1$. To find the base-sequence D_{-1}, D, D_1 valid for some right neighborhood of $\theta > 0$ we formulate the following rates-LP.

For D_{-1} solve the rates-LP,LP* with sign restrictions:

$$\begin{aligned} \dot{x}_k \in \{v', v''\} &\Rightarrow \dot{x}_k \text{ is "U"}, \quad p_k \text{ is "Z"}, \\ \text{else } &\Rightarrow \dot{x}_k \text{ is "P"}, \quad p_k \text{ is "P"}, \\ u_j \in \overline{D}\backslash\{v''\} &\Rightarrow u_j \text{ is "Z"}, \quad \dot{q}_j \text{ is "U"}, \\ \text{else } &\Rightarrow u_j \text{ is "P"}, \quad \dot{q}_j \text{ is "P"}, \end{aligned} \tag{30}$$

For D_1 solve the rates-LP,LP* with sign restrictions:

$$\begin{aligned} \dot{x}_k \in D\backslash\{v'\} &\Rightarrow \dot{x}_k \text{ is "U"}, \quad p_k \text{ is "Z"}, \\ \text{else } &\Rightarrow \dot{x}_k \text{ is "P"}, \quad p_k \text{ is "P"}, \\ u_j \in \{v', v''\} &\Rightarrow u_j \text{ is "Z"}, \quad \dot{q}_j \text{ is "U"}, \\ \text{else } &\Rightarrow u_j \text{ is "P"}, \quad \dot{q}_j \text{ is "P"}. \end{aligned} \tag{31}$$

If D_{-1} is not adjacent to D, or if D_1 is not adjacent to D, one needs to solve a subproblem. The result of this first iteration is a sequence $D_{-M'}, \dots, D, \dots, D_{M''}$. This sequence is an optimal base-sequence for some right neighborhood of $\theta > 0$.

Proposition 16. *The subproblem will be solved by pivot operations, yielding as last a base-sequence $B', D_1, \dots, D_M, B''$ which is optimal in a left neighborhood of $\theta < 1$, and in which v'' leaves the basis before v' (according to (21)).*

Proof. We consider the first iteration. One can easily check the following, by looking at the linear equations (13) for the new sequences: In the cases that $v'' = \dot{x}_{\prime\prime}, v' = u_\prime$, or $v'' = u_{\prime\prime}, v' = \dot{x}_\prime$, it is certainly true that the new base-sequence D_{-1}, D, D_1 (or $D_{-M'}, \dots, D, \dots, D_{M''}$) is optimal for $\theta = 0$ and for some $\theta > 0$. The same may also be true for the cases that $v'' = \dot{x}_{\prime\prime}, v' = \dot{x}_\prime$, or $v'' = u_{\prime\prime}, v' = u_\prime$.

In the case that $v'' = \dot{x}_{\prime\prime}, v' = \dot{x}_\prime$, if the above base-sequence is not optimal for $\theta > 0$, than the sequence D_{-1}, D (or $D_{-M'}, \dots, D$) is optimal for for $\theta = 0$ and for some $\theta > 0$.

In the case that $v'' = u_{\prime\prime}, v' = u_\prime$, if the above base-sequence is not optimal for $\theta > 0$, than the sequence D, D_1 (or $D, \dots, D_{M''}$) is optimal for for $\theta = 0$ and for some $\theta > 0$.

Hence, the first iteration takes us to a solution for $0 < \theta < \theta^{(1)}$. Further iterations will move through validity regions bounded by successive $\theta^{(r)}$. At each iteration, the solution of the rates-LP,LP* with sign restrictions (22-24) is possible. This is because the primal rates-LP is less restricted than B'', and the dual rates-LP* is less restricted than B'^*.

The subproblem will terminate with an optimal base-sequence for $\theta^{(R)} < \theta \leq 1$. this base-sequence will consist of $B', D_1, \dots, D_M, B''$, where as $\theta \nearrow 1$ the corresponding interval lengths satisfy $\tau' \to 1, \tau_1 \searrow 0, \dots, \tau_M \searrow 0, \tau'' \to 1$.

Consider now the order defining inequalities (21). It is easy to check that as $\theta \nearrow$, the left hand sides of the inequality for all cases is increasing. At $\theta = 1$ it is equal to 1. Hence, for $\theta^{(R)} < \theta \leq 1$ the inequalities (21) are reversed. But this means by definition that in the base-sequence $B', D_1, \dots, D_M, B'', v''$ leaves before v'. $\square$

Proposition 17. *Each subproblem is smaller than its calling problem.*

Proof. The proof is based on Proposition 14 and Corollary 9.

We prove the proposition for iterations on $0 < T < \infty$. Assume that $V_{B''} > V_{B_\infty}$. Then there exists $\dot{x}_k \in B''$ such that $\dot{x}_k < 0$, but then $\dot{x}_k \in B' \cap B''$, and it is excluded in the subproblem. Similarly if $V_{B'} < V_{B_\infty}$, there is a u_j which will be excluded.

Assume that $V_{B'} > V_{B_\infty} > V_{B''}$. Then there exists $0 > \dot{x}_{\prime\prime} \in B'$ and $0 > \dot{q}_{\prime} \in B''^*$. If $B' \cap B''$ contains none of the $\dot{x}_k$ and if $B'^* \cap B''^*$ contains none of the $\dot{q}_j$ then $B' \backslash B_\infty = \dot{x}_{\prime\prime}$ and $B_\infty \backslash B'' = u_{\prime}$. But in that case, B', B_∞, B'' are adjacent, B_∞ plays the role of D in the iteration, and there is no subproblem.

