LINEAR SYSTEMS WITH LARGE OFF-DIAGONAL ELEMENTS AND DISCONTINUOUS COEFFICIENTS

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SIAM Conf. on Applied Linear Algebra, Oct. 2009.

OUTLINE

- Linear systems with large off-diagonal elements
- Linear systems with discontinuous coefficients
- Large off-diagonal elements *and* discontinuous coefficients
- Sample results
- Conclusions

LARGE OFF-DIAGONAL ELEMENTS

Some examples:

1. Convection-diffusion(-reaction) PDEs with LARGE convection terms
2. Helmholtz equation with LARGE wave numbers
3. Some circuit problems

Simple solution methods:

- Scaling and/or reordering – may leave some large off-diagonal elements
- Robust but not efficient: Kaczmarz (sequential), and the block-parallel CARP (G&G 2005), type 1 problems
- Robust and efficient results: CGMN algorithm (Björck & Elfving 1979), type 1 problems (G&G 2008)
- Parallel setting: similarly good results with CARP-CG (G&G 2009)
- Good preliminary results also on type 2 problems

L_2-SCALING AND THE NORMAL EQUATIONS

- Given a system $\boxed{Ax = b}$, let $G = \text{diag}(1/\|a_i\|_2)$
- Consider the system $\boxed{GAx = Gb}$ – call it GRS preconditioning
- GRS = Geometric Row Scaling – iterative results depend only on the hyperplanes and not any particular algebraic representation
- Denote $C = GA, \; b' = Gb \quad \Longrightarrow \quad \boxed{Cx = b'}$
- Consider the normal system $\boxed{CC^T y = b', \; x = C^T y}$
- Theorem: Let $D = CC^T$, then $|d_{ii}| = 1$, and $|d_{ij}| < 1$ for $i \neq j$
- Proof: immediate (assuming A is full rank)
- $\boxed{\text{Significance: A method to control large off-diagonal elements}}$

GRS + NORMAL EQUATIONS ARE INHERENT IN:

- Kaczmarz (SOR on normal equations) – inherently sequential
- Cimmino – inherently parallel
- CGMN: CG acceleration of Kaczmarz (Björck & Elfving, 1979)
- CG-acceleration of Cimmino, equivalent to CGNR+GRS
- CARP: a block-parallel version of Kaczmarz (G&G 2005)
- CARP-CG: CG acceleration of CARP (G&G 2009)

DISCONTINUOUS COEFFICIENTS

- "Discontinuous coefficients" – linear systems with large differences between the coefficients of different equations
- Typically arise when PDEs model certain physical phenomena in heterogeneous media, e.g. flow through different materials
- Common approach: Domain Decomposition (DD)
- Problem: May be difficult to implement when:
  - Unstructured grid
  - Complicated boundaries between subdomains

ALTERNATIVE APPROACH: ROW AND/OR COLUMN SCALING

- Not always useful
- Theoretical results, e.g., van der Sluis 1969
- Widlund 1971: "well-scaled ADI methods give good rates of convergence when the coefficients of elliptic problems vary very much in magnitude"
- Duff & van der Vorst 1998: "on vector machines, diagonal scaling is often competitive with other approaches"
- Graham & Hagger 1999: "diagonal scaling has been observed in practical computations to be very effective as a preconditioner for problems with discontinuous coefficients"
- Gambolati et al. 2003: use the least square logarithm scaling on rows and columns for geomechanics problems with discontinuous coefficients

PREVIOUS WORK ON SCALING (G&G 2009):

- Problems considered: nonsymmetric systems w/discontinuous coefficients, from PDEs with small to moderate convection terms
- Considered L_2 scaling of the equations (GRS)
- Results:
  1. improved convergence behavior of Bi-CGSTAB and (restarted) GMRES, both with and without ILU(0)
  2. Improved eigenvalue distribution (large concentration "pushed" away from origin)
  3. Degradation of usefulness of GRS as the convection was increased
- Note: L_1 scaling produced similar results

