Foundation of Cryptography, Lecture 1
One-Way Functions

Handout Mode

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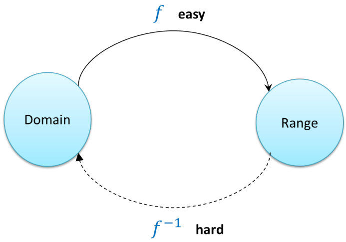
Section 1

One-Way Functions

Informal discussion



A one-way function (OWF) is:

- Easy to compute, everywhere
- Hard to invert, on the average

- Why should we care about OWFs?
- Hidden in (almost) any cryptographic primitive: necessary for "cryptography"
- Sufficient for many cryptographic primitives

"Application": Authentication where server doesn't store the user's password.

Formal definition

Definition 1 (one-way functions (OWFs))

A polynomial-time computable function $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ is one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^n, f(x)) \in f^{-1}(f(x))\right] = \mathrm{neg}(n)$$

for any PPT A.

- polynomial-time computable: there exists polynomial-time algorithm F, such that $F(x) = f(x)$ for every $x \in \{0,1\}^*$.
- neg: a function $\mu\colon \mathbb{N} \mapsto [0,1]$ is a negligible function of n, denoted $\mu(n) = \mathrm{neg}(n)$, if for any $p \in \mathrm{poly}$ there exists $n' \in \mathbb{N}$ such that $\mu(n) < 1/p(n)$ for all $n > n'$
- $x \leftarrow \{0,1\}^n$: x is uniformly drawn from $\{0,1\}^n$
- PPT: probabilistic polynomial-time algorithm.

We typically omit 1^n from the input list of A

Formal definition cont.

1. Is this the right definition?
   - Asymptotic
   - Efficiently computable
   - On the average
   - Only against PPT's
2. OWF $\implies \mathcal{P} \neq \mathcal{NP}$
3. Does $\mathcal{P} \neq \mathcal{NP} \implies$ OWF?
4. (most) Crypto implies OWFs
5. Do OWFs imply Crypto?
6. Where do we find them?
7. Non uniform OWFs

Definition 2 (Non-uniform OWF))

A polynomial-time computable function $f : \{0,1\}^* \mapsto \{0,1\}^*$ is non-uniformly one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[C_n(f(x)) \in f^{-1}(f(x))\right] = \mathrm{neg}(n)$$

for any polynomial-size family of circuits $\{C_n\}_{n \in \mathbb{N}}$.

Length-preserving functions

Definition 3 (length preserving functions)

A function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is length preserving, if $|f(x)| = |x|$ for every $x \in \{0,1\}^*$

Theorem 4

Assume that OWFs exit, then there exist length-preserving OWFs.

Proof idea: use the assumed OWF to create a length preserving one.

Partial domain functions

Definition 5 (Partial domain functions)

Let $m, \ell\colon \mathbb{N} \mapsto \mathbb{N}$ be polynomials. Let $f\colon \{0,1\}^{\ell(n)} \mapsto \{0,1\}^{m(n)}$ denote a function defined over input lengths in $\{m(n)\}_{n\in\mathbb{N}}$, and maps strings of length $\ell(n)$ to strings of length $m(n)$.

Such function is efficient, if it is poly-time computable.

The definition of one-wayness naturally extends to such (efficient) functions.

OWFs imply length-preserving OWFs cont.

Let $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ be a OWF, let $p \in \text{poly}$ be a bound on its computing-time, and assume wlg. that p is monotony increasing (can we?). Note that $|f(x)| \leq p(|x|)$.

Construction 6 (the length preserving function)

Define $g\colon \{0,1\}^{p(n)+1} \mapsto \{0,1\}^{p(n)+1}$ as

$$g(x) = f(x_{1,\ldots,n}), 1, 0^{p(n)-|f(x_{1,\ldots,n})|}$$

Note that g is well defined, length preserving and efficient.

Claim 7

g is one-way.

How can we prove that g is one-way?

Answer: using reduction.