Finally, assume $V_{B'} > V_{B_\infty} = V_{B''}$, which means by the strict monotonicity of V that $B'' = B_\infty$. If $B' \cap B''$ contains none of the $\dot{x}_k$ and if $B'^* \cap B''^*$ contains none of the $\dot{q}_j$ then it follows that B''^* contains none of the $\dot{q}_j$. But then by Corollary 9, from the first time that B'' entered the base-sequence, it will always remain adjacent to B'. The case of $V_{B'} = V_{B_\infty} > V_{B''}$ is similar.

The proof for a subproblem which is called from another subproblem is similar but will be skipped. □

For subproblems called in collision Cases iii$_a$, iii$_b$, the formulation of the subproblems needs to be slightly modified. For Case iii$_a$ we have $v' = u_{\prime} \in D \backslash B''$ but D is not adjacent to B''. The variables excluded are all $x_k, \dot{x}_k, p_k$ for which $x_k^0 > 0$, and all $q_j, \dot{q}_j, u_j$ for which $u_j \notin B'', u_j \neq u_{\prime}$. The boundary values of all the remaining variables except $q_{\prime}$ are set to 0. The boundary values of $q_{\prime}$ are:

$$\text{For } \theta = 0\text{: } q_{\prime}^N = 0. \qquad \text{For } \theta = 1\text{: } q_{\prime}^N = -\dot{q}_{\prime}^{B''^*}.$$

and $T(0) = 1$, $T(1) = 2$.

For Case iii$_b$ we have $v'' = \dot{x}_{\prime\prime} \in B' \backslash D$ but D is not adjacent to B'. The variables excluded are all $q_j, \dot{q}_j, u_j$ for which $q_j^N > 0$, and all $x_k, \dot{x}_k, p_k$ for which $\dot{x}_k \in B', \dot{x}_k \neq \dot{x}_{\prime\prime}$. The boundary values of all the remaining variables except $x_{\prime\prime}$ are set to 0. The boundary values of $x_{\prime\prime}$ are:

$$\text{For } \theta = 0\text{: } x_{\prime\prime}^0 = 0. \qquad \text{For } \theta = 1\text{: } x_{\prime\prime}^0 = -\dot{x}_{\prime\prime}^{B'}.$$

and $T(0) = 1$, $T(1) = 2$.

The remaining details, and the analogous forms of Propositions 15, 16 are similar to Case ii, iii subproblems, and will be skipped.

4.6. Completing all the proofs

Theorem 8. *Assume the feasibility and boundedness and non-degeneracy assumptions 1, 2. The algorithm will reach a solution of the problem for all* $0 < T < \infty$ *in no more than* $2^{2^{K+J}}$ *pivot steps.*

Proof. The total number of bases is bounded by $\binom{K+J}{I+K-L}$ which is bounded by 2^{K+J}. Each of these has a different value of V_B, by assumption 2. Each base-sequence consists of subset of all the bases, strictly ordered by decreasing V_B. The total number of such base-sequences is bounded by $2^{2^{K+J}}$.

What we will show is that each pivot operation of the algorithm is associated with a unique base-sequence, and this will complete the proof.

We shall refer to the iterations of the algorithm over $0 < T < \infty$ as top-level or level 0 iterations. We shall refer to iterations of a subproblem called from the top level, over $0 < \theta_1 < 1$, as level 1 iterations, and inductively, to the iterations of a subproblem called from level $l-1$, and iterating over $0 < \theta_l < 1$, as level l iterations.

Each iteration of any level l involves a unique range of the parameter (T or θ_l), and a unique base-sequence which is optimal for the problem or subproblem for that range, and this base-sequence is not repeated. But we can show more.

Consider an iteration at level l. We have then a hierarchy of base-sequences. For the top level we have $B_1^0, \dots, B_{N_0}^0$, optimal at exactly a single $0 < \hat{T} < \infty$ which is the boundary of two validity ranges of T. Between two of the bases of this sequence we are going to insert the solution of the level 1 subproblem. Next we have a sequence of bases of the level 1 subproblem, $B_1^1, \dots, B_{N_1}^1$, which is optimal at exactly one value $0 < \hat{\theta}_1 < 1$ which is the boundary of two validity ranges of θ_1, and between two bases of this level 1 sequence we are going to insert the solution of the level 2 subproblem. This continues for all levels $< l$. Finally, there is the level l base-sequence $B_1^l, \dots, B_{N_l}^l$. This sequence is the result of the current pivot operation performed at level l. This level l base-sequence is then optimal for a range of values $0 \le \underline{\hat{\theta}}_l < \theta_l < \overline{\hat{\theta}}_l \le 1$.

If we insert the bases of level 1 in their place in the level 0 sequence, the level 2 bases in their place in the level 1 sequence, etc. ending with the insertion of of the level l bases in their place in the level $l-1$ sequence, we end up with a combined sequence of bases, $B_1^0, \dots, B_1^1, \dots, B_1^l, \dots, B_{N_l}^l, \dots, B_{N_1}^1, \dots, B_{N_0}^0$ (note, they are not adjacent). All the bases in this sequence are strictly ordered by their values V_B, and this sequence is uniquely associated with the current pivot operation, because its various parts are optimal uniquely for $\hat{T}, \hat{\theta}_1, \dots, \hat{\theta}_{l-1}, (\underline{\hat{\theta}}_l, \overline{\hat{\theta}}_l)$.

This proves the bound of $2^{2^{K+J}}$ on the total number of pivots. □

We now complete the proofs of all the remaining theorems.

