Foundation of Cryptography, Lecture 2
Pseudorandom Generators

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Part I

Statistical Vs. Computational distance

Section 1

Distributions and Statistical Distance

Distributions and Statistical Distance

Let P and Q be two distributions over a finite set $\mathcal{U}$. Their statistical distance (also known as, variation distance) is defined as

$$\mathrm{SD}(P, Q) := \frac{1}{2} \sum_{x \in \mathcal{U}} |P(x) - Q(x)| = \max_{\mathcal{S} \subseteq \mathcal{U}} (P(\mathcal{S}) - Q(\mathcal{S}))$$

We will only consider finite distributions.

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Claim 1

For any pair of (finite) distribution P and Q, it holds that

$$\mathsf{SD}(P,Q) = \max_{\mathsf{D}}\{\Pr_{x\leftarrow P}[\mathsf{D}(x)=1] - \Pr_{x\leftarrow Q}[\mathsf{D}(x)=1]\},$$

where D is any algorithm.

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Interpretation?

Some useful facts

Let P, Q, R be finite distributions, then

Triangle inequality:

$$\mathsf{SD}(P,R) \leq \mathsf{SD}(P,Q) + \mathsf{SD}(Q,R)$$

Repeated sampling:

$$\mathsf{SD}((P,P),(Q,Q)) \leq 2 \cdot \mathsf{SD}(P,Q)$$

Distribution ensembles and statistical indistinguishability

Definition 2 (distribution ensembles)

$\mathcal{P} = \{P_n\}_{n \in \mathbb{N}}$ is a distribution ensemble, if P_n is a (finite) distribution for any $n \in \mathbb{N}$.
$\mathcal{P}$ is efficiently samplable (or just efficient), if $\exists$ PPT *Samp* with $\text{Sam}(1^n) \equiv P_n$.

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Section 2

Computational Indistinguishability

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- Non uniform variant

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- Can it be different from the statistical case?
- Non uniform variant
- Sometime behaves different then expected!

Repeated sampling

Question 5

Assume that $\mathcal{P}$ and $\mathcal{Q}$ are computationally indistinguishable, is it always true that $\mathcal{P}^2 = (\mathcal{P}, \mathcal{P})$ and $\mathcal{Q}^2 = (\mathcal{Q}, \mathcal{Q})$ are?

Repeated sampling

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Let D be an algorithm and let $\delta(n) = \left| \Delta^{\mathsf{D}}_{(\mathcal{P}^2, \mathcal{Q}^2)}(n) \right|$

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$$\begin{aligned}
\delta(n) &= \left|\Pr_{x\leftarrow P_n^2}[\mathsf{D}(x)=1] - \Pr_{x\leftarrow Q_n^2}[\mathsf{D}(x)=1]\right| \\
&\leq \left|\Pr_{x\leftarrow P_n^2}[\mathsf{D}(x)=1] - \Pr_{x\leftarrow (P_n,Q_n)}[\mathsf{D}(x)=1]\right| \\
&\quad + \left|\Pr_{x\leftarrow (P_n,Q_n)}[\mathsf{D}(x)=1] - \Pr_{x\leftarrow Q_n^2}[\mathsf{D}(x)=1]\right|
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&= \left|\Delta^{\mathsf{D}}_{(\mathcal{P}^2, (\mathcal{P}, \mathcal{Q}))}(n)\right| + \left|\Delta^{\mathsf{D}}_{((\mathcal{P}, \mathcal{Q}), \mathcal{Q}^2)}(n)\right|
\end{aligned}$$

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Let D be an algorithm and let $\delta(n) = \left|\Delta^{\mathsf{D}}_{(\mathcal{P}^2, \mathcal{Q}^2)}(n)\right|$

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&\quad + \left| \Pr_{x \leftarrow (P_n, Q_n)}[\mathsf{D}(x) = 1] - \Pr_{x \leftarrow Q_n^2}[\mathsf{D}(x) = 1] \right| \\
&= \left|\Delta^{\mathsf{D}}_{(\mathcal{P}^2, (\mathcal{P}, \mathcal{Q}))}(n)\right| + \left|\Delta^{\mathsf{D}}_{((\mathcal{P}, \mathcal{Q}), \mathcal{Q}^2)}(n)\right|
\end{aligned}$$

So either $|\Delta^{\mathsf{D}}_{(\mathcal{P}^2, (\mathcal{P}, \mathcal{Q}))}(n)| \geq \delta(n)/2$, or $|\Delta^{\mathsf{D}}_{((\mathcal{P}, \mathcal{Q}), \mathcal{Q}^2)}(n)| \geq \delta(n)/2$

- Assume D is a PPT and that $\left|\Delta^{\mathsf{D}}_{(\mathcal{P}^2,\mathcal{Q}^2)}(n)\right| \geq 1/p(n)$ for some $p \in \mathrm{poly}$ and infinitely many n's, and assume wlg. that $\left|\Delta^{\mathsf{D}}_{\mathcal{P}^2,(\mathcal{P},\mathcal{Q})}(n)\right| \geq 1/2p(n)$ for infinitely many n's.