Proving that g is one-way

Proof: Assume that g is not one-way. Namely, there exists PPT A, $q \in \text{poly}$ and infinite set $\mathcal{I} \subseteq \{p(n)+1 \colon n \in \mathbb{N}\}$, with

$$\Pr_{x \leftarrow \{0,1\}^{n'}}\left[\mathsf{A}(1^{n'}, y) \in g^{-1}(g(x))\right] > 1/q(n') \tag{1}$$

for every $n' \in \mathcal{I}$.

We show how to use A for inverting f.

Claim 8

$w \in g^{-1}(y, 1, 0^{p(n)-|y|}) \implies w_{1,\ldots,n} \in f^{-1}(y)$

Proof: Since $g(w) = f(w_{1,\ldots,n}), 1, 0^{p(n)-|f(w_{1,\ldots,n})|} = y, 1, 0^{p(n)-|y|}$,

it follows that $f(w_{1,\ldots,n}) = y$ (?).$\square$

Algorithm 9 (Inverter B for f)

Input: 1^n and $y \in \{0,1\}^*$

1. Let $x = \mathsf{A}(1^{p(n)+1}, y, 1, 0^{p(n)-|y|})$
2. Return $x_{1,\ldots,n}$

Claim 10

Let $\mathcal{I}' := \{n \in \mathbb{N} \colon p(n)+1 \in \mathcal{I}\}$. Then

1. $\mathcal{I}'$ is infinite
2. $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{B}(1^n, f(x)) \in f^{-1}(f(x))] > 1/q(p(n)+1)$ for every $n \in \mathcal{I}'$

This contradicts the assumed one-wayness of f. $\square$

Proof: (1) is clear, (2)

$$\begin{aligned}
&\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{B}(1^n, f(x)) \in f^{-1}(f(x))\right] \\
&= \Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^{p(n)+1}, f(x), 1, 0^{p(n)-|f(x)|})_{1,\ldots,n} \in f^{-1}(f(x))\right] \\
&= \Pr_{x' \leftarrow \{0,1\}^{p(n)+1}}\left[\mathsf{A}(1^{p(n)+1}, g(x'))_{1,\ldots,n} \in f^{-1}(f(x'_{1,\ldots,n}))\right] \\
&\geq \Pr_{x' \leftarrow \{0,1\}^{p(n)+1}}\left[\mathsf{A}(1^{p(n)+1}, g(x')) \in g^{-1}(g(x'))\right] \geq 1/q(p(n)+1)
\end{aligned}$$

From partial-domain length-preserving OWFs to length-preserving OWFs

Construction 11

Given a function $f\colon \{0,1\}^{\ell(n)} \mapsto \{0,1\}^{\ell(n)}$, define $f_{\mathrm{all}}\colon \{0,1\}^n \mapsto \{0,1\}^n$ as

$$f_{\mathrm{all}}(x) = f(x_{1,\ldots,k}), 0^{n-k}$$

where $n = |x|$ and $k := \max\{\ell(n') \leq n\colon n' \in [n]\}$.

Clearly, f_{all} is length preserving, defined for every input length, and efficient if f is.

Claim 12

Assume f is efficient, f is one-way, and ℓ satisfies $1 \leq \frac{\ell(n+1)}{\ell(n)} \leq p(n)$ for some $p \in \mathrm{poly}$, then f_{all} is one-way function.

Proof: ?

We conclude that the existence of OWF implies the existence of length-preserving OWF that is defined over all input lengths.

Few remarks

More “security-preserving" reductions exits.

Convention for rest of the talk

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a one-way function.

Weak one-way functions

Definition 13 (weak one-way functions)

A poly-time computable function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is α-one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^n, f(x)) \in f^{-1}(f(x))\right] \le \alpha(n)$$

for any PPT A and large enough $n \in \mathbb{N}$.

1. For example consider $\alpha(n) = 0.1$, or $\alpha(n) = 0.99$ or maybe even $\alpha(n) = 1 - 1/n$.
2. (strong) OWF according to Definition 1, are neg-one-way according to the above definition
3. Can we “amplify" weak OWF to strong ones?

Strong to weak OWFs

Claim 14

Assume there exists OWFs, then there exist functions that are $\frac{2}{3}$-one-way, but not (strong) one-way

Proof: For a OWF f, let

$$g(x,b) = \begin{cases} (1, f(x)), & b = 1; \\ (0, x), & \text{otherwise } (b = 0). \end{cases}$$

Weak to strong OWFs

Theorem 15 (weak to strong OWFs (Yao))

Assume there exist $(1-\delta)$-weak OWFs with $\delta(n) \geq 1/q(n)$ for some $q \in \text{poly}$, then there exist (strong) one-way functions.