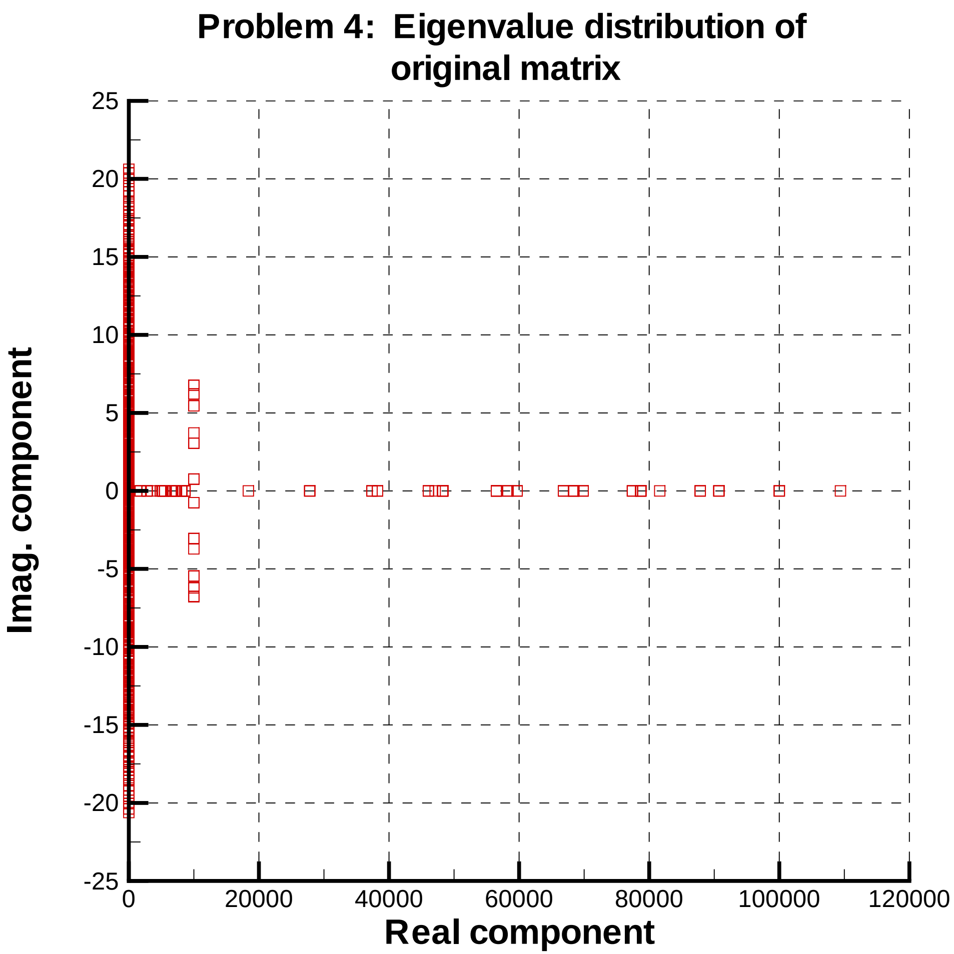
PROBLEM 1

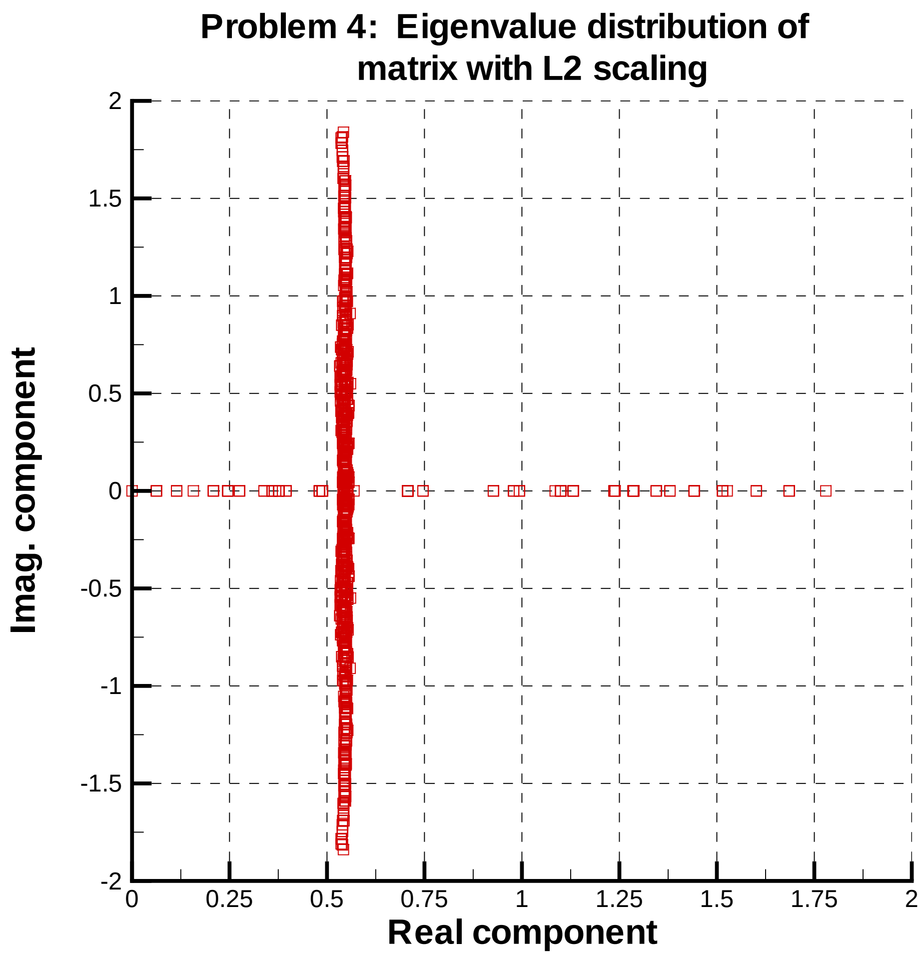
- Based on a 3D (symmetric) problem of Graham & Hagger 1999
- Original problem solved by DD techniques
- We added convection terms to make it nonsymmetric
- PDE: $$-\frac{\partial}{\partial x}(au_x) - \frac{\partial}{\partial y}(au_y) - \frac{\partial}{\partial z}(au_z) + du_x + eu_y + fu_z \;=\; 0$$

  where $a(x,y,z) \;=\; \begin{cases} D & \text{if } \frac{1}{3} < x,\ y,\ z < \frac{2}{3}, \\ 1 & \text{otherwise} \end{cases}$ $\quad D = 10^4$ and $D = 10^6$

- Domain: $[0,1]^3$
- Convection terms: $d = e = f = 100$
- Dirichlet boundary conditions ($u = 1$ on $z = 0$, $u = 0$ elsewhere)
- Discretizations: $40\times 40\times 40$ and $80\times 80\times 80$