Proof of Theorem 2: If the problem satisfies the non-degeneracy assumption 2, this follows immediately from Theorem 8. By continuity this implies that the Theorem also holds without assumption 2. □

Proof of the Converse part of Theorem 4: Follows immediately from Theorem 8. The solution is of the required form unless $T = T^{(r)}$. □

Proof of Theorem 1: If the problem satisfies the non-degeneracy assumption 2, this follows immediately from Theorem 8. If the problem is degenerate, we can perturb a, b, c, d so that assumption 2 holds. If the perturbation is small enough, the perturbed problem has a solution for all $0 < T < \infty$. Furthermore, if the perturbation is small enough then the base-sequences $B_1^{(l)}, \dots, B_{N_l}^{(l)}, l =$

$1, \ldots, R+1$ are optimal for all the successive ranges of values of $0 < T < \infty$ for the unperturbed problem. The solution clearly is complementary slack with no duality gap, and it has piecewise linear x, y, q, r and piecewise constant u, p.

Proof of Theorem 3: This follows from the uniqueness of the solution, Corollary 4.

Proof of Theorem 6 and Corollary 8: By Theorem 2, solutions exist at all of $\mathcal{C}$, and by the converse of Theorem 4 they belong to the validity region of some base-sequence for almost all of $\mathcal{C}$. Since each validity region is a convex polyhedral cone, this implies that $\mathcal{C}$ is tiled by validity regions. Turning to Corollary 8, clearly we can iterate our algorithm and construct the required partition of $0 < \theta < 1$ and the required base-sequences. □

5. Examples

5.1. Demonstration of the Algorithm by a Small Numerical Example

Table 1. *Data for an Example of an Economic Input Output System*

c	2.	7.	3.	5.	2.	7.	2.	4.	6.	3.	4.	3.	α	a
	0	0	0	0	0	0	0	0	0	-2.6	0	0	36.	1.2
	0	0	0	-2.8	0	0	0	3.	-3.7	-1.1	-3.4	8.1	28.	1.1
	2.9	3.1	7.4	8.9	0	0	-3.5	-2.9	-3.7	0	0	0	31.	1.2
	-1.9	0	0	0	0	5.4	8.4	0	4.5	3.6	3.3	-1.6	29.	1.3
G	0	0	0	-1.5	0	-3.4	-2.2	-1.2	0	0	-3.5	-3.2	26.	1.
	-2.2	0	0	0	-2.5	0	0	-2.8	-2.7	0	0	0	30.	1.9
	0	-1.9	0	0	0	0	-3.7	0	0	0	0	0	26.	1.4
	0	0	0	-3.5	5.2	-2.7	0	0	-3.7	-1.9	0	0	34.	1.3
														b
	6.5	8.	6.	6.4	5.4	7.8	6.5	5.6	7.4	3.6	7.3	6.9		106.
	0	3.9	5.8	4.8	0	0	0	7.4	0	7.3	0	3.8		66.
H	0	0	3.1	0	5.9	0	5.8	6.4	0	7.1	5.5	0	$\leq$	115.
	4.9	0	7.5	5.2	4.6	7.4	0	6.9	0	0	0	6.4		86.
	7.	4.3	4.9	0	0	0	3.6	0	7.5	0	6.2	0		112.

The following numerical example illustrates the algorithm for SCLP with linear data. This example consists of an Economic Input Output System, in which there are 8 assets, numbered as $k = 1, \ldots, 8$, and there are 12 activities, numbered $j = 1, \ldots, 12$, and if activity j is used at time t at the rate $u_j(t)$, it will produce assets of type $k(j)$ in a quantity of $G_{k(j),j}$ per unit of $u_j(t)$, and will consume an amount of $-G_{l,j}$ from asset l, where $l = 1, \ldots, 8, l \neq k(j)$, per unit of $u_j(t)$. Also, the use of activity j at time t will consume a quantity $H_{i,j}$ of resource i per unit of $u_j(t)$, for each of the 5 resources numbered $i = 1, \ldots, 5$. The matrices of coefficients G, H define the fluid-model.

The rest of the data needed to formulate the fluid-problem is: The initial level of the assets at time 0 are $\alpha_k, k = 1, \ldots, 8$, the exogenous constant input rates of the assets are $a_k, k = 1 \ldots, 8$, for all $t > 0$, the limits on the resources are the constant values $b_i, i = 1, \ldots, 5$, and the reward rates per unit of activity j at time t, over the remaining time horizon $T - t$, are c_j, $j = 1, \ldots, 12$.

The values of the coefficients of G, H, and the remaining data, α, a, b, c, are given in Table 1. In this example $F, \gamma, d = 0$. Additional primal control functions $u_j(t)$, $j = 13, \ldots, 17$ denote the slacks in the resource constraints with corresponding additional dual state functions $q_j(t)$, $j = 13, \ldots, 17$.

The solution of this problem for an infinite time horizon is described in Figure 1, where the asset levels of the 8 assets are plotted one above the other.

We now solve the problem for all time horizons $0 < T < \infty$. The boundary values of the solution are $x(0) = \alpha$, $q(0) = 0$. The initial optimal basis is

$$B_0 = \{\text{all } \dot{x}, u_2, u_6, u_{14}, u_{15}, u_{17}\}.$$

In this initial basis one is using activities 2,6, which uses up all the capacity of resources 1,4, while resources 2,3,5 have slack capacities given by u_{14}, u_{15}, u_{17}. To describe each iteration we state the collision which happens, and list the pivots from $B_{n-1} \to B_n, n = 1, \ldots, N$.

Iteration 1: The single basis sequence B_0 is optimal for $0 < T < T^{(1)} = .472$

At $T^{(1)}$ a collision Case iii$_b$, $x_4 = 0$, occurs at time t_1. To obtain the next solution a single basis is inserted at t_1.