- Idea: parallel repetition (i.e., direct product): Consider $g(x_1, \ldots, x_t) = f(x_1), \ldots, f(x_t)$ for large enough t
- Motivation: if something is somewhat hard, than doing it many times is (very) hard
- But, is it really so?

  Consider matrix multiplication: Let $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$

  Computing Ax takes $\Theta(n^2)$ times, but computing $A(x_1, x_2, \ldots, x_n)$ takes …only $O(n^{2.3\ldots}) < \Theta(n^3)$
- Fortunately, parallel repetition does amplify weak OWFs :-)

Amplification via parallel repetition

Theorem 16

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a $(1-\delta)$-weak OWF for $\delta(n) = 1/q(n)$ for some (positive) $q \in \mathrm{poly}$, and let $t(n) = \left\lceil \frac{\log^2 n}{\delta(n)} \right\rceil$. Then $g\colon (\{0,1\}^n)^{t(n)} \mapsto (\{0,1\}^n)^{t(n)}$ defined by $g(x_1,\ldots,x_{t(n)}) = f(x_1),\ldots,f(x_{t(n)})$, is a one-way function.

Clearly g is efficient. Is it one-way? Proof via reduction: Assume $\exists$ PPT A violating the one-wayness of g, we show there exists a PPT B violating the weak hardness of f.

Difficultly: We need to use an inverter for g with low success probability, e.g., $\frac{1}{n}$, to get an inverter for f with high success probability, e.g., $\frac{1}{2}$ or even $1-\frac{1}{n}$

In the following we fix (an assumed) PPT A, $p \in \mathrm{poly}$ and infinite set $\mathcal{I} \subseteq \mathbb{N}$ s.t.

$$\Pr_{w \leftarrow \{0,1\}^{t(n)\cdot n}}[\mathsf{A}(g(w)) \in g^{-1}(g(w))] \geq 1/p(n)$$

for every $n \in \mathcal{I}$. We also “fix" $n \in \mathcal{I}$ and omit it from the notation.

Proving that g is One-Way – the Naive approach

Assume A attacks each of the t outputs of g independently: $\exists$ PPT A$'$ such that $A(z_1,\ldots,z_t) = A'(z_1)\ldots,A'(z_t)$

It follows that A$'$ inverts f with probability greater than $(1-\delta)$. Otherwise

$$\Pr_{w\leftarrow\{0,1\}^{t\cdot n}}[A(g(w)) \in g^{-1}(g(w))] = \prod_{i=1}^{t} \Pr_{x\leftarrow\{0,1\}^n}\left[A'(f(x)) \in f^{-1}(f(x))\right]$$
$$\le (1-\delta)^t \le e^{-\log^2 n} \le n^{-\log n}$$

Hence A$'$ violates the weak hardness of f

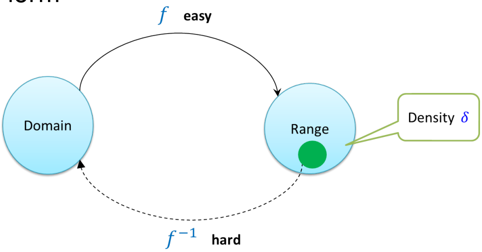
A less naive approach would be to assume that A goes over the inputs sequentially.

Unfortunately, we can assume none of the above.

Any idea?

Hardcore sets

Assume f is of the form



Definition 17 (hardcore sets)

$\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$ is a δ-hardcore set for $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, if:

1. $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}] \geq \delta(n)$ for large enough n, and
2. For any PPT A and $q \in$ poly: for large enough n, it holds that $\Pr\left[\mathsf{A}(y) \in f^{-1}(y)\right] \leq \frac{1}{q(n)}$ for every $y \in \mathcal{S}_n$.

