Compact fourth order scheme for the elastic wave equation in the frequency domain using a first order formulation

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Abstract

We develop a compact fourth order scheme for the elastic wave equation in the frequency domain using a first order formulation of the equation. We use a 3D staggered grid and apply our minimalistic "Gradient Method" concept to the BGT absorbing boundary condition. The equations are solved in a partitioned domain using the block-parallel CARP-CG algorithm. The results are compared with the analytic solution.

Keywords: Elastic wave equation, frequency domain, first order formulation, compact scheme, fourth order accuracy, gradient method absorbing boundary conditions.

1 The compact fourth order scheme

The elastic wave equation in the frequency domain can be expressed as a first order system of velocities and stresses. The stresses are denoted by σ_{ij} and form a symmetric matrix, so there are only six different values of σ_{ij}. For convenience, we denote $D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$. Then the system (including a forcing term) is given by the following nine equations:

$$
\begin{aligned}
-\rho i\omega u &= \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{13}}{\partial z} + F^u \\
-\rho i\omega v &= \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + \frac{\partial \sigma_{23}}{\partial z} + F^v \\
-\rho i\omega w &= \frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial y} + \frac{\partial \sigma_{33}}{\partial z} + F^w \\
-\rho i\omega \sigma_{11} &= \lambda D + 2\mu \frac{\partial u}{\partial x} \\
-\rho i\omega \sigma_{22} &= \lambda D + 2\mu \frac{\partial v}{\partial y} \\
-\rho i\omega \sigma_{33} &= \lambda D + 2\mu \frac{\partial w}{\partial z} \\
-\rho i\omega \sigma_{12} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
-\rho i\omega \sigma_{13} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\
-\rho i\omega \sigma_{23} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)
\end{aligned}
$$

$u(x,y,z), v(x,y,z), w(x,y,z)$ are the displacements in the x,y,z directions, respectively, and $F(x,y,z) = (F^x, F^y, F^z)$ is the vector of displacements in the x,y,z directions at a point of the domain. Additional parameters: $\omega = 2\pi f$, where f is the frequency, λ and μ are the Lamé parameters and ρ is the density (which are all assumed to be constant),. The elastic equations give rise to two wave speeds, the compression or P-wave v_p, and the shear or S-wave v_s, given by

$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad v_s = \sqrt{\frac{\mu}{\rho}}.$$

The wave numbers associated with the two wave speeds are $k_p = \omega/v_p$ and $k_s = \omega/v_s$.

2 Discretization

We use the standard discretization notations on a staggered grid, but we forgo the development steps of the discretization. This yields

$$u_{xxx} = -i\omega\left(\frac{\sigma_{11}}{\lambda+2\mu}\right)_{xx} - \left(\frac{\lambda v_y}{\lambda+2\mu}\right)_{xx} - \left(\frac{\lambda w_z}{\lambda+2\mu}\right)_{xx}$$

with corresponding expressions for v_{yyy} and w_{zzz}. We also obtain appropriate 4th order expressions for σ_{ij}, which depend on these third order derivatives. The derivatives in the $O(h^2)$ terms are replaced by second order central differences. These can all be computed on a compact $3\times3\times3$ stencil.

3 Absorbing boundary conditions

The absorbing boundary conditions (ABC) of [1] (BGT) was originally developed for a sphere and used radial derivatives (which are also normal to the boundary direction). We made the following adaptation of BGT for the elastic case, which we call BGTE:

$$\left(\frac{\partial}{\partial r} + \frac{1}{r} - ik_p\right)\left(\frac{\partial}{\partial r} + \frac{1}{r} - ik_s\right)u = 0,$$

where r is the distance from the source and the derivatives are in the radial direction.

In [3] we developed the "Gradient Method" concept for ABCs, which is based on the principle that the directional derivatives in any ABC should be in the direction of the *gradient* of the wavefront, without regard to the orientation of the boundary. According to this concept, BGT and BGTE can be used in any convex domain with an interior source. The advantage of this approach is that the ABCs take up just one extra grid point on each side, and this is big advantage over PMLs.

4 Preliminary results

We use a well-known example from the literature, see, for example, [5]. The domain is of size 2000^3 meters, with a source at the center, discretized by 142^3 grid points (including the extra points for the ABC). Since the Green's function of the solution is known, the source of impact was simulated by placing a small cube at the center with values of the Green's function. We made tests with the BGTE ABC and also with Dirichlet BC (with boundary values from the Green's function). The density was $\rho = 1000$ Kg/m^3, and the frequency was $f = 10$. The acoustic case and the elastic case were tested, with parameters $v_p = 2500$ m/s, $v_s = 0$ m/s for the acoustic case, and $v_p = 5000$ m/s, $v_s = 2500$ m/s for the elastic case.

The acoustic case was solved with GMRES, with a restart of 20. For the elastic case, we used the CARP-CG algorithm [2], which has been shown to be especially useful on strongly convection-dominated PDEs and the Helmholtz equation at high frequencies. CARP-CG was used in various wave problems e.g. [3–5].

Fig. 1 shows sample plots for the acoustic and elastic cases. Each plot compares the Green's function with the solutions with the ABC and with Dirichlet boundary conditions. The acoustic case is the Helmholtz equation for the 3 stresses σ_{ii}. The good correspondence between the plots is in line with our previous Helmholtz results. The results in the elastic case are not as good, and work on improving the ABC for this case is still in progress.

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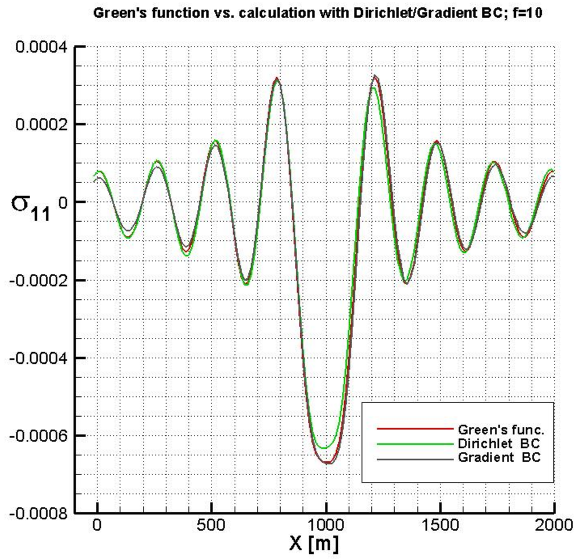

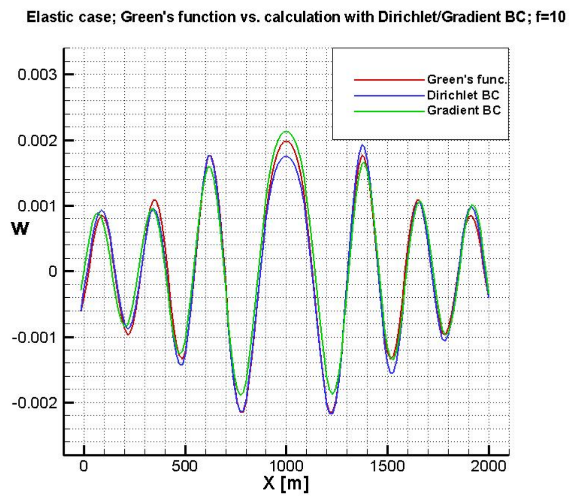

Figure 1: Comparison of the Green's function with the solutions obtained with Dirichlet boundary condition and the ABC. Top: the acoustic case. Bottom: the elastic case.