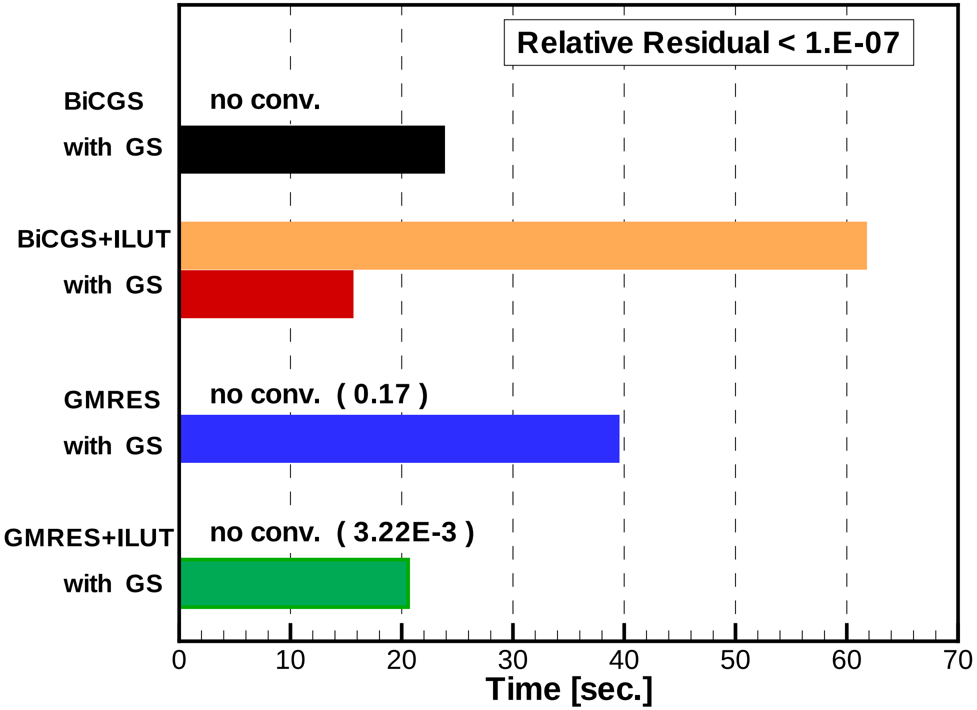
PROBLEM 1: EIGENVALUE DISTRIBUTION

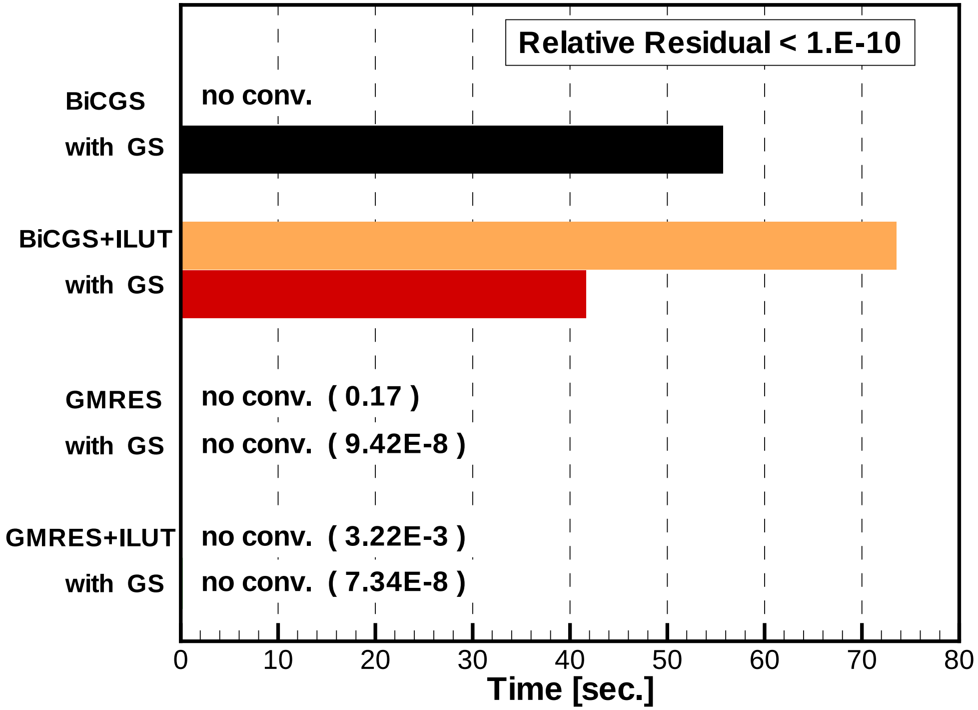




PROBLEM 1: RUNTIMES (SEC.)

$D = 10^6$, grid: $80 \times 80 \times 80$





PROBLEM 1: DEGRADATION WITH INCREASED CONVECTION

Method / Convection:	**100**	**200**	**500**	**1000**
Bi-CGSTAB	—	—	—	—
with GRS	1.2E-12	1.7E-11	3.4E-13	—
Bi-CGSTAB+ILU(0)	2.5E-14	3.2E-5	—	—
with GRS	2.5E-14	4.1E-5	—	—
GMRES	—	—	—	—
with GRS	1.8E-5	2.7E-5	5.6E-5	—
GMRES+ILU(0)	—	—	—	—
with GRS	1.2E-5	—	—	—
Notes: '—' means no convergence. Numbers are the best relative error obtained.				