Assuming f has such a δ-HC set seems like a good starting point :-)

Unfortunately, we do not know how to prove that f has hardcore set :-<

Failing sets

Definition 18 (failing sets)

$f\colon \{0,1\}^n \mapsto \{0,1\}^n$ has a δ-failing set for a pair (A, q) of algorithm and polynomial, if exists $\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$, such that the following holds for large enough n:

1. $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}_n] \geq \delta(n)$, and
2. $\Pr\left[\mathsf{A}(y) \in f^{-1}(y)\right] \leq 1/q(n)$, for every $y \in \mathcal{S}_n$

Claim 19

Let f be a $(1-\delta)$-OWF, then f has a $\delta/2$-failing set, for any pair of PPT A and $q \in \text{poly}$.

High level idea: Define $\mathcal{S}_n := \{y \in \{0,1\}^n \colon \Pr\left[\mathsf{A}(y) \in f^{-1}(y)]\right] < 1/q(n)\}$.

1. If this set is small, show that A inverts f very well.
2. If this set is large, then it is by definition a fooling set.

Proof: Assume $\exists$ PPT A and $q \in \text{poly}$, such that for any $\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$ at least one of the following holds:

1. $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}_n] < \delta(n)/2$ for infinitely many n's, or
2. For infinitely many n's: $\exists y \in \mathcal{S}_n$ with $\Pr\left[\mathsf{A}(y) \in f^{-1}(y)\right] \geq 1/q(n)$.

We'll use A to contradict the hardness of f.

Using A to invert f

For $n \in \mathbb{N}$, let $\mathcal{S}_n := \{y \in \{0,1\}^n \colon \Pr\left[\mathsf{A}(y) \in f^{-1}(y)\right] < 1/q(n)\}$.
The second item cannot hold, therefore the first item must hold, meaning that:

Claim 20

$\exists$ infinite $\mathcal{I} \subseteq \mathbb{N}$ with $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}_n] < \delta(n)/2$ for every $n \in \mathcal{I}$.

Algorithm 21 (The inverter B on input $y \in \{0,1\}^n$)

Do (with fresh randomness) for $n \cdot q(n)$ times:
If $x = \mathsf{A}(y) \in f^{-1}(y)$, return x

Clearly, B is a PPT

Claim 22

For $n \in \mathcal{I}$, it holds that $\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{B}(f(x)) \in f^{-1}(f(x))\right] > 1 - \frac{\delta(n)}{2} - 2^{-n}$

Proof: ?

Hence, for large enough $n \in \mathcal{I}$: $\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{B}(f(x)) \in f^{-1}(f(x))\right] > 1 - \delta(n)$.

Namely, f is not $(1-\delta)$-one-way $\square$

g is not one-way $\implies$ f has no $\delta/2$ failing set

We show: g is not one way $\implies$ f has no $\delta/2$ failing-set for some PPT B and $q \in \text{poly}$.

Claim 23

Assume $\exists$ PPT A, $p \in \text{poly}$ and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that

$$\Pr_{w \leftarrow \{0,1\}^{t(n)\cdot n}}\left[\mathsf{A}(g(x)) \in g^{-1}(g(w))\right] \geq \frac{1}{p(n)}$$

for every $n \in \mathcal{I}$. Then $\exists$ PPT B such that

$$\Pr_{x \leftarrow \{0,1\}^n \mid y=f(x) \in \mathcal{S}_n}\left[\mathsf{B}(y) \in f^{-1}(y)\right] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$$

for every $n \in \mathcal{I}$ and every $\mathcal{S}_n \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}_n] \geq \delta(n)/2$.

Thm follows: Fix $\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$. By Claim 23, for every $n \in \mathcal{I}$, either

- $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}_n] < \delta(n)/2$, or
- $\Pr_{x \leftarrow \{0,1\}^n \mid y=f(x) \in \mathcal{S}_n}\left[\mathsf{B}(y) \in f^{-1}(y)\right] \geq \frac{1}{t(n)p(n)} - n^{-\log n}$
  $\overset{\text{(for large enough } n)}{\geq} \frac{1}{2t(n)p(n)}$
  $\overset{\text{(for large enough } n)}{\implies} \exists y \in \mathcal{S}_n\colon \Pr\left[\mathsf{B}(y) \in f^{-1}(y)\right] \geq \frac{1}{2t(n)p(n)}$.