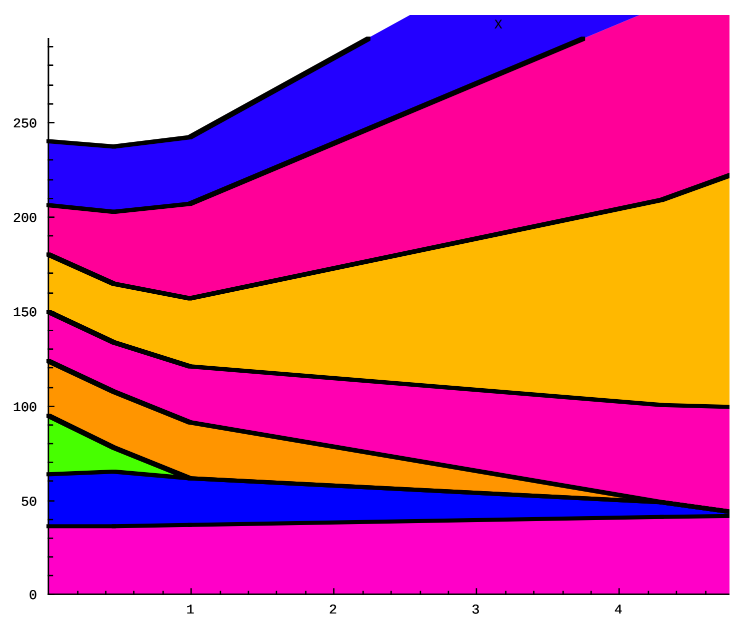

Fig. 3. Optimal Asset levels of an Economic Input Output System

Iteration 2: The sequence of pivots: $\begin{matrix}\dot{x}_4\\ \downarrow\\ u_{16}\end{matrix}$ is optimal for $.472 < T < T^{(2)} = 1.206$.

At $T^{(2)}$ a collision Case iii$_b$, $x_3 = 0$, occurs at time t_2. To obtain the next solution a subproblem is solved, and 4 new bases are inserted at t_2.

Iteration 3: The sequence of pivots: $\begin{matrix}\dot{x}_4 & u_6 & u_{14} & \dot{x}_3 & \dot{x}_4\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ u_{16} & \dot{x}_4 & u_8 & u_9 & u_{12}\end{matrix}$

is optimal for $1.206 < T < T^{(3)} = 1.373$.

At $T^{(3)}$ a collision Case i, $\tau_2 = 0$ occurs. To obtain the next solution a single basis is deleted between t_1, t_2.

Iteration 4: The sequence of pivots: $\begin{matrix}u_6 & u_{14} & \dot{x}_3 & \dot{x}_4\\ \downarrow & \downarrow & \downarrow & \downarrow\\ u_{16} & u_8 & u_9 & u_{12}\end{matrix}$

is optimal for $1.373 < T < T^{(4)} = 2.180$.

At $T^{(4)}$ a collision Case ii, $\tau_3 = 0$ occurs. To obtain the next solution a single basis is exchanged between t_2, t_3.

Iteration 5: The sequence of pivots: $\begin{matrix}u_6 & \dot{x}_3 & u_{14} & \dot{x}_4\\ \downarrow & \downarrow & \downarrow & \downarrow\\ u_{16} & u_9 & u_8 & u_{12}\end{matrix}$

is optimal for $2.180 < T < T^{(5)} = 3.681$.

At $T^{(5)}$ a collision Case i$_a$, $\tau_1 = 0$ occurs. To obtain the next solution a single basis is deleted between t_0, t_1.

Iteration 6: The sequence of pivots: $\begin{matrix}\dot{x}_3 & u_{14} & \dot{x}_4\\ \downarrow & \downarrow & \downarrow\\ u_9 & u_8 & u_{12}\end{matrix}$

is optimal for $3.681 < T < T^{(6)} = 4.353$.

At $T^{(6)}$ a collision Case iii$_b$, $x_2 = 0$, occurs at t_4. To obtain the next solution a subproblem is solved, and 2 new bases are inserted at t_4.

Iteration 7: The sequence of pivots: $\begin{matrix}\dot{x}_3 & u_{14} & \dot{x}_4 & u_{12} & \dot{x}_2\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ u_9 & u_8 & u_{12} & u_1 & u_{14}\end{matrix}$

is optimal for $4.353 < T < T^{(7)} = 4.589$.

At $T^{(7)}$ a collision Case ii, $\tau_2 = 0$ occurs. To obtain the next solution a single basis is exchanged between t_1, t_2.

Iteration 8: The sequence of pivots: $\begin{matrix}u_{14} & \dot{x}_3 & \dot{x}_4 & u_{12} & \dot{x}_2\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ u_8 & u_9 & u_{12} & u_1 & u_{14}\end{matrix}$

is optimal for $4.589 < T < T^{(8)} = 5.015$.

At $T^{(8)}$ a collision Case i, $\tau_4 = 0$ occurs. To obtain the next solution a single basis is deleted between t_3, t_4.

Iteration 9: The sequence of pivots: $\begin{array}{cccc} u_{14} & \dot{x}_3 & \dot{x}_4 & \dot{x}_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ u_8 & u_9 & u_1 & u_{14} \end{array}$

is optimal for $5.015 < T < \infty$.
Algorithm terminates.