PROBLEM 1: SOLUTIONS FOR LARGE CONVECTION

Convection:	**100**			**200**			**500**			**1000**		
Convergence goal:	10^{-4}	10^{-7}	10^{-10}	10^{-4}	10^{-7}	10^{-10}	10^{-4}	10^{-7}	10^{-10}	10^{-4}	10^{-7}	10^{-10}
Bi-CGSTAB+GRS	1.0	2.8	3.6	2.8	7.2	9.9	9.7	21.0	38.2	—	—	—
Bi-CGSTAB+ILU(0)	1.4	1.6	1.7	2.3	—	—	—	—	—	—	—	—
with GRS	**0.8**	**1.4**	**1.7**	**1.9**	—	—	—	—	—	—	—	—
GMRES+GRS	2.1	—	—	2.2	—	—	2.8	—	—	—	—	—
GMRES+ILU(0)+GRS	**0.8**	—	—	—	—	—	—	—	—	—	—	—
CGNR+GRS (note 2)	5.1	10.4	11.3	4.8	9.5	10.2	6.0	12.4	13.4	7.3	15.0	16.2
CGNR+GCS (note 3)	9.4	10.4	11.3	8.7	9.4	10.2	11.3	12.4	13.4	13.7	14.9	16.1
CGMN (note 4)	2.0	4.3	4.6	**1.9**	**4.6**	**5.0**	**2.0**	**5.8**	**6.3**	**2.5**	**6.9**	**7.6**

Notes:

1. Times in seconds; minimal times in boldface
2. CGNR+GRS is CG-accelerated Cimmino
3. GCS = L_2 Column Scaling
4. CGMN is also CARP-CG on one processor

PROBLEM 2

- A 3D problem of Gerardo-Giorda, Le Tallec and Nataf, 2004
- System(s) with large off-diagonal elements and discontinuous coefficients.
- Solved originally by DD techniques
- PDE: $-\text{div}(\nu(x)\nabla u) + \vec{b} \cdot \nabla u + u = 0$, with

$$\nu(x) \;=\; \begin{cases} 10^{-1} & \text{if } \; x < \frac{1}{2}, \\ 10^{-5} & \text{otherwise.} \end{cases}$$

- $\vec{b} = (1,0,0) \;/\; (0,1,1) \;/\; (1,3,5) \;/\; (0,1,0)$
- Domain: unit cube
- Dirichlet boundary conditions ($u = 1$ on $z = 0$, $u = 0$ elsewhere)
- Discretizations: $40 \times 40 \times 40$ and $80 \times 80 \times 80$

PROBLEM 2 – SUMMARY OF RESULTS

- When $\vec{b} = (1,0,0)$ or $(0,1,0)$ (only one large convection term):
  - Bi-CGSTAB+ILU(0) and GMRES+ILU(0) very efficient
  - GRS helped very little
  - CGMN relatively slow

- When $\vec{b} = (0,1,1)$ or $(1,3,5)$ (two or more large convection terms):
  - GMRES converged very slowly, GRS was quite useful
  - CGMN was very efficient

- Conclusion: the difficulties with large convection terms occur when there are more than two large off-diagonal elements (each convection term contributes two off-diagonal elements with second-order finite difference schemes)

SUMMARY

- Two tools for nonsymmetric systems with discontinuous coefficients:

- Small to moderate off-diagonal elements: L_2-scaling with your favorite algorithm/preconditioner

- Also: one or two large off-diagonal elements

- Three or more large off-diagonal elements: CGMN (sequential), CARP-CG (parallel), CGNR+GRS (CG-accelerated Cimmino, sequential or parallel)

SOME LITERATURE

- Å. Björck & T. Elfving. Accelerated projection methods for computing pseudoinverse solutions of systems of linear equations. *BIT, 19:145-163, 1979*.

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THANK YOU