The no failing-set algorithm: Proof of main claim

Algorithm 24 (Inverter B on input $y \in \{0,1\}^n$)

1. Choose $w \leftarrow (\{0,1\}^n)^{t(n)}$, $z = (z_1, \ldots, z_t) = g(w)$ and $i \leftarrow [t]$
2. Set $z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t)$
3. Return $\mathsf{A}(z')_i$

Fix $n \in \mathcal{I}$ and a set $\mathcal{S}_n \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}] \geq \delta(n)/2$.

Claim 25

$\Pr_{x \leftarrow \{0,1\}^n | y = f(x) \in \mathcal{S}_n}\left[\mathsf{B}(y) \in f^{-1}(y)\right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}$

Proving $\Pr_{x \leftarrow \{0,1\}^n \mid y=f(x) \in \mathcal{S}_n}\left[\mathsf{B}(y) \in f^{-1}(y)\right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}$

Algorithm 26 (Inverter B on input $y \in \{0,1\}^n$)

1. Choose $w \leftarrow (\{0,1\}^n)^{t(n)}$, $z = (z_1, \ldots, z_t) = g(w)$ and $i \leftarrow [t]$
2. Set $z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t)$
3. Return $\mathsf{A}(z')_i$

- For $Typ = \{v \in \{0,1\}^{t \cdot n} \colon \exists i \in [t] \colon v_i \in \mathcal{S}_n\}$, it holds $\Pr_z[Typ] \geq 1 - n^{-\log n}$
- $\forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n}$:
$$\Pr_z[\mathcal{L}' = \mathcal{L} \cap Typ] = \sum_{\ell \in \mathcal{L}'} \Pr[z = \ell] \leq \sum_{\ell \in \mathcal{L}'} \frac{\Pr[z' = \ell]}{t} = \frac{\Pr_{z'}[\mathcal{L}']}{t}.$$
- Hence $\forall \mathcal{L} \subseteq \{0,1\}^{t(n) \cdot n}$: $\Pr_{z'}[\mathcal{L}] \geq \frac{\Pr_z[\mathcal{L} \cap Typ]}{t(n)} \geq \frac{\Pr_z[\mathcal{L}] - n^{-\log n}}{t(n)}$.
- Assume A is *deterministic* and let $\mathcal{L}_{\mathcal{A}} = \{v \in \{0,1\}^{t \cdot n} \colon \mathsf{A}(v) \in g^{-1}(v)\}$.

$$\Pr_{x \leftarrow \{0,1\}^n \mid y=f(x) \in \mathcal{S}_n}\left[\mathsf{B}(y) \in f^{-1}(y)\right] \geq \Pr[z' \in \mathcal{L}_{\mathcal{A}}] \geq \frac{\Pr[z \in \mathcal{L}_{\mathcal{A}}] - n^{-\log n}}{t(n)}$$
$$\geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}$$

Proving $\Pr_{x \leftarrow \{0,1\}^n | y=f(x) \in \mathcal{S}_n}\left[\mathsf{B}(y) \in f^{-1}(y)\right] \geq \frac{1}{t(n)\cdot p(n)} - n^{-\log n}$, cont.

In the case that A is randomized, let

- A_r — A whose coins fixed to r
- $\alpha_r(n)$ — the inversion probability of A_r, for a uniform input for g

Note that $\mathsf{E}_r\left[\alpha_r(n)\right] \geq 1/p(n)$.

It follows that

$$\begin{aligned}
\Pr_{x \leftarrow \{0,1\}^n | y=f(x) \in \mathcal{S}_n}\left[\mathsf{B}(y) \in f^{-1}(y)\right] &\geq \mathop{\mathsf{E}}_r\left[\frac{\alpha_r(n)}{t(n)} - n^{-\log n}\right] \\
&= \mathop{\mathsf{E}}_r\left[\alpha_r(n)\right]/t(n) - n^{-\log n} \\
&\geq \frac{1}{t(n)\cdot p(n)} - n^{-\log n}.
\end{aligned}$$

Closing remarks

- Weak OWFs can be amplified into strong one

- Can we give a more security preserving amplification?
- Similar hardness amplification theorems for other cryptographic primitives (e.g., Captchas, general protocols)?
- What properties of the weak OWFs have we used in the proof?