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- Can we use D to contradict the fact that $\mathcal{P}$ and $\mathcal{Q}$ are computationally close?
- Assuming that $\mathcal{P}$ and $\mathcal{Q}$ are efficiently samplable
- Non-uniform settings

Repeated sampling cont.

Given $t = t(n) \in \mathbb{N}$ and a distribution ensemble $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$, let $\mathcal{P}^t = \{P_n^{t(n)}\}_{n\in\mathbb{N}}$.

Repeated sampling cont.

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Question 6

Let $t = t(n) \leq \mathrm{poly}(n)$ be an eff. computable integer function. Assume that $\mathcal{P}$ and $\mathcal{Q}$ are eff. samplable and computationally indistinguishable, does it mean that $\mathcal{P}^t$ and $\mathcal{Q}^t$ are?

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Proof:

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Proof:

- Induction?

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Proof:

- Induction?
- Hybrid

Hybrid argument

Let D be an algorithm and let $\delta(n) = \left|\Delta^{\mathsf{D}}_{(\mathcal{P}^t,\mathcal{Q}^t)}(n)\right|$.

- Fix $n \in \mathbb{N}$, and for $i \in \{0,\ldots,t = t(n)\}$, let $H^i = (p_1,\ldots,p_i,q_{i+1},\ldots,q_t)$, where the p's [resp., q's] are uniformly (and independently) chosen from P_n [resp., from Q_n].

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- Since $\delta(n) = \left|\Delta^{\mathsf{D}}_{H^t,H^0}(t)\right| = \left|\sum_{i\in[t]}\Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right|$, there exists $i \in [t]$ with $\left|\Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right| \geq \delta(n)/t(n)$.

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- Since $\delta(n) = \left|\Delta^{\mathsf{D}}_{H^t,H^0}(t)\right| = \left|\sum_{i\in[t]} \Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right|$, there exists $i \in [t]$ with $\left|\Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right| \geq \delta(n)/t(n)$.
- How do we use it?

Using hybrid argument via estimation

Algorithm 7 (D′)

Input: 1^n and $x \in \{0,1\}^*$

1. Find $i \in [t]$ with $\left|\Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right| \geq \delta(n)/2t(n)$
2. Let $(p_1,\ldots,p_i,q_{i+1},\ldots,q_t) \leftarrow H^i$
3. Return $\mathsf{D}(1^t,p_1,\ldots,p_{i-1},x,q_{i+1},\ldots,q_t)$, .

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1. how do we find i? why $\delta(n)/2t(n)$

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1. how do we find i? why $\delta(n)/2t(n)$
2. Easy in the non-uniform case

Using hybrid argument via sampling

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Input: 1^n and $x \in \{0,1\}^*$

1. Sample $i \leftarrow [t = t(n)]$
2. Let $(p_1, \ldots, p_i, q_{i+1}, \ldots, q_t) \leftarrow H^i$
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$$\left|\Delta^{\mathsf{D}'}_{(\mathcal{P},\mathcal{Q})}(n)\right| = \left|\Pr_{p \leftarrow P_n}[\mathsf{D}'(p) = 1] - \Pr_{q \leftarrow Q_n}[\mathsf{D}'(q) = 1]\right|$$

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&= \left|\frac{1}{t}\left(\Pr_{x\leftarrow H_t}[\mathsf{D}(x)=1] - \Pr_{x\leftarrow H_0}[\mathsf{D}(x)=1]\right)\right|
\end{aligned}
$$

Using hybrid argument via sampling

Algorithm 8 (D')

Input: 1^n and $x \in \{0,1\}^*$

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2. Let $(p_1, \ldots, p_i, q_{i+1}, \ldots, q_t) \leftarrow H^i$
3. Return $D(1^t, p_1, \ldots, p_{i-1}, x, q_{i+1}, \ldots, q_t)$.