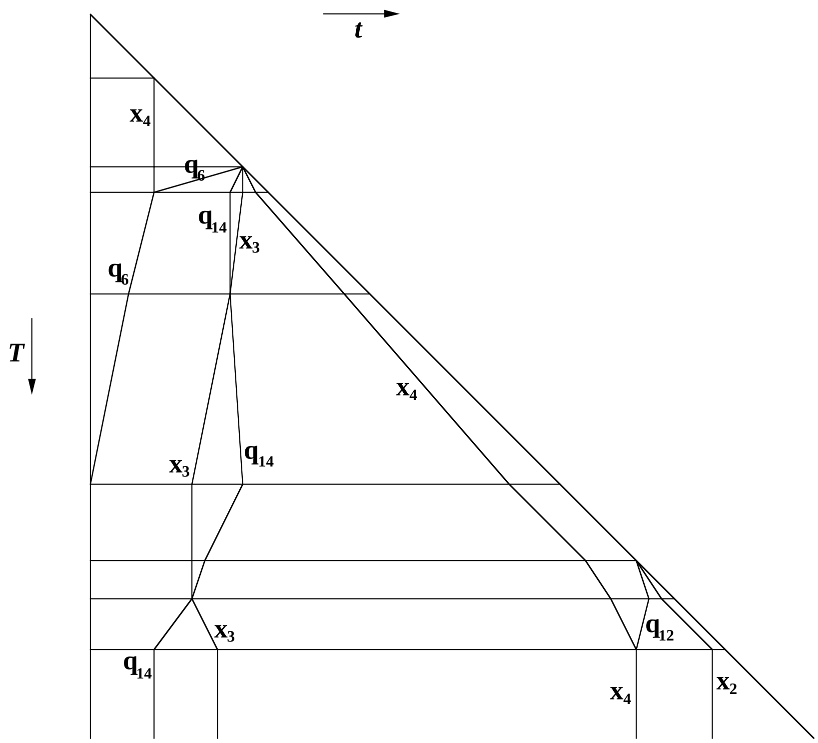

Fig. 4. Iterations of the Algorithm for the Economic Input Output System

Figure 2 provides a graphical description of the evolution of the solution by the algorithm, for this example. The vertical axis represents the time horizon T, and the horizontal axis the time $0 < t < T$. A horizontal cut of the figure at T gives the breakpoints of the solution at time t. We marked the line segments by x_k if the event at that point is that $\dot{x}_k$ leaves the basis. This means that x_k hits zero at that time, with $x_k > 0$ on the left and $x_k = 0$ on the right. We marked the line segment q_j if the event at that point is that u_j leave the basis. This means that q_j hits zero at that time, with $q_j > 0$ on the right and $q_j = 0$ on the left.

6. Discussion and Extensions

6.1. Some insights

When a problem is solved after 50 years it is sometimes instructive to speculate what made it work. We feel that 3 ideas helped here: The non-degeneracy assumption, the formulation of the dual in reversed time, and the parametric iterations of the algorithm.

It is quite clear how these serve in our algorithm: The non-degeneracy implies the uniqueness on which the verification of the algorithm hinges. The reversed time leads to symmetry of primal and dual. The parametric iterations, in particular starting from $T = 0$, provide a way to reach the solution.

Yet the significance of these 3 ideas may not be apparent at first glance: Clearly, once we know how to solve the non-degenerate problem we can solve any problem by perturbation, the reversed time seems to be no more than a notational device, and the choice of parametric iterations seems arbitrary.

In fact this is not so, all 3 ideas are essential to the solution, and nothing else seems to work. We try to explain this now.

We start with the 3rd idea — the use of parametric iterations. In our analysis we find that an extreme point of SCLP is given by a base-sequence (Theorem 4). Such a base sequence is an extreme point by virtue of being a sequence of primal dual complementary slack bases. The LP analog of this would be a pair of complementary slack primal and a dual bases, which is optimal for the LP with the corresponding sign restrictions. In SCLP we do not have the analog of the primal feasible polytope of standard LP. Hence we do not have the analog of a primal feasible extreme point, and we cannot use a primal simplex algorithm which will move along edges of a primal extreme polytope. Similarly we do not have a dual simplex algorithm for SCLP. We do have the notion of neighboring exterme point base sequences, defined through their validity regions. An analog in standard LP is the convex polyhedral cone of r.h.s. and objective vectors for which a primal basis and its complementary slack dual basis are optimal. Changing the r.h.s. or the objective along a line leads to a paramertic simplex algorithm for standard LP. Our algorithm performs the analogous parametric iterations for SCLP.

We move on to the 2nd idea — the use of reversed time for the dual. If we replace p, q, r, γ, c by $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{\gamma}, \tilde{c}$, where: $\tilde{p}(t) = p(T-t), \tilde{r}(t) = r(T-t), \tilde{q}(t) = q(T-t), \tilde{c}(t) = -c, \tilde{\gamma} = \gamma + cT$, then SCLP, SCLP* of (4,5) emerge in the form:

$$
\text{SCLP} \qquad \begin{aligned}
&\max \int_0^T (\tilde{\gamma} + \tilde{c}t)' u(t) + d' y(t) dt \\
&\text{s.t.} \int_0^t G u(s) ds + F y(t) + x(t) = \alpha + a t, \\
&\qquad H u(t) \qquad\qquad\qquad\quad = b, \\
&\qquad x(t), u(t) \geq 0, \quad t \in [0, T].
\end{aligned}
$$

$$
\text{SCLP}^* \qquad \begin{array}{ll} \min & \displaystyle\int_0^T ((\alpha + at)'\tilde{p}(t) + b'\tilde{r}(t))dt \\ \text{s.t.} & \displaystyle\int_t^T G'\tilde{p}(s)ds + H'\tilde{r}(t) - \tilde{q}(t) = \tilde{\gamma} + \tilde{c}t, \\ & F'\tilde{p}(t) \qquad\qquad\qquad\qquad = d, \\ & \tilde{p}(t), \tilde{q}(t) \geq 0, \quad t \in [0, T]. \end{array}
$$

where complementary slackness is $\tilde{p}_k(t)x_k(t) = 0$, $\tilde{q}_j(t)u_j(t) = 0$ almost everywhere, and the two problems run in forward time. Note that the symmetry is lost in the dual constraint integral $\int_t^T$. But even worse: if we keep γ fixed and change T, the objective of the primal and the right hand side of the dual now include $\tilde{\gamma} = \gamma + cT$ which depends on T. So the parametric family of direct time problems with fixed $\alpha, \tilde{\gamma}$ and varying T is different from the reversed time family of problems with fixed α, γ and varying T. Hence, the use of reversed time is not just an elegant notational device — in the parametric iterations we would move along a different path.

By the use of reversed time, the boundary values obtained from the solution of the boundary-LP,LP* (8) are independent of T, they remain the same for the complete parametric family. It is remarkable that the resulting validity region of a base-sequence extreme point of SCLP can be described by a convex polyhedral cone in Euclidean space of dimension $1 + K + J$ — the space of the values of α, γ, T. This is no larger than the dimension of the standard LP validity region of a basis.

But the most crucial idea for our algorithm is the non-degeneracy assumption. It is quite remarkable that non-degeneracy of SCLP in functional space can be reduced to non-degeneracy of the rates-LP,LP* (9). Of course the non-degeneracy assured us of the uniqueness which we used to verify the pivot operation, and of the strict monotonicity of the objective V_B which proved that the algorithm cannot cycle. But it is much more crucial than that. In fact without non-degeneracy we would not have our pivot operation.

Recall that at the boundary of the validity regions of two base-sequences a collision occurs, as we move parametrically to the boundary from either side. Our pivot takes the collision which is reached on one side of the boundary, and uses it to reconstruct the collision on the other side, and to reconstruct the base-sequence on the other side of the boundary. This is possible because under non-degeneracy the collisions on both sides occur at the same breakpoint time t_n, and involve the same two state variables which hit zero. This tight connection disappears without non-degeneracy: If the two base-sequences on both sides of the validity region boundary are degenerate, then they are not unique, and the optimal solutions are not unique. It is then possible that the collisions from the two sides occur at different breakpoint times, and involve different state variables. Hence approaching the collision from one side we would have no way to reconstruct the base-sequence on the other side.

To see how complicated things get without non-degeneracy, the reader may wish to look at [56–58], where earlier attempts are made to drain a re-entrant line, which is a special degenerate SCLP.

6.2. Some extensions

We discuss briefly three extensions of the results of this paper: (i) Singular control of primal and dual at $t = 0$ (ii) Solution of SCLP with piecewise constant data (iii) Solution of SCLP with analytic right hand side and objective functions. These are the subject of a follow up paper [60].

Singular control of primal and dual at $t = 0$ If we do not make assumption 1, then as we let T increase we can reach a time horizon $T^{(R)}$ with collision Case iii$_b$, in which the rates-LP under sign restriction (24) is infeasible, and its dual rates-LP* is unbounded with an unbounded extremal ray of $p, \dot{r}, \dot{q}$. There are now two possibilities: If all the unbounded $\dot{q}_j$ along the ray are ≥ 0 then SCLP is infeasible and SCLP* is unbounded for $T > T^{(R)}$. If however some of the unbounded $\dot{q}_j$ along the ray are < 0 then a singular dual control p at dual time 0 will reduce q_j^N to 0, and the solution of SCLP and of SCLP* can be continued beyond $T^{(R)}$. Similarly, a collision Case iii$_a$ may yield a rates-LP under sign restriction (23) which is unbounded, with dual rates-LP* which is infeasible, and a singular control u may then yield an instantaneous change of x_k^0 to 0, and allow the solution to be extended for $T > T^{(R)}$.

The need for singular dual controls at the time horizon T has been noted by Pullan [47]. In fact, it is the motivation for his choice of non-decreasing $\pi(t)$ which allows atomic jumps, to replace our $\int_t^T p(s)ds$. Pullan has avoided singular jumps in $u(t)$ at time 0 by requiring that u be measurable and bounded, and assumed that the feasible region of $Hu(t) = b, u(t) \geq 0$ is bounded.

If we allow such singular controls for the primal SCLP at time 0 and for the dual SCLP* at reversed time 0, then strong duality still holds. In fact, we can show that SCLP, SCLP* possess an optimal solution for all $0 < T < \infty$ if and only if the rates-LP,LP* (9) are both feasible under the sign restrictions $\dot{x} \geq 0, \dot{q} \geq 0$. Our follow up work in [60] will prove this and show how to extend our algorithm to solve such problems.

Piecewise constant data In practice we may have that instead of constant problem data G, F, H, a, b, c, d, we have piecewise constant $G^m, F^m, H^m, a^m, b^m, c^m, d^m$, in time intervals (t^{m-1}, t^m), $m = 1, \dots, M$. In such problems the solution will still be given by base-sequences, however, all the data change points t^m will automatically be breakpoints of the solution, and at these breakpoints singular primal controls u and singular dual controls p may be needed. The extension of our algorithm to solve such problems will also be described in [60].