$$
\begin{aligned}
\left|\Delta^{D'}_{(\mathcal{P},\mathcal{Q})}(n)\right| &= \left|\Pr_{p \leftarrow P_n}[D'(p) = 1] - \Pr_{q \leftarrow Q_n}[D'(q) = 1]\right| \\
&= \left|\frac{1}{t}\sum_{i\in[t]} \Pr_{x \leftarrow H_i}[D(x) = 1] - \frac{1}{t}\sum_{i\in[t]} \Pr_{x \leftarrow H_{i-1}}[D(x) = 1]\right| \\
&= \left|\frac{1}{t}\left(\Pr_{x \leftarrow H_t}[D(x) = 1] - \Pr_{x \leftarrow H_0}[D(x) = 1]\right)\right| \\
&= \delta(n)/t(n)
\end{aligned}
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Part II

Pseudorandom Generators

Pseudorandom generator

Definition 9 (pseudorandom distributions)

A distribution ensemble $\mathcal{P}$ over $\{\{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ is pseudorandom, if it is computationally indistinguishable from $\{U_{\ell(n)}\}_{n\in\mathbb{N}}$.

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- Do such distributions exit?

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An efficiently computable function $g\colon\{0,1\}^n \mapsto \{0,1\}^{\ell(n)}$ is a pseudorandom generator, if

- g is length extending (i.e., $\ell(n) > n$ for any n)
- $g(U_n)$ is pseudorandom

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- Imply one-way functions (homework)

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- Do such generators exist?
- Imply one-way functions (homework)
- Do they have any use?

Section 3

Hardcore Predicates

Hardcore predicates

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An efficiently computable function $b : \{0,1\}^n \mapsto \{0,1\}$ is a hardcore predicate of $f : \{0,1\}^n \mapsto \{0,1\}^n$, if

$$\Pr_{x \leftarrow \{0,1\}^n}[P(f(x)) = b(x)] \leq \frac{1}{2} + \text{neg}(n),$$

for any PPT P.

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for any PPT P.

- Does the existence of a hardcore predicate for f, implies that f is one way?

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- Building blocks in constructions of PRGS from OWF

Section 4

PRGs from OWPs

OWP to PRG

Claim 12

Let $f : \{0,1\}^n \mapsto \{0,1\}^n$ be an eff. permutation and let $b : \{0,1\}^n \mapsto \{0,1\}$ be a hardcore predicate for f, then $g(x) = (f(x), b(x))$ is a PRG.

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$$\left|\Delta^{\mathsf{D}}_{g(U_n),U_{n+1}}\right| > \varepsilon(n) = 1/p(n)$$

for any $n \in \mathcal{I}$. We use D for breaking the hardness of b.

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- We assume wlg. that $\Pr[\mathsf{D}(g(U_n)) = 1] - \Pr[\mathsf{D}(U_{n+1}) = 1] \geq \varepsilon(n)$ for any $n \in \mathcal{I}$ (?), and fix $n \in \mathcal{I}$.

OWP to PRG cont.

- Let $\delta(n) = \Pr[\mathsf{D}(U_{n+1}) = 1]$ (note that $\Pr[\mathsf{D}(g(U_n)) = 1] = \delta + \varepsilon$).

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Hence,

$$\Pr[\mathsf{D}(f(U_n), \overline{b(U_n)}) = 1] = \delta - \varepsilon \qquad (2)$$

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Input: $y \in \{0,1\}^n$

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OWP to PRG cont.

Remark 14

- Prediction to distinguishing (homework)

OWP to PRG cont.

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- PRG from any OWF: (1) Regular OWFs, first use pairwise hashing to convert into “almost" permutation. (2) Any OWF, harder

PRG Length Extension

Construction 15 (iterated function)

Given $g\colon \{0,1\}^n \mapsto \{0,1\}^{n+1}$ and $i \in \mathbb{N}$, define $g^i\colon \{0,1\}^n \mapsto \{0,1\}^{n+i}$ as

$$g^i(x) = g(x)_1, g^{i-1}(g(x)_{2,\ldots,n+1}),$$

where $g^0(x) = x$.

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Let $g\colon \{0,1\}^n \mapsto \{0,1\}^{n+1}$ be a PRG, then $g^{t(n)}\colon \{0,1\}^n \mapsto \{0,1\}^{n+t(n)}$ is a PRG, for any $t \in \text{poly}$.

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Algorithm 17 (D′)

Input: 1^n and $y \in \{0,1\}^{n+1}$

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