General analytic exogenous rates Our algorithm can be used also for the solution of SCLP, SCLP* with general analytic functions $a(t), b(t), c(t), d(t)$ (instead of $\alpha + at, b, \gamma + ct, d$). For these problems a solution will still be described by a finite base-sequence. The breakpoints between those bases will again be determined by such events as $x_k(t_n) = 0, q_j(t_{n'}) = 0$, but these will now form a set of non-linear equations for the interval lengths. To determine range of validity will require the solution of these equations parametrically, to decide when intervals or slacks shrink to zero. Furthermore, an additional type of collision can occur: In the interior of an interval (t_{n-1}, t_n), corresponding to a basis B_n, it is possible for $u_j \in B_n$ or for $p_k \in B_n^*$ to hit zero, $u_j(t) \searrow 0$ or $p_k(t) \searrow 0$, for some $t_{n-1} < t < t_n$, as $\theta \nearrow \theta^{(r)}$. Nevertheless, the pivot operations for this more general problem are similar to the linear data case. The extension of our algorithm to solve such problems will also be described in [60].

Acknowledgments

This paper is the result of several years of research. I was introduced to the topic by Florin Avram, and together we first solved the two machine three buffer problem of Example as an SCLP. I am grateful for many discussions to John Vande Vate, Eddie Anderson, Malcolm Pullan, Frank Kelly, Arcady Nemirovsky, Michal Penn, Rolf Möhring, Marc Uetz, Alexander Shapiro, Jim Dai, Sean Meyn, Dimitris Bertsimas, Gil Kalai, and for inspiration to George B. Dantzig.

References

1. Anderson, E. J. (1978). *A Continuous Model for Job-Shop Scheduling.* Ph.D. Thesis, University of Cambridge, Cambridge, U. K.
2. Anderson, E. J. (1981). A new continuous model for job-shop scheduling. *International J. Systems Science* **12**, 1469–1475.
3. Anderson, E. J. and Nash, P. (1987). *Linear Programming in Infinite Dimensional Spaces.* Wiley-Interscience, Chichester.
4. Anderson, E. J., Nash, P. and Perold, A.F. (1983). Some properties of a class of continuous linear programs. *SIAM J. Control Optimization* **21**, 758–765.
5. Anderson, E. J., Nash, P. and Philpott, A.B. (1982) A class of continuous network flow problems. *Mathematics of Operations Research* **7**, 501–514.
6. Anderson, E. J. and Philpott, A.B. (1984) Duality and an algorithm for a class of continuous transportation problems. *Mathematics of Operations Research* **9**, 222–231.
7. Anderson, E. J., Philpott, A.B. (1989) A continuous time network simplex algorithm. *Networks* **19**, 395–425.
8. Anderson, E. J., Philpott, A.B. (1989) Erratum, a continuous time network simplex algorithm. *Networks* **19**, 823–827.
9. Anderson, E. J. and Pullan, M. C. (1996). Purification for separated continuous linear programs. *Mathematical Methods of Operations Research* **43**, 9–33.
10. Anstreicher, K.M. (1983) Generation of feasible descent directions in continuous time linear programming. Technical Report, SOL 83-18, Department of Operations Research, Stanford University, Stanford, CA.
11. Avram, F., Bertsimas, D., Ricard, M. (1995). Fluid models of sequencing problems in open queueing networks; an optimal control approach. In *Stochastic Networks — IMA Volumes in Mathematics and its Applications*, Editors: F.P. Kelly and Ruth Williams, Springer-Verlag, New York, 199–234.

12. Bellman, R. (1953) Bottleneck problems and dynamic programming. *Proc National Academy of Science* **39**, 947–951.
13. Bellman, R. (1957) *Dynamic Programming*. Princeton University Press, Princeton NJ.
14. Bertsimas, D., Gamarnik, D. (1998). Asymptotically optimal algorithms for job shop scheduling and packet routing. Preprint.
15. Bertsimas, D., Gamarnik, D., Sethuraman, J. (2002) From fluid relaxation to practical algorithms for high multiplicity job shop scheduling: the holding cost objective. Preprint
16. Bertsimas, D., Sethuraman, J. (2002) From fluid relaxation to practical algorithms for job shop scheduling: the makespan objective. *Mathematical Programming, Series A* 92:61–102.
17. Bertsimas, D., Tsitsiklis, J.N. (1997) *Introduction to Linear Optimization*, Athena Scientific, Belmont, Massachusetts.
18. Boudoukh, T., Penn, M. and Weiss, G. (2001) Scheduling jobshops with some identical or similar jobs. *J. of Scheduling* 4:177-199.
19. Buie, R.N., Abraham, J. (1973) Numerical solution to continuous linear programming problems. *Z. Operations Research* **17**, 107–111.
20. Dai, J.G. 1995. On positive Harris recurrence of multi-class queueing networks: a unified approach via fluid limit models. *Annals of Applied Probability* **5**, 49–77.
21. Dai, J.G., Weiss G. 2002. A fluid heuristic for minimizing makespan in job-shops. *Operations Research* 50:692-707.
22. Dantzig, G.B. (1963) *Linear Programming and Extensions* Princeton University Press, Princeton, NJ.
23. Dorfman, R., Samuelson, P.A., Solow, R.M. (1958) *Linear Programming and Economic Analysis* MacGraw Hill, New York, 1958 (Dover edition, 1987).
24. Drews, W.P. (1974) A simplex like algorithm for continuous time linear optimal control problems. In *Optimization Methods for Resource Allocation* R.W. Cottle, J. Krarup, editors. Crane Russak and Co. New York.
25. Fukuda, K. and Terlaky, T. (1997) Criss cross methods, a fresh view on pivot algorithms. *Mathematical Programming* **79**, 369–395.
26. Grinold, R.C. (1970) Symmetric duality for continuous linear programs. *SIAM J. Applied Mathematics* **18**, 32–51.
27. Hajek, B., Ogier, R.G. (1984) Optimal dynamic routing in communications networks with continuous traffic. *Networks* **14**, 457–487.
28. Hartberger, R.J. (1974) Representation extended to continuous time. In *Optimization Methods for Resource Allocation* R.W. Cottle, J. Krarup, editors. Crane Russak and Co. New York.
29. Ilyutovich, A.E. (1976) Iterative optimization methods for linear programming problems in functional spaces. In: *Studies in Linear and Non-Linear Programming*, Stanford University Press, Stanford, CA.
30. Ilyutovich, A.E. (1976) Piecewise continuous solutions of linear dynamic problems in economic planning. *Automat Remote Control* **41**, 501–508.
31. Ito, C. Kelley, T. and Sachs, E.W. (1995) Inexact primal dual interior point iteration for linear program in function spaces. Comput. Optim. Appl. **4**, 189–202.
32. Koopmans, T.C. (1951), editor, in Cooperation with Alchien, A., Dantzig, G.B., Georgescu-Roegen, N., Samuelson, P.A., Tucker, A.W. *Activity Analysis of Production and Allocation*. Wiley, New York, Chapman and Hall London, 1951.
33. Lehman, R.S. (1954) On the continuous simplex method. Technical report RM-1386, Rand Corporations, Santa Monica.
34. Leontief, W.W. (1941) *The Structure of the American Economy, 1919–1929* Harvard University Press, Cambridge, Mass. (new, enlarged edition, Oxford University Press, New York, 1951).
35. Levinson, N. (1966) A class of continuous linear programming problems. *J. Mathematical Analysis Applications* **16**, 73–83.
36. X. Luo and D. Bertsimas (1998) A new algorithm for state constrained separated continuous linear programs. *SIAM J. Control and Optimization*, **37**, 177–210.
37. Meyn, S.P (1995) The policy improvement algorithm for Markov decision processes with general state space. *IEEE Transactions on Automatic Control*, to appear.
38. Meyn, S.P. (2001) Sequencing and routing in multiclass queueing networks. part I: feedback regulation, *SIAM J. Control and Optimization*, 40:741–776.
39. Meyn, S.P. (2001) Sequencing and Routing in Multiclass Queueing Networks. Part II: Workload Relaxations, *SIAM J. Control and Optimization* to appear.
40. Chen, M., Pandit, C., Meyn, S.P. (2001) In Search of Sensitivity in Network Optimization. Preprint.

41. Perold, A.F. (1978) Fundamentals of a continuous time simplex method. Technical Report, SOL 78-26, Department of Operations Research, Stanford University, Stanford, CA.
42. Perold, A.F. (1981) Extreme points and basic feasible solutions in continuous time linear programming. *SIAM J. Control Optimization* **19**, 52–63.
43. Philpott, A.B. and Craddock, M. (1995) An adaptive discretization algorithm for a class of continuous network programs. *Networks* **26**, 1–11.
44. Pullan, M. C. (1993). An algorithm for a class of continuous linear programs. *SIAM J. Control and Optimization* **31**, 1558–1577.
45. Pullan, M. C. (1995). Forms of optimal solutions for separated continuous linear programs. *SIAM J. Control and Optimization* **33**, 1952–1977.
46. Pullan, M. C. (1996). A duality theory for separated continuous linear programs. *SIAM J. Control and Optimization* **34** 931–965.
47. Pullan, M. C. (1997). Existence and duality theory for separated continuous linear programs. *Mathematical Modeling of Systems* **3**, 219–245.
48. Pullan, M. C. (2000). Convergence of a general class of algorithms for separated continuous linear programs. *SIAM Journal on Optimization* **10**, 722-731
49. Pullan, M. C. (2002). An extended algorithm for separated continuous linear programs *Mathematical Programming* To appear.
50. Segers, R.G. (1974) A generalized function setting for dynamic optimal control problems. In *Optimization Methods for Resource Allocation* R.W. Cottle, J. Krarup, editors. Crane Russak and Co. New York.
51. Shapiro, A. (2000) On duality theory of conic linear problems. Preprint.
52. Sourd, F. (2002) The continuous assignment problem and its application to preemptive and non-preemptive scheduling with irregular cost functions. Preprint
53. Tyndall, W.F. (1965), Duality theorem for a class of continuous linear programming problems. *SIAM J. Applied Mathematics* **13**, 644–666.
54. Vanderbei, R.J. (1997) *Linear Programming, Foundations and Extensions*, Kluwer, Boston.
55. Verhaegh, W.F.J. (2001) Capacity Scheduling for Data Services over Digital Networks. Unpublished report.
56. Weiss, G. (1995) On optimal draining of fluid re-entrant lines. In *Stochastic Networks — IMA Volumes in Mathematics and its Applications*, Editors: F.P. Kelly and Ruth Williams, Springer-Verlag, New York, 93–105.
57. Weiss, G. (1996) Optimal draining of fluid re-entrant lines: Some solved examples. *Stochastic Networks: Theory and Applications*, F. P. Kelly, I. Ziedins, S. Zachary, Editors, Oxford University Press.
58. Weiss, G. (1999) Algorithm for minimum wait draining of two station re-entrant line. *Annals of Operations Research* **92**, 65-86.
59. Weiss, G. (1999) Scheduling and control of manufacturing systems — a fluid approach *Proceedings of the 37 Allerton Conference, 21–24 September, 1999, Monticello, Illinois*, 577-586.
60. Weiss, G. (2002) A simplex algorithm for separated continuous linear programs, part II In preparation.